Title: TQFTs in Nature and Topological Quantum Computation

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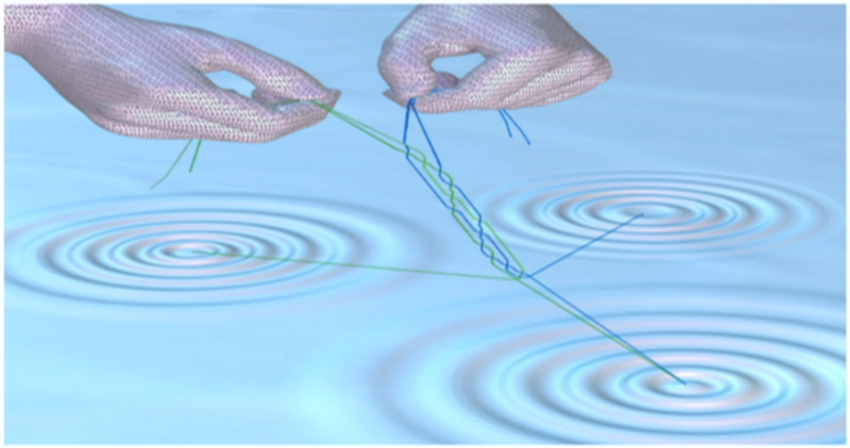
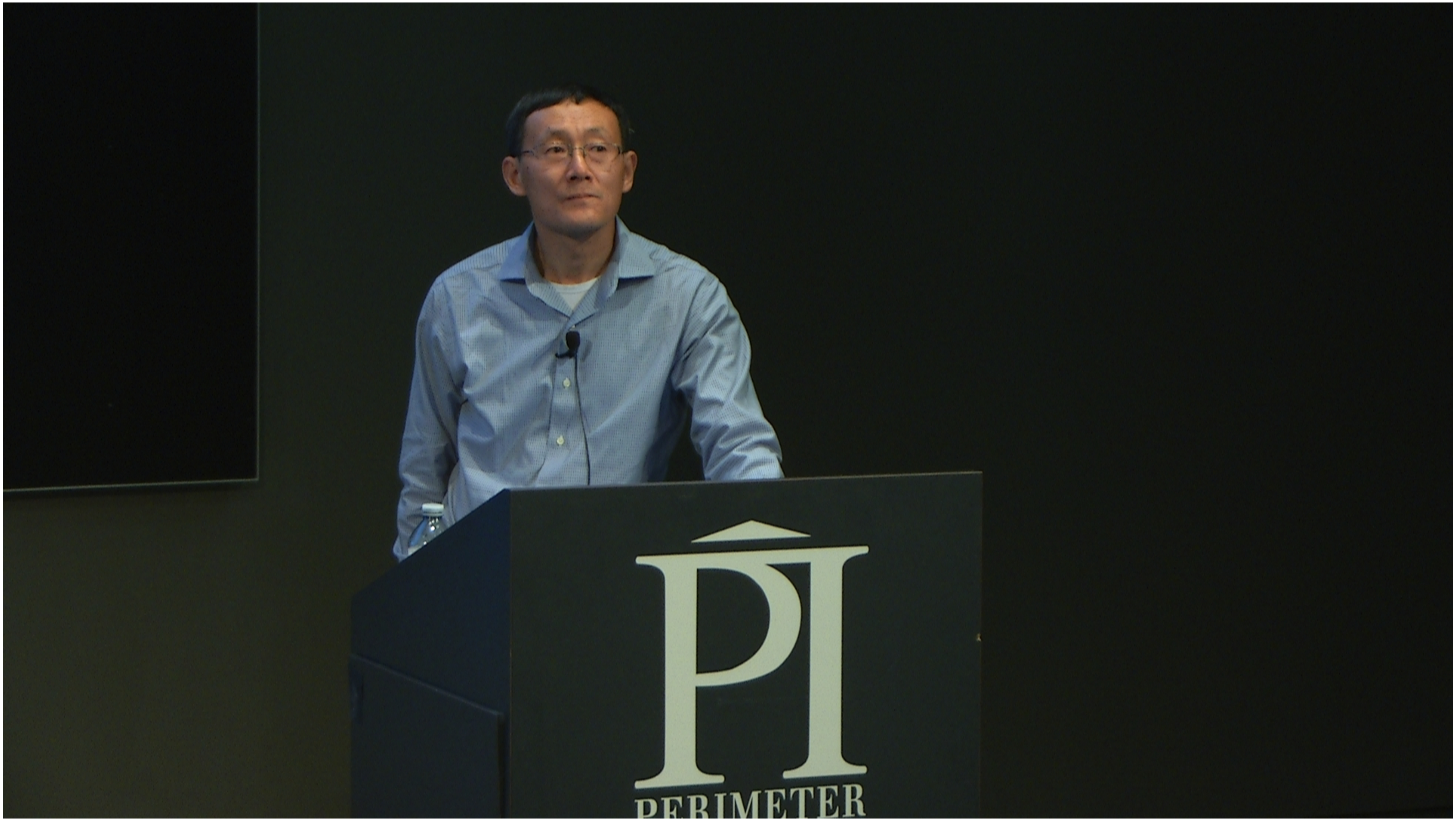
Abstract: <p>Topological quantum computation is based on the possibility of the realization of some TQFTs in Nature as topological phases of quantum matter. Theoretically, we would like to classify topological phases of matter, and experimentally, find non-abelian objects in Nature. We will discussion some progress for a general audience.

</p>

TQFTs in Nature and Topological Quantum Computation



Zhenghan Wang
Microsoft Station Q & UC Santa Barbara
PI, Oct. 21, 2015


PERIMETER

DREAM: "Periodic Table" of Quantum Phases of Matter

Classification of symmetry enriched topological order in all dimensions

Too hard!!!

Special Cases:

Symmetry	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$	Z	Z_2	Z_2	Z_2^2
Z_2^T	Z_1	Z_2	Z_1	Z_2
$U(1)$	Z	Z_1	Z	Z_1
$SO(3)$	Z_1	Z_2	Z	Z_1
$SO(3) \times Z_2^T$	Z_1	Z_2^2	Z_2	Z_2^3
Z_n	Z_n	Z_1	Z_n	Z_1
$Z_2^T \times D_2 = D_{2h}$	Z_2^2	Z_2^4	Z_2^6	Z_2^9

Symmetry classes	Physical realizations	$d=1$	$d=2$	$d=3$
D	SC	p-wave SC	$(p+ip)$-SC	0
DIII	TRI SC	Z_2	$(p+ip)(p-ip)$-SC	He^3-B
AII	TRI ins.	0	HgTe Quantum well	$Bi_{1-x}Sb_x$, Bi_2Se_3, etc.
CII	Bipartite TRI ins.	Carbon nanotube	0	Z_2
C	Singlet SC	0	$(d+id)$-SC	0
CI	Singlet TRI SC	0	0	Z
AI	TRI ins. w/o SOC	0	0	0
BDI	Bipartite TRI ins. w/o SOC	Carbon nanotube	0	0

1): short-range entangled (or SPT) including topological insulators and topological superconductors: X.-G. Wen (Group Cohomology), …, and A. Kitaev (K-theory)---generalized cohomologies.

