Title: 2+1D topological orders and braided fusion category

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Abstract:

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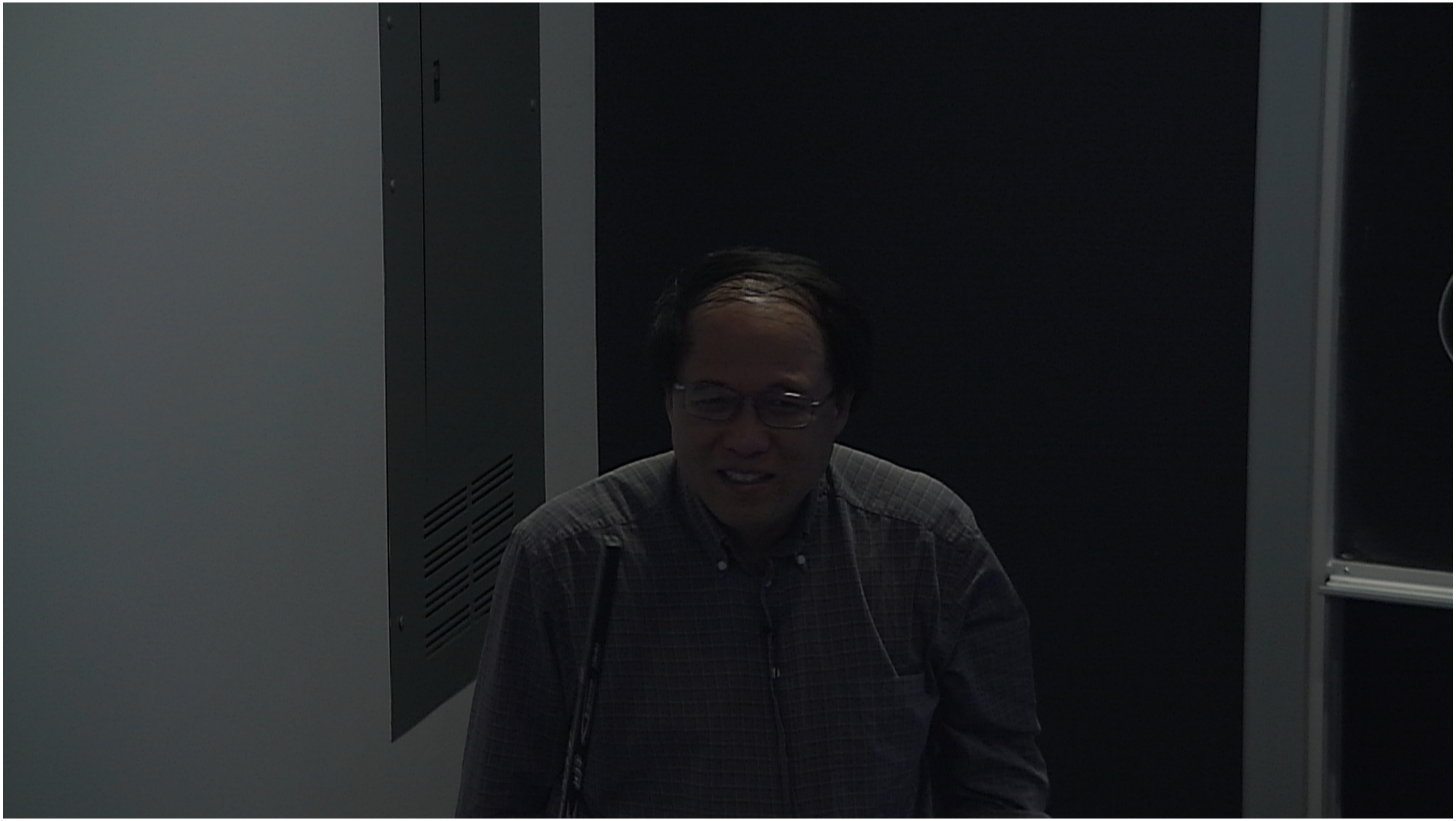
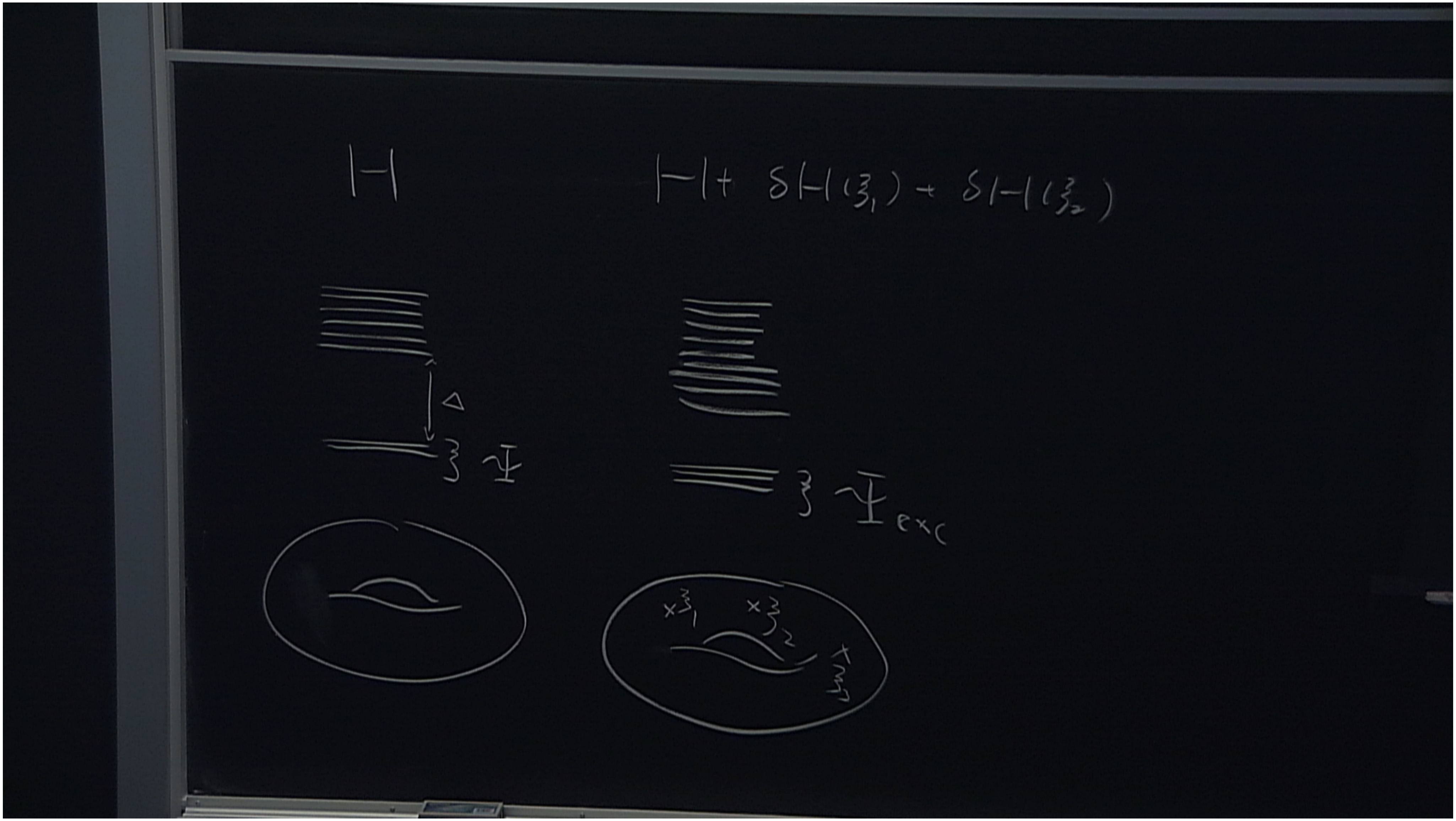
Xiao-Gang Wen
Oct., 2015

There are gapped phases beyond symmetry breaking order
→ **topological orders**




H
H + δH(ξ1) + δH(ξ2)
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ξ Ψexc
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→ **topological orders**

Examples of topo. orders – long range entanglement (LRE)

Abelian topological order: $\to$ fractional statistics $N \to \infty$

- IQH and Laughlin state Laughlin PRL **50** 1395 (1983)

$$\Psi^F_{\nu=1} = \prod_{1\le i<j\le N}(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \quad \Psi^{F,B}_{\nu=1/m} = \prod(z_i - z_j)^m \mathrm{e}^{-\frac{m}{4}\sum |z_i|^2} = (\Psi^F_{\nu=1})^m$$

where $z_i = x_i + \mathrm{i}\, y_i$.

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$$\in \mathcal{V}_{N=0} \oplus \mathcal{V}_{N=1} \cdots = \otimes_z \mathcal{V}_z^{\{|0\rangle,|1\rangle\}} \qquad = (\Psi^F_{\nu=1})^m$$

where $z_i = x_i + \mathrm{i}\, y_i$. $\quad \Psi^{F,IQH}_{\nu=n}$ *has an invertible topological order*

- $\Psi^{B/F}$ symmetric/anti-symmetric $\to$ bosonic/fermionic topo. order; Wen PRB 40, 7387 (89)

Non-abelian topological order: $\to$ non-abelian statistics

- $SU(N)_2$ state via slave-particle — Wen PRL **66** 802 (Feb. 1991)

$$\Psi^B_{SU(2)_2} = (\Psi^F_{\nu=2})^2,\ \nu = 1; \quad \Psi^F_{SU(3)_2} = (\Psi^F_{\nu=2})^3,\ \nu = \frac{2}{3};$$

$\to SU(N)_2$ Chern-Simons effective theory $\to$ non-abelian statistics

- Pfaffien state via CFT — Moore-Read NPB **360** 362 (Aug. 1991)

$$\Psi^B_{Pfa} = \mathcal{A}\Big[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\Big]\prod(z_i - z_j)\,\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \qquad \nu = 1$$