2): **Low dimensional: spatial dimensions D=1, 2, 3, n=d=D+1**

2a: classify 2D topological orders without symmetry

2b: enrich them with symmetry

2c: **3D much harder**

Classification of Unitary Modular Categories

rank = 2, 3, 4 with Rowell and Stong, rank = 5 with Bruillard, Ng, Rowell

	A 1 Trivial			
	A 2 Semion		NA 2 Fib BU	
	A 2 Z_3	NA 3 Ising	NA 2 (SO(3),5) BU	
A 3 Toric Code	A 4 Z_4	NA 4 Fib × Semion BU	NA 2 (SO(3),7) BU	NA 3 DFib BU

The ith-row lists all rank = i unitary modular tensor categories.

Middle symbol: the fusion rule.
Upper left corner: A = abelian theory, NA = non-abelian.
Upper right corner number = the number of distinct theories.
Lower left corner BU = there is a universal braiding anyon.

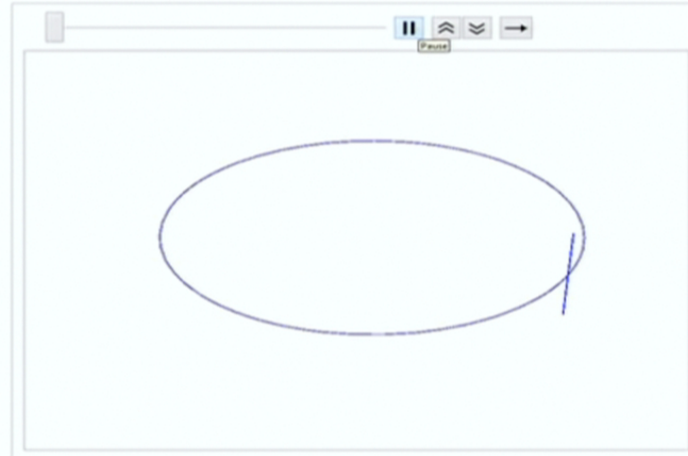
Topology Protects Against Noise

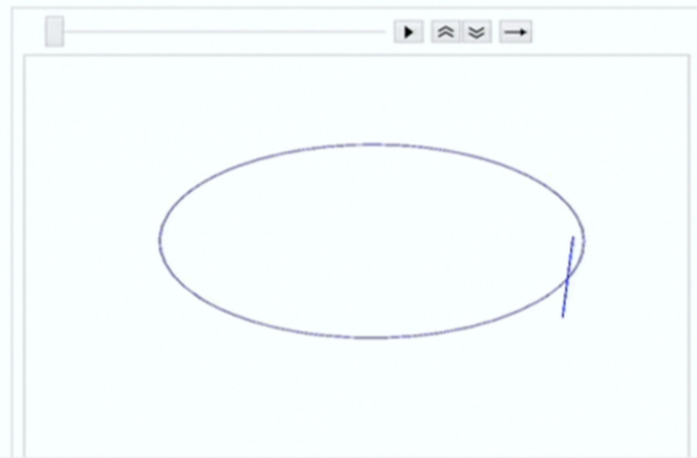
Topological precision:

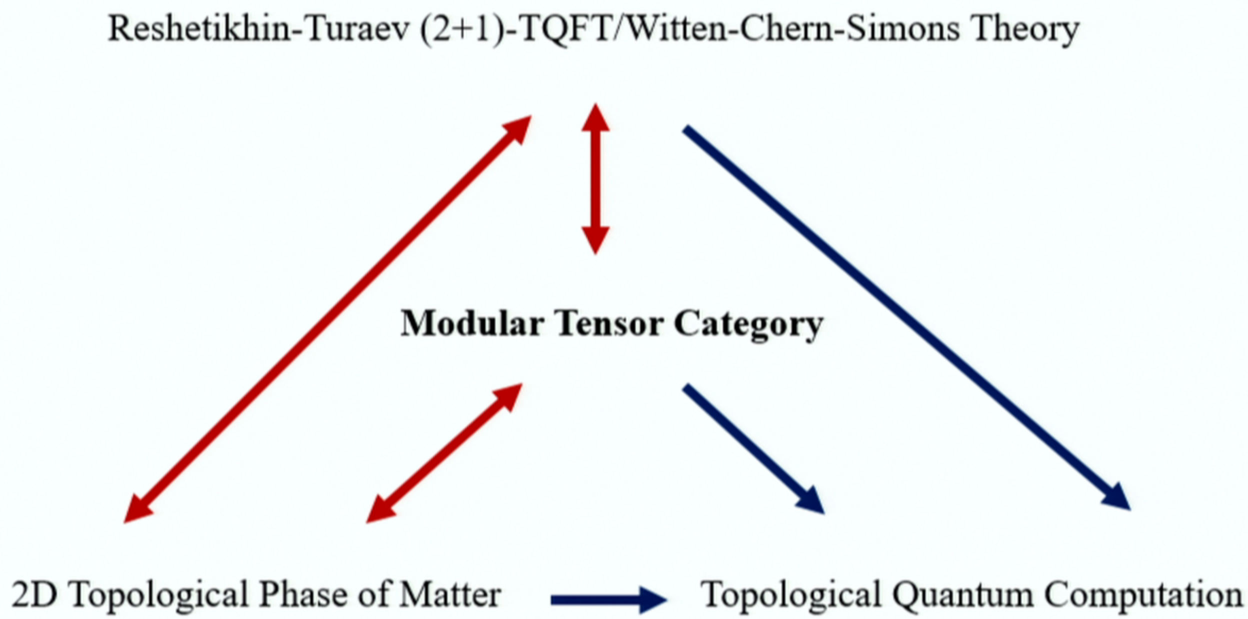
Topological theory (IQHE) is confirmed by experiment to 10 decimal places:

$$\alpha^{-1} = 137.035999074(44)$$

- Feynman and other pioneers taught us that the universal is the ultimate quantum computer
- Quantum information is notoriously fragile
- Quantum information can be locked into topology such as knots to be protected
- Station Q pursue topological protection of qubits





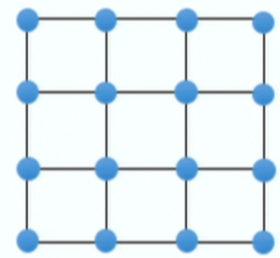


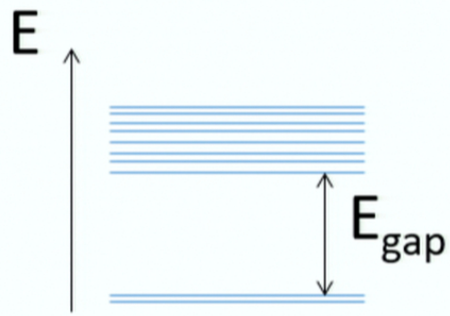
2D topological phases of matter are (2+1)-TQFTs in Nature and hardware for hypothetical topological quantum computers.