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- Given vector Ψ_1, $\exists$ other LI $\Psi_2, \cdots$.
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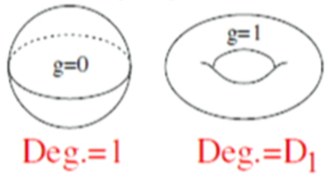

Wen PRB 40, 7387 (89), Wen-Niu PRB 41, 9377 (90)

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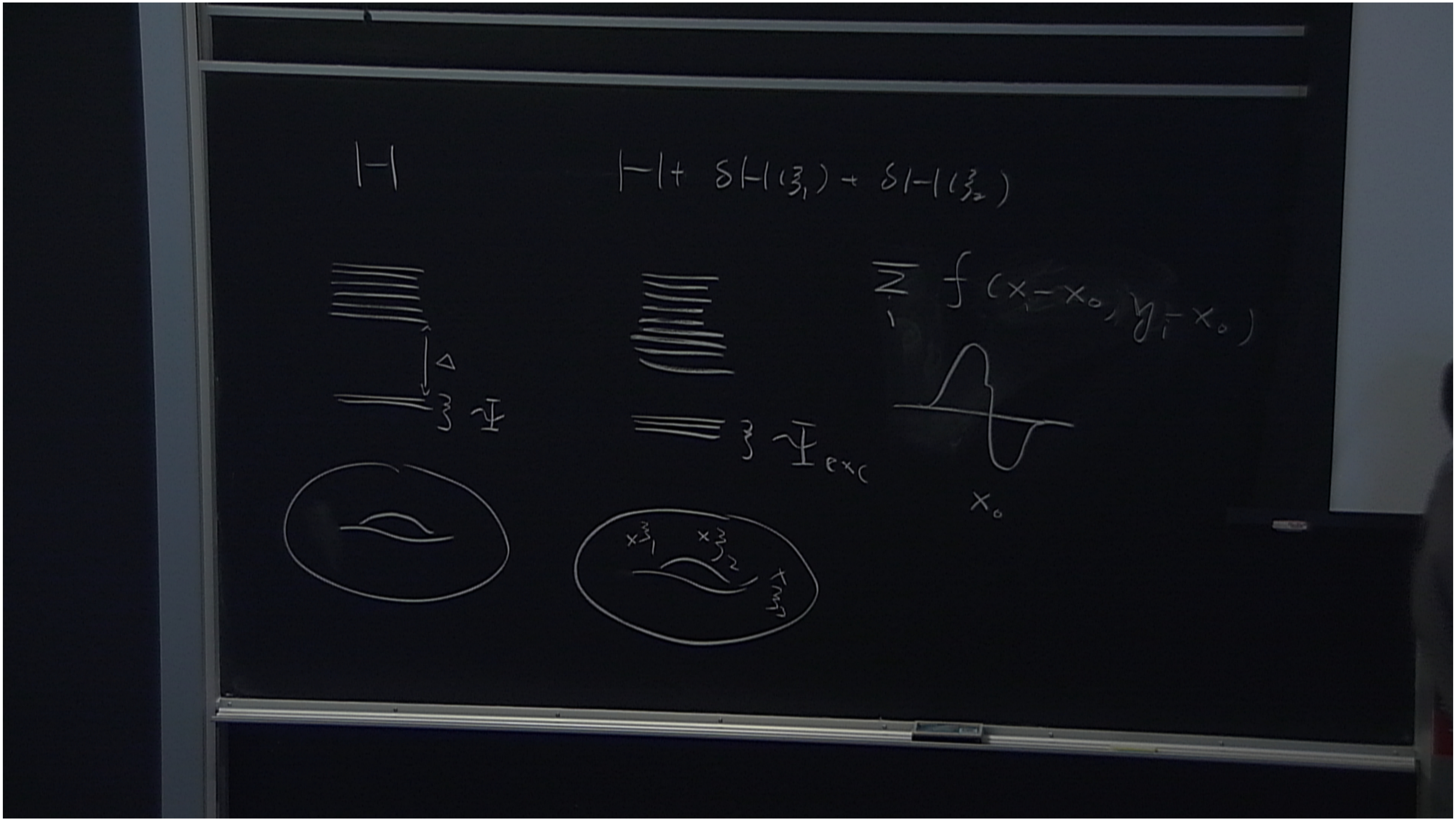
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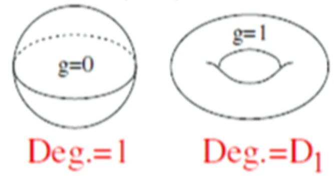

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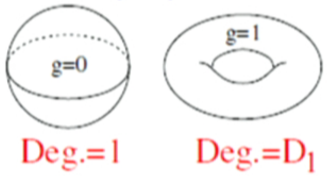

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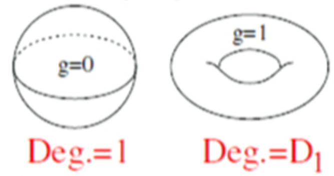

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- Loops $\pi_1(\mathcal{M}) = SL(2,\mathbb{Z})$: 90° rotation $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$

Dehn twist: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$

S, T generate a rep. of modular group: $S^2 = (ST)^3 = C, C^2 = 1$

Wen IJMPB 4, 239 (90); KeskiVakkuri-Wen IJMPB 7, 4227 (93)

Classify 2+1D topo. orders (*ie* patterns of entanglement)

via the topological invariants (S, T, c)

- A 2+1D topological order $\to$ a (S, T, c)
- An arbitary (S, T, c) $\not\to$ a 2+1D topological order

- (S, T, c)'s satisfying **a set of conditions** $\leftrightarrow$ 2+1D topo. orders

Bosonic invertible topological orders

invertible topological orders have no topological bluk excitation (particle-like, string-like, ...), but have non-trivial boundary with **gauge/gravitaional anomalies**. $(S, T, c) = (1, 1, c)$.

- SPT orders $\subset$ invertible topological orders with symmetry G.
- Construct SPT orders or more general invertible topological orders:

$$S = \int \mathrm{d}^d x \ \lambda |(\partial + \mathrm{i}A + \mathrm{i}\Gamma)g|^2 + \mathrm{i}2\pi \int W[(\partial + \mathrm{i}A + \mathrm{i}\Gamma)g]$$

$$g \in G \times SO(\infty)$$

$\rightarrow$ SPT orders are labeled by $\mathcal{H}^d[G \times SO(\infty), U(1)]/\sim$.

$G \setminus d =$	0+1	1+1	2+1	3+1	4+1	5+1	6+1
iTO^d	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	0
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0	$\mathbb{Z}_n \oplus \mathbb{Z}_n$	$\mathbb{Z}_{\langle n,2\rangle}$	$\mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_{\langle n,2\rangle}$
Z_2^T	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$
$U(1) \rtimes Z_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\mathbb{Z} \oplus 2\mathbb{Z}_2 \oplus \mathbb{Z} \oplus 2\mathbb{Z}_2$
$U(1) \times Z_2^T$	0	$2\mathbb{Z}_2$	0	$3\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$4\mathbb{Z}_2 \oplus 3\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus 3\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Wen arXiv:1410.8477

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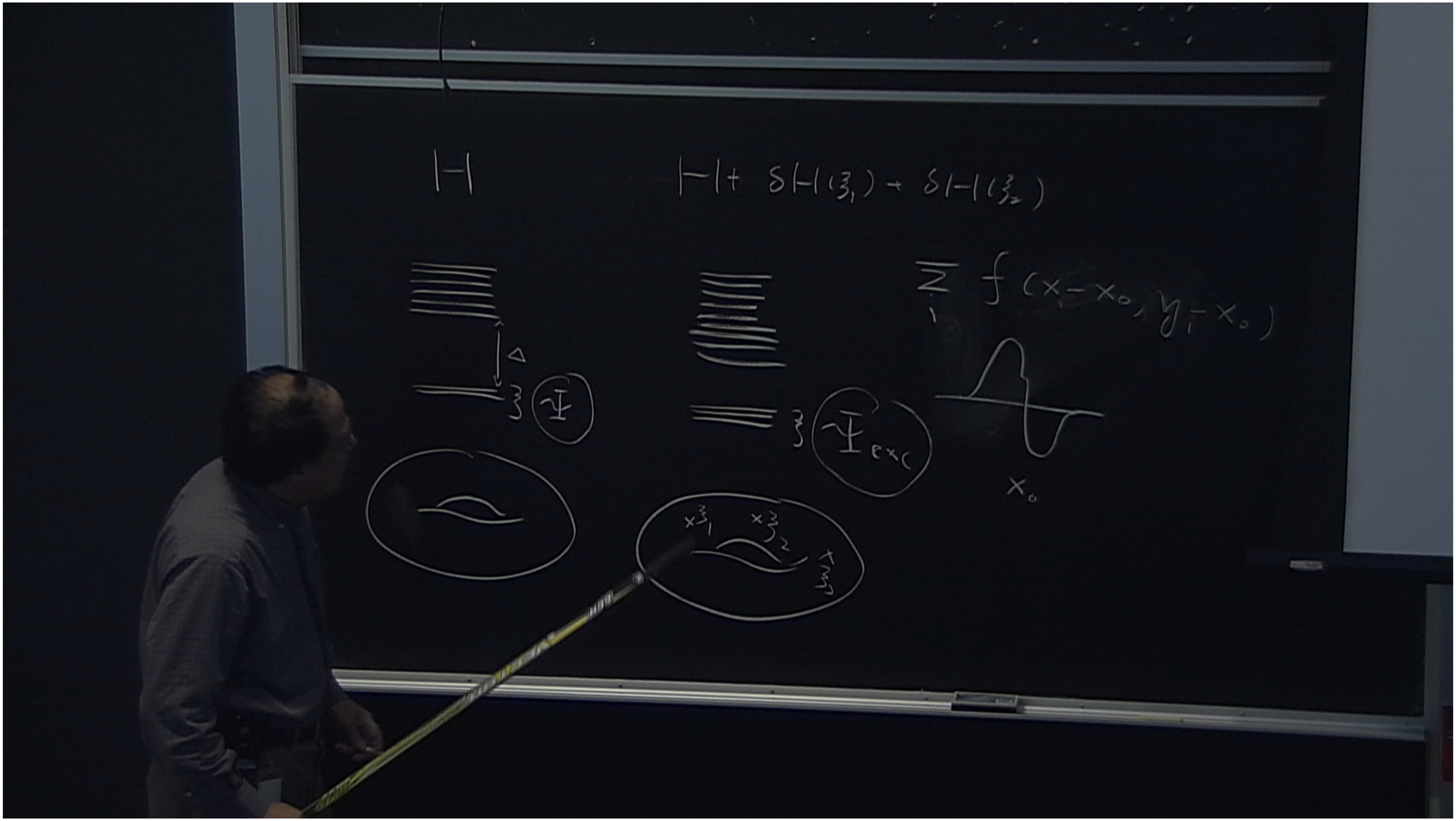
- (S, T, c)'s satisfying **a set of conditions** $\leftrightarrow$ 2+1D topo. orders assuming each $(S, T, c) \to$ one topological order, otherwise (S, T, c)'s satisfying **a set of conditions** $\leftrightarrow$ several topo. orders