Topological Phases of Quantum Matter

Local Hilbert Space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$

Local, Gapped Hamiltonian $H : \mathcal{H} \rightarrow \mathcal{H}$





Two gapped Hamiltonians H_1, H_2 realize the same topological phase of matter if there exists a continuous path connecting them without closing the gap/a phase transition.

A topological phase, to first approximation, is a class of gapped Hamiltonians that realize the same phase. Topological order in a 2D topological phase is encoded by a TQFT or anyon model.

Atiyah-Segal Type (2+1)-TQFT: Codim=1

A symmetric monoidal functor (V, Z):
category of 2-3-mfds → Vec
2-mfd Y → vector space $V(Y)$
3-bord X from Y_1 to Y_2 → Z(X): $V(Y_1) \to V(Y_2)$

- $V(\emptyset) = \mathbb{C}$
- $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2)$
- $V(-Y) \cong V^*(Y)$
- $Z(Y \times I) = \mathrm{Id}_{V(Y)}$
- $Z(X_1 \cup X_2) = Z(X_1) \cdot Z(X_2)$ (anomaly-free)

Realization of TQFTs as Topological Phases

A gapped quantum Hamiltonian schema represents a topological phase of matter if the functor $Y \rightarrow V(Y)$ (ground states) is a TQFT.

Hamiltonian schema:

A recipe to cook up a quantum system from any celluation of Y.

Pachner theorem organizes all celluations Δ into an inverse system,
so the ground states $V(Y;\Delta)$ have a limit V(Y).

Haldane Hamiltonian for Semion Theory or $\mu = \frac{1}{2}$ Bosonic FQH

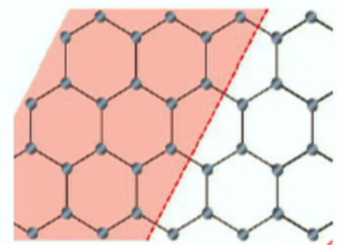
$$H_{\text{Hal.}} = -t \sum_{\langle rr' \rangle} b_r^\dagger b_{r'} - t' \sum_{\langle\langle rr' \rangle\rangle} b_r^\dagger b_{r'} e^{i\phi_{rr'}} - t'' \sum_{\langle\langle\langle rr' \rangle\rangle\rangle} b_r^\dagger b_{r'} + \text{H.c.},$$

Cincio, Vidal, PRL (2013)

set $\phi = 0.4\pi$ and $(t, t', t'') = (1, 0.6, -0.58)$

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{10^{-3}}{\sqrt{2}} \begin{bmatrix} -1.4 & 0.2 \\ -1.4 & 4+4i \end{bmatrix},$$

$$U = e^{-i\frac{2\pi}{24}} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \times \left(e^{i\frac{2\pi}{24}0.01} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i0.007} \end{bmatrix} \right),$$



Local Gapped Hamiltonian

$$L = \bigotimes_i C^r, H = \sum_j P_j,$$

where $P_j's$ are local commuting Hermitian projectors (LCPs)

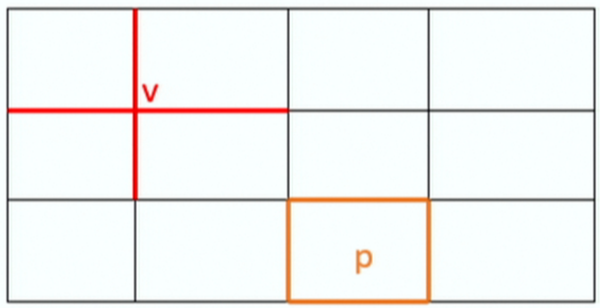
1. $P_j^+ = P_j$, $P_j^2 = P_j$, $[P_j, P_l] = 0$

2. k-local: P_j is of the form $O_k \otimes Id$ for some operator O_k on k qudits

⇨ Gapped!

Toric Code---Kitaev

Local Commuting $\mathbf{H=-\sum_v A_v - \sum_p B_p}$



$=T^2$

$L = \otimes_{edges} \mathbb{C}^2$

$A_v = \otimes_{e \varepsilon v}\ \sigma^z \otimes_{others} Id_e,$

$B_p = \otimes_{e \varepsilon p}\ \sigma^x \otimes_{others} Id_e,$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Ground States Form TQFT

For each surface Y and a triangulation Δ_Y, the ground state manifold $V_0(Y, \Delta_Y)$ of the toric code Hamiltonian is canonically equivalent to the Z_2-homology TQFT vector space V(Y).

The toric code represents a topological phase of matter whose low energy physics is modeled by the Z_2-homology TQFT.

The Z_2-homology TQFT has a Hamiltonian realization by the toric code Hamiltonian.

Realization of Unitary TQFTs by Local Commuting Projectors

Which unitary TQFT has a LCP Hamiltonian realization?

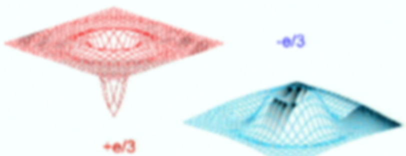
Conjecture: only doubles or Drinfeld centers

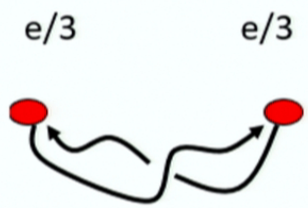
- Turaev-Viro unitary TQFTs---2D Atiyah-Segal type
  String-net/loop condensation---Levin-Wen/Kitaev models
  Mathematically well-understood, Physically not clear.

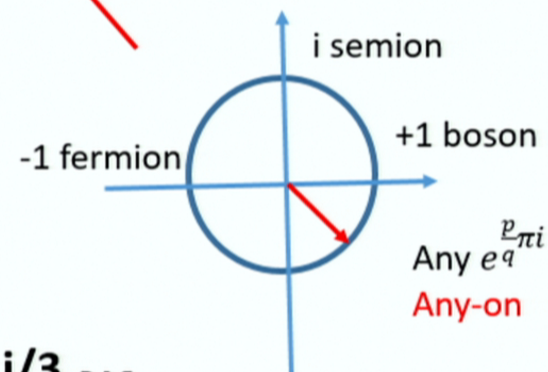
- Reshetikhin-Turaev/Witten-Chern-Simons unitary TQFTs---not Atiyah-Segal type in 2D, yes in 3D
  Trial wave functions---chiral TQFTs
  Physically in better shape (FQH states), mathematically not quite.