- How to find the conditions?
  Study topological excitations above the ground states
  $\to$ unitary modular tensor category theory (UMTC)

Theory of topological excitations = category theory

- Local excitations: 1) Ψ_{exc} and Ψ are LI except near a few points ξ_I.
  2) Ψ_{exc} = ground-state subspace of $H_{\text{trap}} = H + \delta H_{\xi_1} + \delta H_{\xi_2} \cdots$.
  Ex. $\Psi = \prod(z_i - z_j)^m$, $\Psi_{\text{exc}} = \prod_{i,I}(z_i - \xi_I)\prod(z_i - z_j)^m \in \otimes_z \mathcal{V}_z$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- **Trivial excitation**: can be created by local operators
  $O(\xi)\Psi \subset \Psi_{\text{exc}} : \Psi \to \Psi_{\text{exc}}$. $O'(\xi)\Psi_{\text{exc}} \subset \Psi : \Psi_{\text{exc}} \to \Psi$. $O(\xi)$ acts on $\mathcal{V}_\xi$
  "$\to$" (contained in) = morphism in category. $\Psi_{\text{trivial exc}} \leftrightarrow \Psi$
- **Topological excitation** if cannot be created by local operators
  (or more precisely, $\Psi_{\text{topo. exc}}(\xi_1, \xi_2) \not\to \Psi$, $\Psi \not\to \Psi_{\text{topo. exc}}(\xi_1, \xi_2)$)

Theory of topological excitations = category theory

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- **Topological type** = equivalence class of Ψ_{exc}: $\Psi_{\text{exc}} \sim \Psi'_{\text{exc}}$ iff
  $\Psi_{\text{exc}} \to \Psi'_{\text{exc}}$ and $\Psi'_{\text{exc}} \to \Psi_{\text{exc}}$ *isomorphic in category*

- **simple type**: $\Psi_{\text{exc}} \to \Psi^{\text{simple}}_{\text{exc}}$ implies $\Psi^{\text{simple}}_{\text{exc}} \to \Psi_{\text{exc}}$
- **composite type**: $k = i \oplus j$, $i \to k$, $j \to k$. *Accidental degeneracy*

- Topo. excitations (topo. types) $\leftrightarrow$ objects in category $\mathcal{C}$:
  A category $\mathcal{C}$ = a set of topo. exc.: $\mathcal{C} = \{i\}_{\text{simple}} + \{i \oplus j, \cdots\}$
  Example: $\mathcal{C} = \{\textit{spin-0}, \textit{spin-1}, ..\}_{\textit{simple}} + \{\textit{spin-1} \oplus \textit{spin-2}, \cdots\}$

Theory of topological excitations = tensor category theory

- Topological excitations can fuse (form bound states) →
  { Topological excitations } ↔ A tensor (fusion) category $\mathcal{C}$.

Data to describe fusion of simple types:

- $i \otimes j = k$ → maps $\{i\}_{\text{simple}} \times \{i\}_{\text{simple}} \to \{i\}_{\text{simple}}$
  $\to k_{ij} \in \{i\}_{\text{simple}}, \ \forall i, j \in \{i\}_{\text{simple}}$. **Wrong!**
- Bound state of simple types i, j may correspond to serveral k's with accidental degeneracy: $i \otimes j = k_1 \oplus k_1 \oplus k_2 \oplus k_3 \oplus \cdots = \oplus_k N_k^{ij} k$
  Ex. for H with $SO(3)$: spin-1 ⊗ spin-1 = spin-0 ⊕ spin-1 ⊕ spin-2
- Associativity condition:
  $(i \otimes j) \times k = i \otimes (j \otimes k) \to \sum_m N_m^{ij} N_l^{mk} = \sum_m N_l^{im} N_m^{jk}$

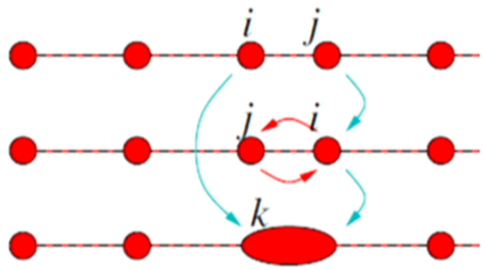
N_k^{ij} **are the data to describe fusion of the tensor category.**

N_k^{ij} = topological inv. → fusion ring of a tensor category.

N_k^{ij} → quantum dimension d_i for topological type-i.

Theory of topological excitations = braided fusion category

- Particles can also braid → unitary braided fusion category
- Braiding requires that $N^{ij}_k = N^{ji}_k$.

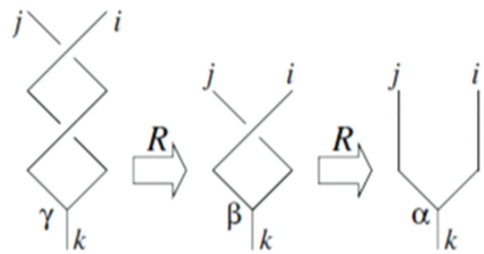


- Braiding → **mutual statistics** $e^{i\theta^{(k)}_{ij}}$ and non-trivial **spin** s_i

  2π rotation of (i,j) = 2π rotation of k

  2π rotation of (i,j) = 2π rotation of i and j and exchange i,j twice

$$e^{i2\pi s_i} e^{i2\pi s_j} e^{i\theta^{(k)}_{ij}} = e^{i2\pi s_k}$$



A unitary braided fusion category (UBFC) is a set of quasiparticles with fusion and braiding, which is described by data (N^{ij}_k, s_i)

Relation between (S, T, c) and (N^{ij}_k, s_i, c)

Conjecture: **A bosonic topological order [*ie* a non-degenerate UBFC $\equiv$ an unitary modular tensor category (UMTC)] is fully characterized by data (S, T, c) or by data (N^{ij}_k, s_i, c) .**

- From (S, T, c) to (N^{ij}_k, s_i, c): Verlinde formula

$$N^{ij}_k = \sum_l \frac{S_{li} S_{lj} (S_{lk})^*}{S_{1l}}, \qquad \mathrm{e}^{\mathrm{i} 2\pi s_i} \mathrm{e}^{-\mathrm{i} 2\pi \frac{c}{24}} = T_{ii}.$$

- From (N^{ij}_k, s_i, c) to (S, T, c):

$$S_{ij} = \frac{1}{\sqrt{\sum_i d_i^2}} \sum_k N^{ij}_k \mathrm{e}^{2\pi \mathrm{i} (s_i + s_j - s_k)} d_k, \qquad T_{ii}. = \mathrm{e}^{\mathrm{i} 2\pi s_i} \mathrm{e}^{-\mathrm{i} 2\pi \frac{c}{24}}$$

Conditions on $(N^{ij}_k, s_i, c) \leftrightarrow$ **Conditions on** (S, T, c)
$\rightarrow$ **A theory of unitary modular tensor category (UMTC)**

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simplified theory of UMTC Rowell-Stong-Wang arXiv:0712.1377

- The standard point of view:
UMTC's are fully characterized by $(N^{ij}_k, F^{ijm;\alpha\beta}_{kln;\gamma\lambda}, R^{ij;\alpha}_{k;\beta})$ (but not one-to-one). Conditions on those data + the equivalent relations $\rightarrow$ a theory of UMTC. *hard to work with*

A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

- **Fusion ring**: N^{ij}_k are non-negative integers that satisfy

$$N^{ij}_k = N^{ji}_k, \quad N^{1i}_j = \delta_{ij}, \quad \sum_{k=1}^{N} N^{ik}_1 N^{kj}_1 = \delta_{ij},$$

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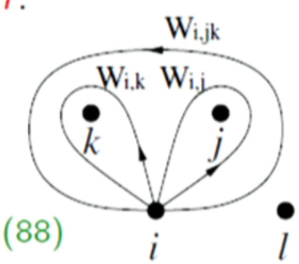

We refer N as the rank.