Elementary Excitations=Anyons

Quasi-holes/particles in ν=1/3 FQH are abelian anyons







$$\Psi_{1/3} = \Pi_k(\eta_0 - z_j)^3 \ \Pi_{i<j}(z_i - z_j)^3 \ e^{-\Sigma_i |z_i|^2/4}$$

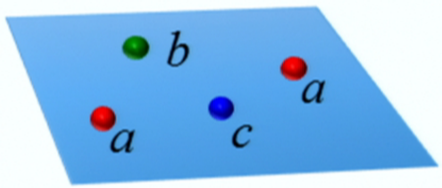
$$= \Pi_k(\eta_1 - z_j)\, \Pi_k(\eta_2 - z_j)\ \ \Pi_k(\eta_3 - z_j)\ \Pi_{i<j}(z_i - z_j)^3 \ e^{-\Sigma_i |z_i|^2/4}$$

n anyons at well-separated η_i, i=1,2,.., n,
there is a **unique** ground state

$$\psi \ \longrightarrow e^{\pi\, i/3}\ \psi$$

Anyon Model

Finite-energy elementary excitations=anyons



Anyons a, b, c

Anyons are of the same type if they differ only by local operators

Anyons in 2D topological phase described mathematically by a Unitary Modular Category = Anyon Model = 2D Topological Order

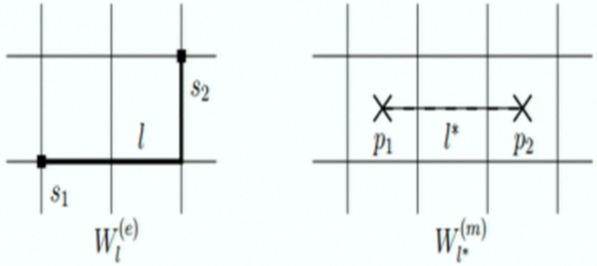
Anyons in Toric Code

- 4 types of anyons $\{1, e, m, \psi\}$:

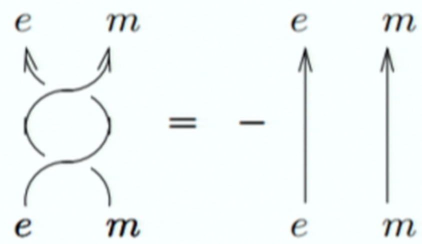
1=ground state or vaccum, e, m=bosons, ψ=fermion,

$e \otimes e = 1, m \otimes m = 1, e \otimes m = \psi$

The fusion rule same as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

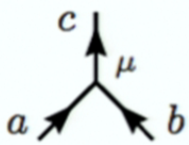


- The anyons form the modular tensor category $D(Z_2)$:

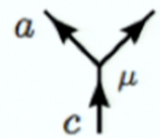


Anyon Model = Unitary Modular Category

- Anyon types {a, b, c, ...} **The number of anyon types called the rank**
- Fusion Rules $a \times b = \sum_c N_{ab}^c c$ $N_{ab}^c \geq 0$ integer
- Fusion/Splitting spaces:

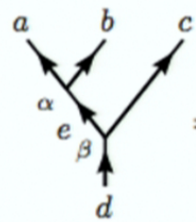


$\propto \langle a, b; c, \mu| \in V_{ab}^c$

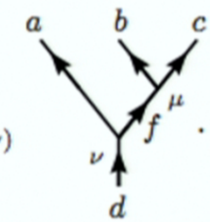


$\propto |a, b; c, \mu\rangle \in V_c^{ab}$

- F-Symbols



$= \sum_{f,\mu,\nu} \left[F_d^{abc}\right]_{(e,\alpha,\beta)(f,\mu,\nu)}$



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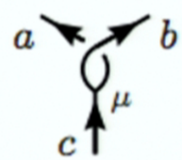
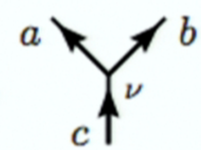
- Braiding (R-Symbols)



$= \sum_\nu \left[R_c^{ab}\right]_{\mu\nu}$



Rank-Finiteness for Modular Categories

Theorem (Bruillard-Ng-Rowell-W., JAMS (to appear)):

For a fixed rank, there are only finitely many equivalence classes of modular categories.

Remarks:

1. Refinement of Ocneanu rigidity: fix the fusion rule, finite**.**
2. Rank-finiteness for fusion/spherical fusion categories open.
3. An explicit bound and effective algorithm.
4. **Feasible to classify by rank.**

Classification of Unitary Modular Categories

rank = **2, 3, 4** with Rowell and Stong, rank = **5** with Bruillard, Ng, Rowell

	A 1 Trivial			
	A 2 Semion		NA 2 Fib BU	
	A 2 $\mathbb{Z}_3$	NA 8 Ising	NA 2 $(SO(3), 5)$ BU	
A 5 Toric Code	A 4 $\mathbb{Z}_4$	NA 4 Fib × Semion BU	NA 2 $(SO(3), 7)$ BU	NA 3 DFib BU

The ith-row lists all rank $= i$ unitary modular tensor categories.

Middle symbol: the fusion rule.
Upper left corner: A = abelian theory, NA = non-abelian.
Upper right corner number = the number of distinct theories.
Lower left corner BU = there is a universal braiding anyon.

TQFTs and Higher Categories

Basic Principle:

Physics is local, so realistic TQFTs are determined by local data.

(D+1)-Topological Quantum Field Theories ← – – → (D+1)-Categories

(2+1)-TQFTs ⟷ Modular Tensor Categories

Quantum Finite Group Algebras

Remarks:

1. Not fully extended. Not covered by Lurie's cobordism hypothesis.
2. Frontiers are in d=3+1 both mathematically and physically:

(2+1)-TQFTs are unemployed---no major topological problems to solve in d=2+1,

(3+1)-TQFTs that can detect smooth structures are highly desired.

Quantum Computation

- There is a serious prospect for quantum physics to change the face of information science, and vice versa.
- Theoretically, the story is quite compelling:
  - Shor's factoring algorithm (1994)
  - Fault tolerance ~1996-1997 (Shor, Steane, Kitaev)
- But for the last twenty years the most interesting progress has been to build a quantum computer.
- **Why? Can? How? When?...**

Quantum Speedup:

Factoring is in BQP (Shor's algorithm), but not known in FP (although Primality is in P).