- **Vafa's theorem**: N^{ij}_k and s_i satisfy Vafa PLB 206, 421 (88)

$$\det(W_{i,j}) \det(W_{i,k}) = \det(W_{i,jk}) \quad \to \quad \sum_r V^r_{ijkl} s_r = 0 \bmod 1$$

$$V^r_{ijkl} = N^{ij}_r N^{kl}_{\bar{r}} + N^{il}_r N^{jk}_{\bar{r}} + N^{ik}_r N^{jl}_{\bar{r}} - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N^{ij}_m N^{kl}_{\bar{m}}$$

2+1D bosonic topo. orders (up to E_8-states) via (N^{ij}_k, s_i, c)

$\zeta_n^m = \frac{\sin(\pi(m+1)/(n+2))}{\sin(\pi/(n+2))}$ Rowell-Stong-Wang arXiv:0712.1377; Wen arXiv:1506.05768

N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.	N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.
1_1^B	1	0					
2_1^B	$1, 1$	$0, \frac{1}{4}$	$\prod(z_i - z_j)^2$	2_{-1}^B	$1, 1$	$0, -\frac{1}{4}$	$\prod(z_i^* - z_j^*)^2$
$2_{14/5}^B$	$1, \zeta_3^1$	$0, \frac{2}{5}$		$2_{-14/5}^B$	$1, \zeta_3^1$	$0, -\frac{2}{5}$	
3_2^B	$1, 1, 1$	$0, \frac{1}{3}, \frac{1}{3}$	(221) double-layer	3_{-2}^B	$1, 1, 1$	$0, -\frac{1}{3}, -\frac{1}{3}$	
$3_{8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, -\frac{1}{7}, \frac{2}{7}$		$3_{-8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, \frac{1}{7}, -\frac{2}{7}$	
$3_{1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{1}{16}$		$3_{-1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{1}{16}$	
$3_{3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{3}{16}$	Ψ_{Pfaffian}	$3_{-3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{3}{16}$	
$3_{5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{5}{16}$	$\Psi^2_{\nu=2}$ $SU(2)_2$	$3_{-5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{5}{16}$	
$3_{7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{7}{16}$		$3_{-7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{7}{16}$	
$4_0^{B,a}$	$1, 1, 1, 1$	$0, 0, 0, \frac{1}{2}$	Z_2 gauge	4_4^B	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	
4_1^B	$1, 1, 1, 1$	$0, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$	$\prod(z_i - z_j)^4$	4_{-1}^B	$1, 1, 1, 1$	$0, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{2}$	
4_2^B	$1, 1, 1, 1$	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	(220) double-layer	4_{-2}^B	$1, 1, 1, 1$	$0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}$	
4_3^B	$1, 1, 1, 1$	$0, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}$		4_{-3}^B	$1, 1, 1, 1$	$0, -\frac{3}{8}, -\frac{3}{8}, \frac{1}{2}$	
$4_0^{B,b}$	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, -\frac{1}{4}$	double semion	$4_{9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{3}{20}, \frac{2}{5}$	
$4_{-9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{3}{20}, -\frac{2}{5}$		$4_{19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{7}{20}, \frac{2}{5}$	
$4_{-19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{7}{20}, -\frac{2}{5}$	$\Psi^2_{\nu=3}$ $SU(2)_3$	$4_0^{B,c}$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, -\frac{2}{5}, 0$	Fibonacci^2
$4_{12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, -\frac{2}{5}, -\frac{2}{5}, \frac{1}{5}$		$4_{-12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{1}{5}$	
$4_{10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, \frac{1}{3}, \frac{2}{9}, -\frac{1}{3}$		$4_{-10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, -\frac{1}{3}, -\frac{2}{9}, \frac{1}{3}$	
5_0^B	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	(223) DL	5_4^B	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	
$5_2^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$		$5_2^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	
$5_{-2}^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$		$5_{-2}^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$	
$5_{16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$		$5_{-16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	
$5_{18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$		$5_{-18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	

Remote detectability: why those (N^{ij}_k, s_i, c) are realizable

- The list cover all the 2+1D bosonic topological orders. But the list might contain fake entries that are not realizable. Schoutens-Wen arXiv:1508.01111 used simple current algebra to construct many-body wave functions for all the entries in the list.
  All the topological order in the table can be realized in multilayer FQH systems

A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

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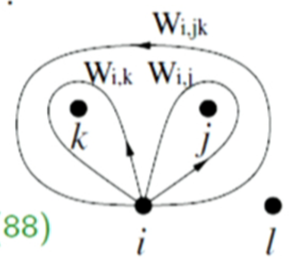

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A simplified theory of UMTC based on (N^{ij}_k, s_i, c)

From $(N^{ij}_k, s_i, c) \to (S, T)$

- Let d_i be the largest eigenvalue of the matrix $\mathbf{N}_i$. Let

$$S_{ij} = \frac{1}{D} \sum_k N^{ij}_k \mathrm{e}^{2\pi \mathrm{i} (s_i + s_j - s_k)} d_k, \quad D^2 = \sum_i d_i^2.$$

Then, S satisfies

$$S_{11} > 0, \quad \sum_k S_{kl} N^{ij}_k = \frac{S_{li} S_{lj}}{S_{1l}}, \quad S = S^\dagger C, \quad C_{ij} \equiv N^{ij}_1.$$

- Let $T_{ij}. = \mathrm{e}^{\mathrm{i} 2\pi s_i} \mathrm{e}^{-\mathrm{i} 2\pi \frac{c}{24}} \delta_{ij}$ then (rep. of modular group $SL(2, \mathbb{Z})$)

$$S^2 = (ST)^3 = C.$$

- Let $\nu_i = \frac{1}{D^2} \sum_{jk} N^{jk}_i d_j d_k \mathrm{e}^{4\pi \mathrm{i} (s_j - s_k)}$. Then $\nu_i = 0$ if $i \neq \bar{i}$, and $\nu_i = \pm 1$ if $i = \bar{i}$.