Given an n bit integer $N \sim 2^n$

Classically $\sim e^{c\, n^{1/3} \text{poly} (\log n)}$
Quantum mechanically $\sim n^2$ poly (log n)
For $N \approx 2^{1000}$, classically ~ billion years
Quantum computer ~ minutes

25195908475657893494027183240048398571429282126
20403202777713783604366202070759555626401852588
07844069182906412495150821892985591491761845028
08489120072844992687392807287776735971418347270

RSA-2048
Challenge
Problem

Classical: 1 billion years
Quantum: 100 seconds

19676256133844143603833904414952634432190114657
54445417842402092461651572335077870774981712577
24679629263863563732899121548314381678998850404
45364023527381951378636564391212010397122822120
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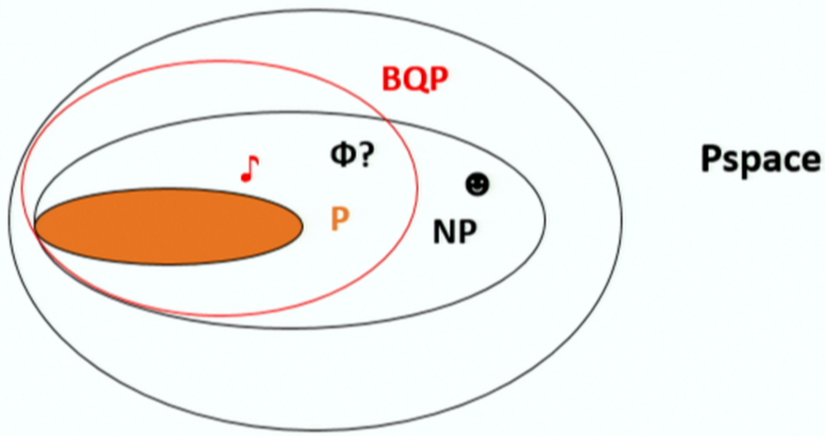


Why Quantum More Powerful?

- Superposition

A (classical) **bit** is given by a physical system that can exist in one of two distinct states:
0 or 1

A **qubit** is given by a physical system that can exist in a linear combination of two distinct quantum states: $|0\rangle$ or $|1\rangle$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$\alpha, \beta \in \mathbb{C}$
$|\alpha|^2 + |\beta|^2 = 1$ $\quad |\psi> \in CP^1$

- Entanglement

Quantum states need not be products. For example:

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$
$$\neq |\psi_A\rangle \otimes |\phi_B\rangle$$

This is the property that enables quantum state teleportation and Einstein's "spooky action at a distance."

MIT Technology Review

Microsoft's Quantum Mechanics

Can an aging corporation's adventures in fundamental physics research open a new era of unimaginably powerful computers?

By Tom Simonite on October 10, 2014

QUANTA MAGAZINE illuminating science

QUANTUM COMPUTING

Forging a Qubit to Rule Them All

Construction is now under way on a new information-storing device that could become the building block of a robust, scalable quantum computer.

Quantum Projects

COMPANY	TECHNOLOGY	WHY IT COULD FAIL
IBM	Makes qubits from superconducting metal circuits.	The error rate of the qubits is too high to operate them together in a useful computer.
Microsoft	Building a new kind of "topological qubit" that in theory should be more reliable than others.	The existence of the subatomic particle used in this qubit remains unproven. Even if it is real, there isn't yet evidence it can be controlled.
Alcatel-Lucent	Inspired by Microsoft's research, it is pursuing a topological qubit based on a different material.	Same as above.
D-Wave Systems	Sells computers based on superconducting chips with 512 qubits.	It's not clear that its chips harness quantum effects. Even if they do, their design is limited to solving a narrow set of mathematical problems.
Google	After experimenting with D-Wave's computers since 2009, it recently opened a lab to build chips like D-Wave's.	Same as above. Plus, Google is trying to adapt technology first developed for a different kind of qubit to the kind used by D-Wave.

Alibaba

Key "Post-Shor" Idea

Peter Shor
Shor's Factoring Algorithm

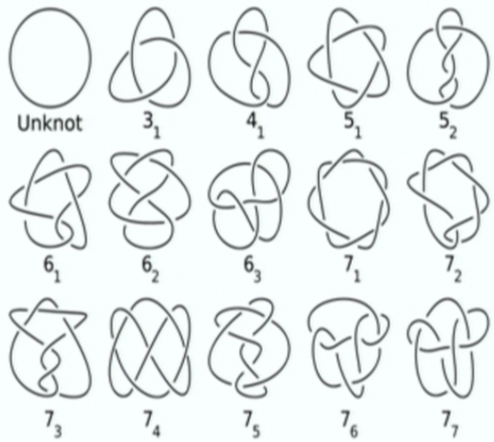
To use topology to protect quantum information

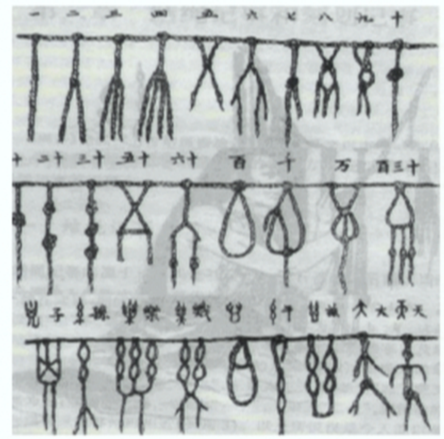
Michael Freedman

Alexei Kitaev

Why Topology?

- **Topology** is usually conceived of as that part of geometry which survives deformation.





- But, equally, **topology** is that part of quantum physics which is robust to deformation (error).

P/NP, and the quantum field computer

MICHAEL H. FREEDMAN

Abstract

The central problem in computer science is the conjecture that two complexity classes, P (polynomial time) and NP (nondeterministic polynomial time—roughly those decision problems for which a proposed solution can be checked in polynomial time), are distinct in the standard Turing model of computation: $P \neq NP$. As a generality, we propose that each physical theory supports computational models whose power is limited by the physical theory. It is well known that classical physics supports a multitude of implementation of the Turing machine. Non-Abelian topological quantum field theories exhibit the mathematical features necessary to support a model capable of solving all $\#P$ problems, a computationally intractable class, in polynomial time. Specifically, Witten [Witten, E. (1989) *Commun. Math. Phys.* 121, 351–391] has identified expectation values in a certain $SU(2)$-field theory with values of the Jones polynomial [Jones, V. (1985) *Bull. Am. Math. Soc.* 12, 103–111] that are $\#P$-hard [Jaeger, F., Vertigen, D. & Welsh, D. (1990) *Math. Proc. Comb. Philos. Soc.* 108, 35–53]. This suggests that some physical system whose effective Lagrangian contains a non-Abelian topological term might be manipulated to serve as an analog computer capable of solving NP or even $\#P$-hard problems in polynomial time. Defining such a system and addressing the accuracy issues inherent in preparation and measurement is a major unsolved problem.

Fault-tolerant quantum computation by anyons

A.Yu. Kitaev

Abstract

A two-dimensional quantum system with anyonic excitations can be considered as a quantum computer. Unitary transformations can be performed by moving the excitations around each other. Measurements can be performed by joining excitations in pairs and observing the result of fusion. Such computation is fault-tolerant by its physical nature.

Classical Physics	**Classical Computing**
Quantum Mechanics	**Quantum Computing**
Quantum Field Theory	**?**
String Theory	**???**

Quantum field computing is the same as quantum computing.

True for TQFTs
(Freedman, Kitaev, Larsen, W.)

CFT? Topological string theory?

A Revolutionary Idea

If a physical system were to have quantum *topological (necessarily nonlocal) degrees of freedom,* which were *insensitive to local probes*, then information contained in them would be *automatically protected against errors* caused by *local interactions with the environment*.