Rowell-Stong-Wang arXiv:0712.1377

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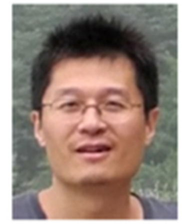

  Levin arXiv:1301.7355, Kong-Wen arXiv:1405.5858

- **Remote detectable = Realizable (anomaly-free)**:
  Every non-trivial topo. excitation i can be remotely detected by at least one other topo. excitation j via the non-zero mutual braiding $\theta^{(k)}_{ij} \neq 0 \to S_{ij} = \frac{1}{D}\sum_k N^{ij}_k \mathrm{e}^{-\mathrm{i}\theta^{(k)}_{ij}} d_k$ is unitary (one of conditions) $\to$ the topological order is realizable in the same dimension.
- The M-center of BFC $\mathcal{C}$ = the set of particles with trivial mutual statistics respecting to all others: $Z_M(\mathcal{C}) \equiv \{i \mid \theta^{(k)}_{ij} = 0, \ \forall j, k\}$.
  Remote detectable $\leftrightarrow Z_M(\mathcal{C}) = \{1\} \leftrightarrow$ Realizable (anomaly-free)

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Rowell-Stong-Wang arXiv:0712.1377

2+1D fermionic topo. orders (up to $p + \mathrm{i}p$) via (N^{ij}_k, s_i, c)

Classified by **UBFC's with M-center** $\{1, f\}$.

Lan-Kong-Wen arXiv:1507.04673


$N^F_c\binom{\vert\Theta_2\vert}{\angle\Theta_2/2\pi}$	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	com
$2^F_0\binom{\zeta_2^1}{0}$	2	$1, 1$	$0, \frac{1}{2}$	trivial $\mathcal{F}_0$
$4^F_0\binom{0}{0}$	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 2^B_1\binom{0}{0}$, $K = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$
$4^F_{1/5}\binom{\zeta_2^1\zeta_3^1}{3/20}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2^B_{-14/5}\binom{\zeta_3^1}{3/20}$
$4^F_{-1/5}\binom{\zeta_2^1\zeta_3^1}{-3/20}$	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2^B_{14/5}\binom{\zeta_3^1}{-3/20}$
$4^F_{1/4}\binom{\zeta_6^3}{1/2}$	13.656	$1, 1, \zeta_6^2, \zeta_6^2 = 1 + \sqrt{2}$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_{(A_1, 6)}$
$6^F_0\binom{\zeta_2^1}{1/4}$	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3^B_{-2}\binom{1}{1/4}$, $K = (3)$, $\Psi_{1/3}(z_i)$
$6^F_0\binom{\zeta_2^1}{-1/4}$	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3^B_2\binom{1}{-1/4}$, $K = (-3)$, $\Psi^*_{1/3}(z_i)$
$6^F_0\binom{\zeta_6^3}{1/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1 = \sqrt{2}$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{1/2}\binom{\zeta_6^1}{1/16}$, $\mathcal{F}_{U(1)_2/\mathbb{Z}_2}$
$6^F_0\binom{\zeta_6^3}{-1/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{16}, \frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{-1/2}\binom{\zeta_6^1}{-1/16}$
$6^F_0\binom{1.0823}{3/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{3/2}\binom{0.7653}{3/16}$
$6^F_0\binom{1.0823}{-3/16}$	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{16}, \frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3^B_{-3/2}\binom{0.7653}{-3/16}$
$6^F_{1/7}\binom{\zeta_2^1\zeta_5^2}{-5/14}$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, \frac{5}{14}, -\frac{1}{7}, -\frac{3}{14}, \frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3^B_{8/7}\binom{\zeta_5^2}{-5/14}$
$6^F_{-1/7}\binom{\zeta_2^1\zeta_5^2}{5/14}$	18.591	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, -\frac{5}{14}, \frac{1}{7}, \frac{3}{14}, -\frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3^B_{-8/7}\binom{\zeta_5^2}{5/14}$
$6^F_0\binom{2\zeta_{10}^1}{-1/12}$	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, 0, \frac{1}{2}$	$\mathcal{F}_{(A_1, -10)}$
$6^F_0\binom{2\zeta_{10}^1}{1/12}$	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, -\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{2}$	$\mathcal{F}_{(A_1, 10)}$