This would be fault tolerance guaranteed by physics at the hardware level, with no further need for quantum error correction, i.e. topological protection.

Alexei Kitaev

2D Topological Phases in Nature

- Quantum Hall States

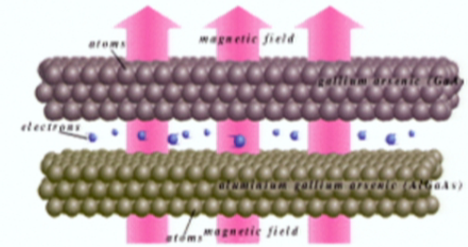
  1980 Integral Quantum Hall Effect ---von Klitzing

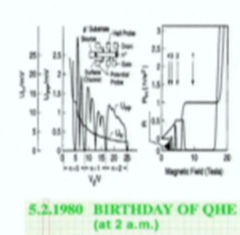
  (1985 Nobel)

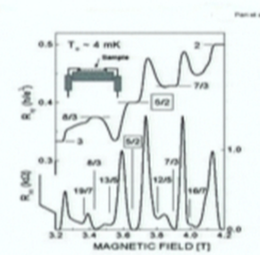
  1982 Fractional QHE---Stormer, Tsui, Gossard at $\nu = \frac{1}{3}$

  (1998 Nobel for Stormer, Tsui and Laughlin)

  1987 Non-abelian FQHE**???**---R. Willet et al at $\nu = \frac{5}{2}$
- Topological superconductors and insulators
- Topological Nanowires---Kouwenhoven and Marcus
- …

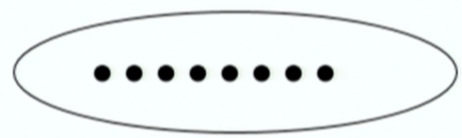


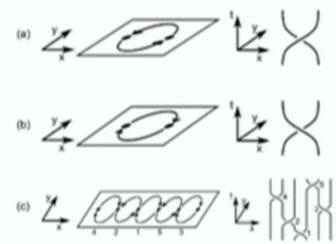




Quantum Dimension

Given n anyons of type x on the sphere S^2, then the ground state degeneracy $V(S^2, x, \ldots, x) \sim d_x^n$ for some $d_x \geq 1$.





If $d_x = 1$, then x is abelian.

If $d_x > 1$, then x is non-abelian,

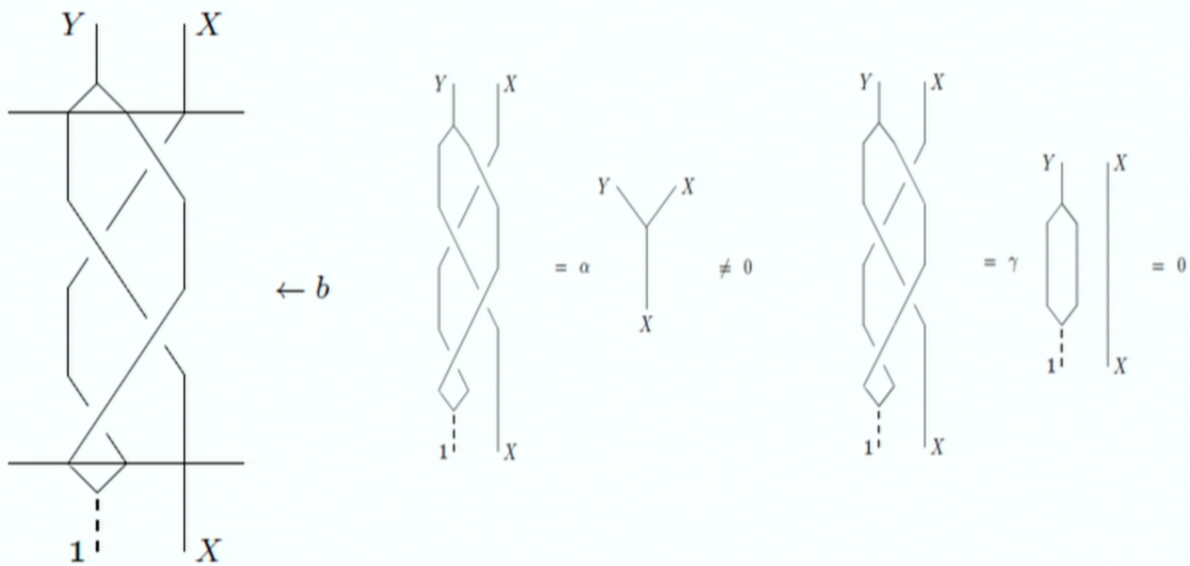
which leads to degeneracy and non-abelian statistics.

Degeneracy Implies Non-abelian Statistics

X self-dual with fusion rule

$$X \otimes X = 1 \oplus Y \oplus \cdots,$$

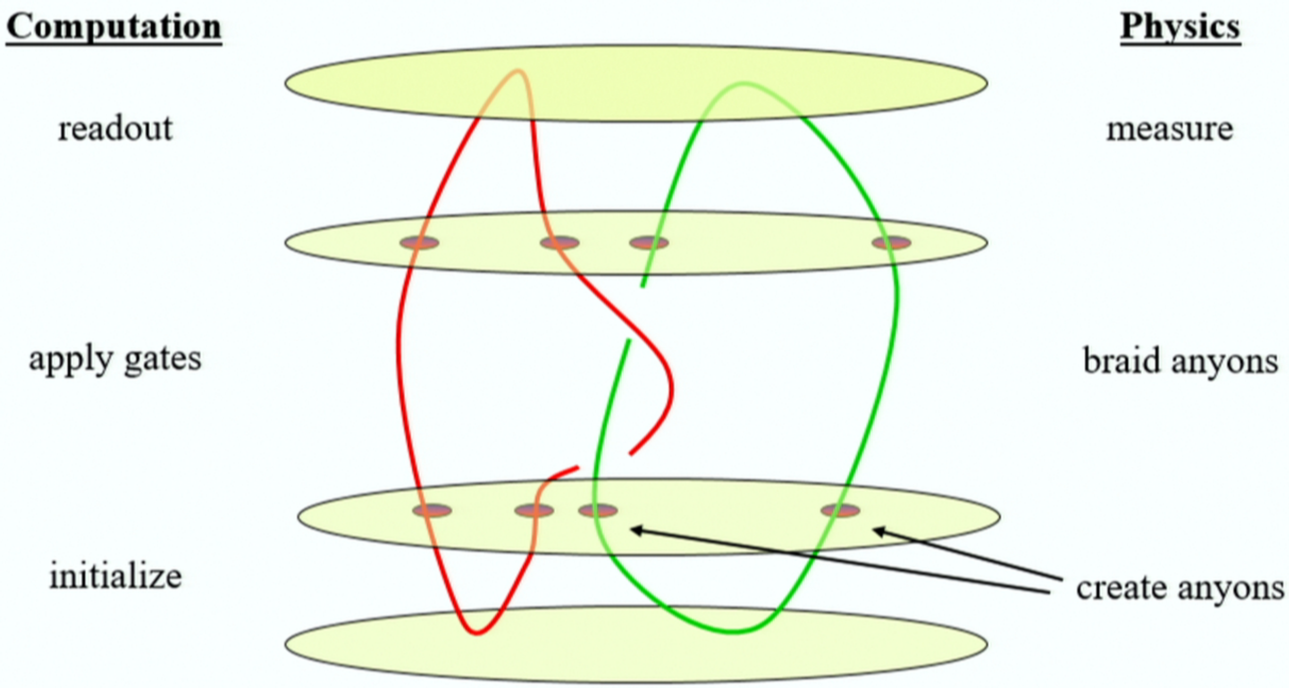
The braid b is non-trivial.



Rowell, W. (2015)

Topological Quantum Computation

Freedman 97, Kitaev 97, FKW 00, FLW 00



Mathematical Theorems

Theorem 1 (Freedman-Kitaev-W.): Any unitary (2+1)-TQFT can be efficiently simulated by the quantum circuit model.

There are efficient additive approximation algorithms of quantum invariants by the quantum circuit model.

Theorem 2 (Freedman-Larsen-W.): Anyonic quantum computers based on RT/WCS $\mathrm{SU}(2)$-TQFTs at level k are braiding universal except $k = 1, 2, 4$.

The approximation of Jones poly of links at the $(k+2)^{\text{th}}$ root of unity ($k \neq 1, 2, 4$) is a BQP-complete problem.

Theorem 3 (Cui-W., Levaillant-Bauer-Freedman-W.-Bonderson): Anyonic model based on $\mathrm{SU}(2)$ at level $k = 4$ is universal for quantum computation if braidings are supplemented with measurements in the middle of computation.

Majoranas in Nature

Of all non-abelian objects believed to exist in topological quantum physics, we are the closest to detecting and harnessing Majoranas.

perspective

Majorana returns

F. Wilczek, Nature Physics'09

Physics Today / Volume 64 / Issue 3 / SEARCH AND DISCOVERY

Physics Today - **March 2011**

The expanding search for Majorana particles

Barbara Goss Levi

Science, April (2011)

NEWS

Search for Majorana Fermions Nearing Success at Last?

Researchers think they are on the verge of discovering weird new particles that borrow a trick from superconductors and could give a big boost to quantum computers

32

1D Kitaev Chain – Majorana Zero Mode

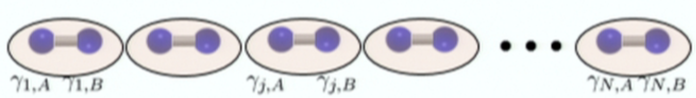
$$H = -\mu \sum_{j=1}^{N} c_j^\dagger c_j - \sum_{j=1}^{N-1} (t c_j^\dagger c_{j+1} + \Delta c_j c_{j+1} + h.c.)$$

Kitaev (2001)

$$c_j = \frac{\gamma_{jA} + i\gamma_{jB}}{2}$$

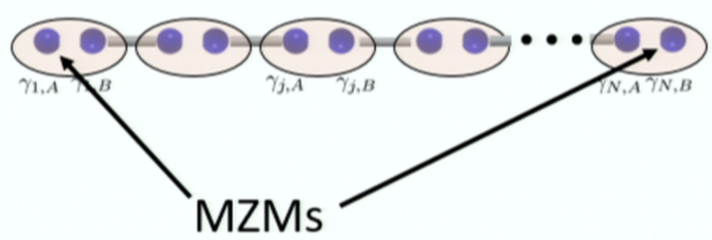
trivial: $t = 0$ and $\Delta = 0$ and $\mu < 0$

$$H = i\mu \sum_j \gamma_{B,j}\gamma_{A,j}$$

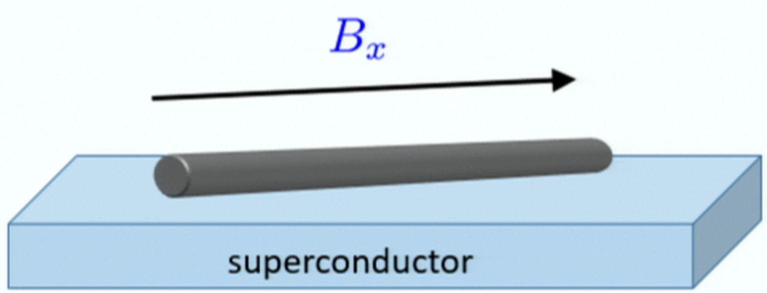


topological: $\mu = 0$ and $t = \Delta$

$$H = it \sum_{j=1}^{N-1} \gamma_{B,j}\gamma_{A,j+1}$$

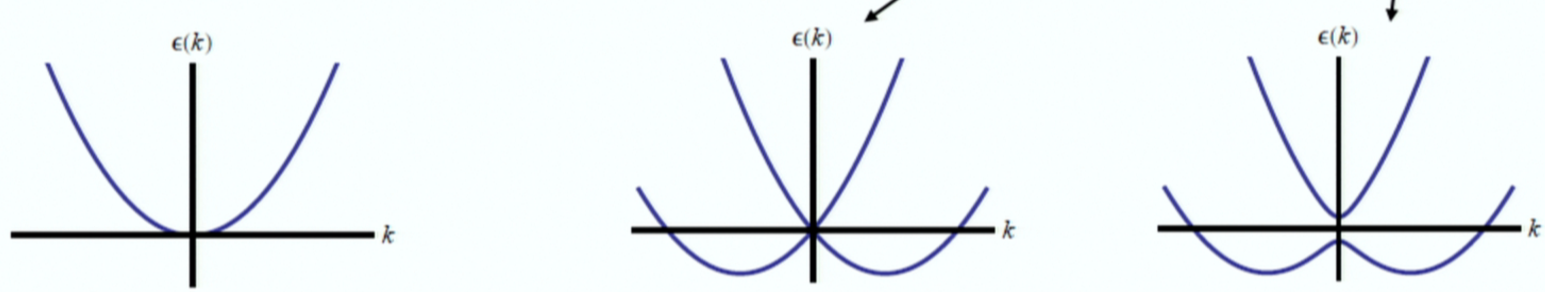


Majorana Wires



Lutchyn, Sau & Das Sarma (2010)
Oreg, Refael & von Oppen (2010)

$$\mathcal{H} = \int dx \, [\psi^\dagger (- \frac{\partial_x^2}{2m} - \mu + i\alpha\sigma_y\partial_x + V_x\sigma_x)\psi + (|\Delta\psi_\uparrow\psi_\downarrow + \text{h.c.})]$$



Majorana Qubits---Ising Theory

- Three anyon types: $\{1, \sigma, \psi\}$
- **Fusion rules**:

$$\sigma \otimes \sigma = 1 + \psi, \sigma \otimes \psi = \sigma, \psi \otimes \psi = 1.$$

1=vacuum,

$\psi =$ Majorana fermion,

$\sigma =$ Ising anyon or **Majorana zero mode**

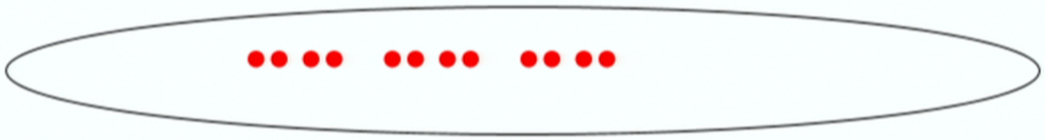
- Majorana systems:

$\nu = \frac{5}{2}$ FQH, nanowires, …

Majorana Quantum Computer

For n qubits, consider 4n MZMs

$\rho: B_{4n} \longrightarrow U(N_{4n})$



Given a quantum circuit on n qubits

$U_L: (C^2)^{\otimes n} \longrightarrow (C^2)^{\otimes n}$

Topological compiling: find a braid $b \in B_{4n}$ so that the following commutes for any U_L:

$$\begin{tikzcd} (C^2)^{\otimes n} \arrow[r] \arrow[d, "U_L"] & V_{4n} \arrow[d, "\rho(b)"] \\ (C^2)^{\otimes n} \arrow[r] & V_{4n} \end{tikzcd}$$

V_{4n}-gs of 4n MZMs

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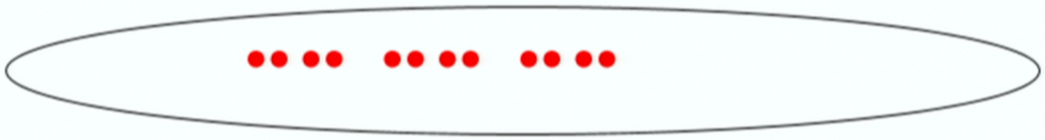

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V_{4n}-gs of 4n MZMs

Ising Braiding Gates

$$\longrightarrow \quad e^{-\pi i/8}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Knill-Gottesman Thm

$$\longrightarrow \quad e^{-\pi i/8}\begin{pmatrix} (1-i)/2 & (1+i)/2 \\ (1+i)/2 & (1-i)/2 \end{pmatrix}$$

$\sigma_1\, \sigma_2 \neq \sigma_2\, \sigma_1$

$$\longrightarrow \quad e^{-\pi i/4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

4 σ's

NOT Gate

$\frac{\pi}{8}$-gate cannot be realized

CNOT can be realized

Symmetry and 2D Topological Phases of Matter

A general framework to classify 2D topological phases of matter with symmetry by introducing G-crossed braided fusion category.

Given a 2D topological phase $\mathcal{C}$ and a global symmetry G of $\mathcal{C}$, **three intertwined themes** on the interplay of symmetry group G and intrinsic topological order of $\mathcal{C}$

- **Symmetry Fractionalization**---topological quasi-particles carry fractional quantum numbers of the underlying constituents
- **Defects**---extrinsic point-like defects. Many are non-abelian objects harboring zero modes
- **Gauging**---deconfine defects by promoting the global symmetry G to a local G gauge theory

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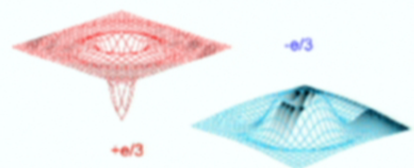
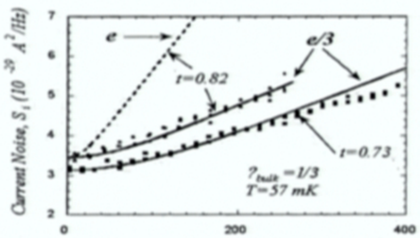
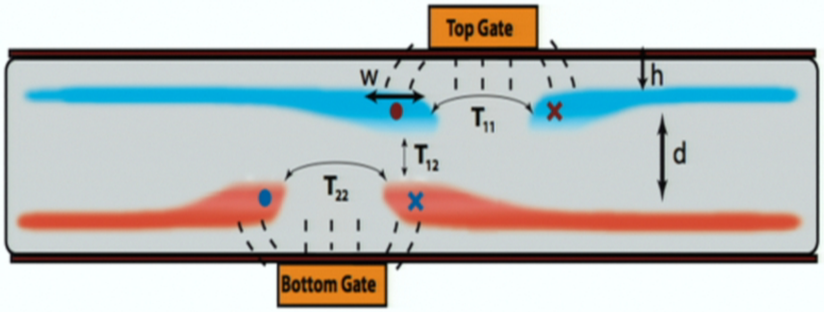
Possible Experimental Tests

Charge fractionalization in $\nu=1/3$ FQH liquid: e/3-Laughlin quasi-particles

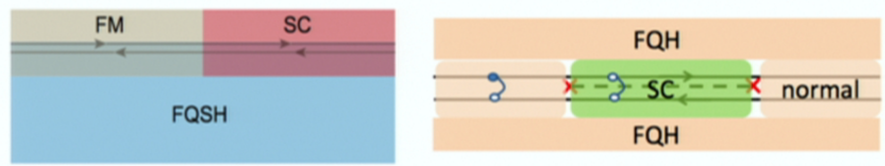




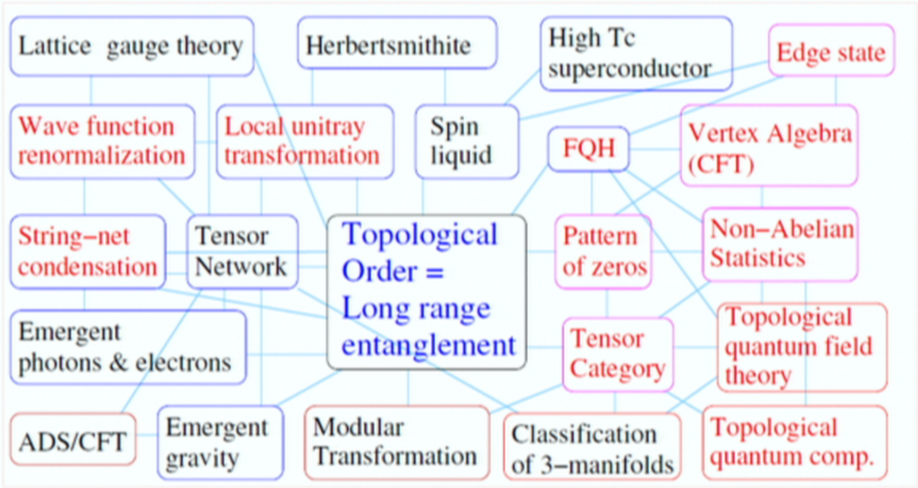
Symmetry defects in bilayer FQH system (two layers of $\nu= 1/3$-FQH liquids)



Normal/SC domain walls in FQH / FQSH states



Rich World of Many-body Entanglement Physics, Topological Materials, and Quantum Mathematics



X.-G. Wen