Title: Reflection positivity and the classification of invertible topological phases

Date: Oct 21, 2015 11:30 AM

URL: http://pirsa.org/15100103

Abstract:

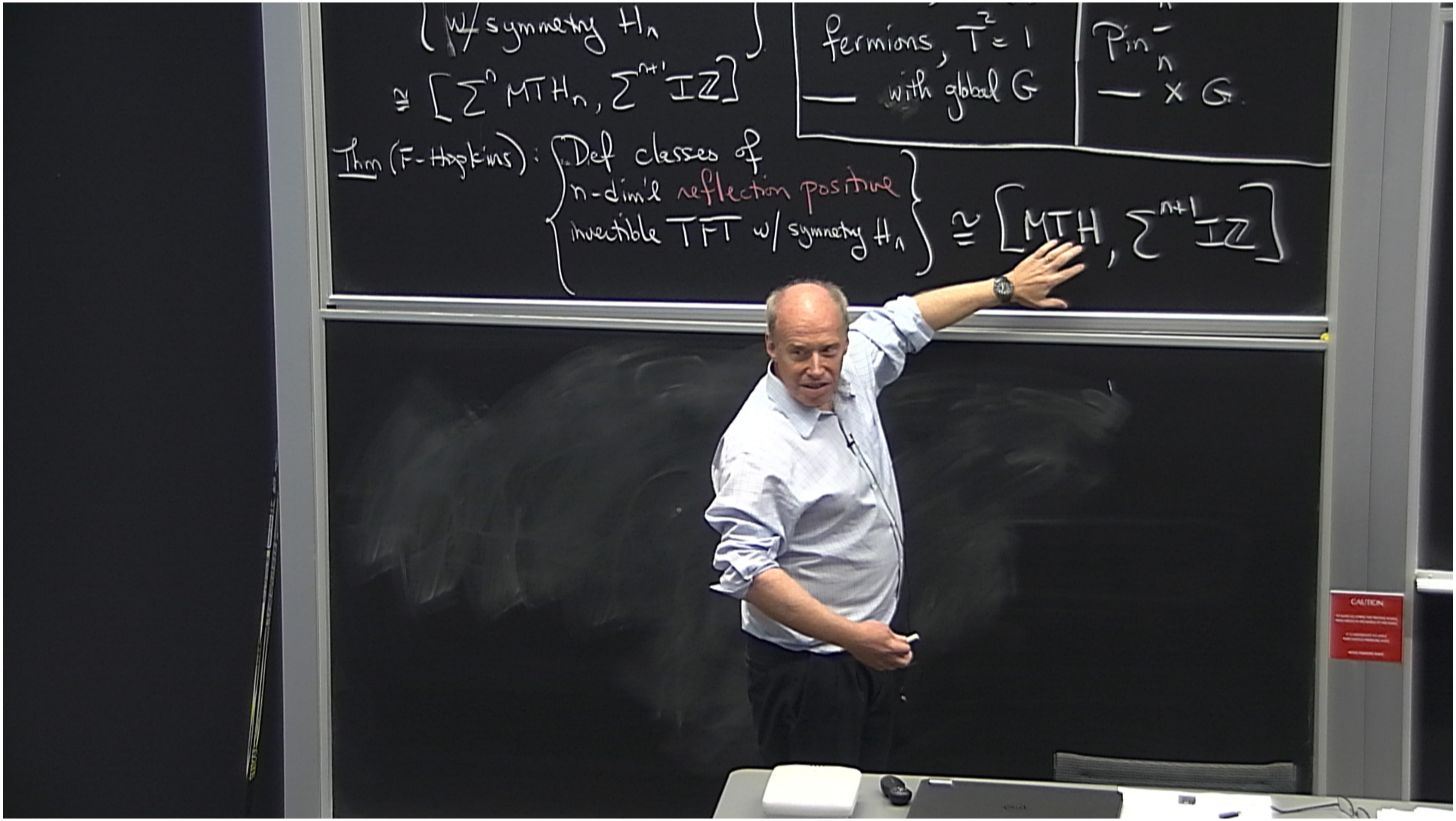
n = spacetime dim

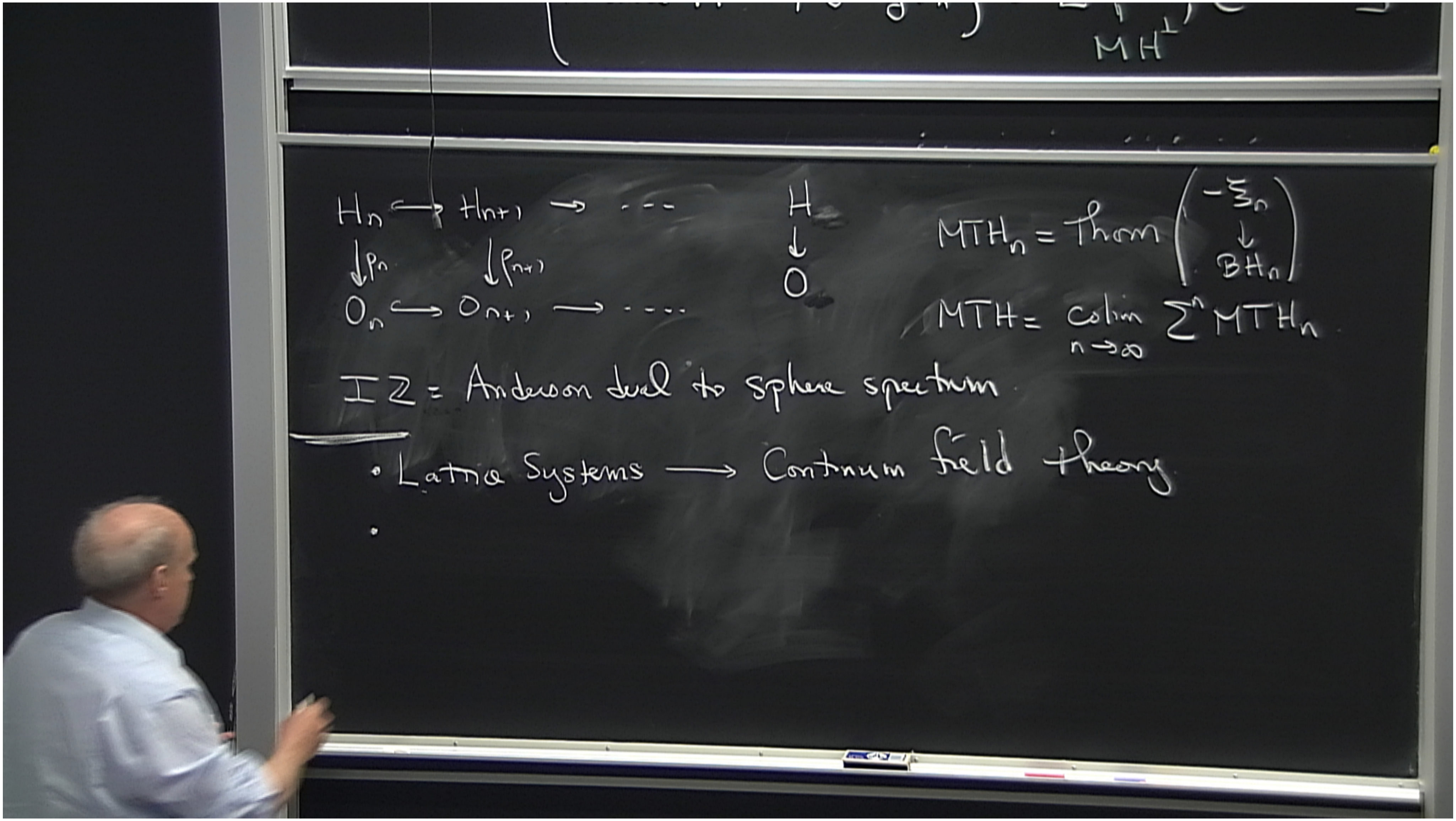
H_n = symmetry group

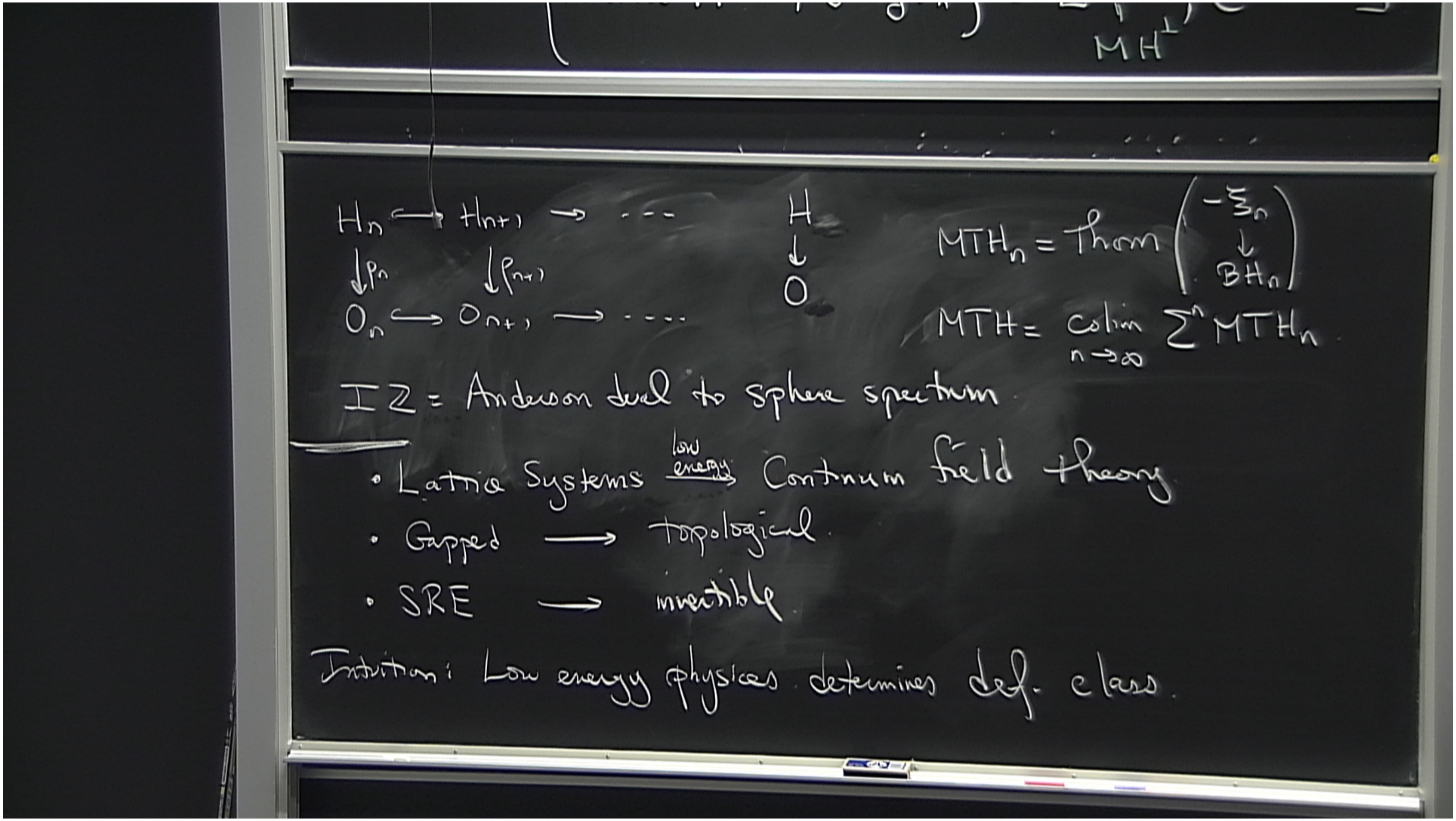
<u>Fact</u>: $\left\{\begin{array}{l}\text{Deformation classes of} \\ n\text{-dim'l invertible TFT} \\ \text{w/ symmetry } H_n\end{array}\right\} \cong [\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}]$

	H_n
bosons only, no TR	SO_n
fermions allowed, no TR	$Spin_n$
bosons, TR	O_n
fermions, $T^2 = (-1)^F$	Pin^+_n
fermions, $T^2 = 1$	Pin^-_n
— with global G	— $\times G$

<u>Thm</u> (F-Hopkins): $\left\{\begin{array}{l}\text{Def classes of} \\ n\text{-dim'l reflection positive} \\ \text{invertible TFT w/ symmetry } H_n\end{array}\right\} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}]$


w/ symmetry H_n
$\cong [\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}]$
fermions, $T^2 = 1$
Pin^-_n
— with global G
— × G
Thm (F-Hopkins): {Def classes of n-dim'l reflection positive invertible TFT w/ symmetry H_n} $\cong [MTH, \Sigma^{n+1} I\mathbb{Z}]$
CAUTION


MH⊥
Hn ↪ Hn+1 → · · ·   H
↓pn   ↓pn+1   ↓
On ↪ On+1 → · · ·   O
MTHn = Thom (−ξn → BHn)
MTH = colim n→∞ Σ^n MTHn
IZ = Anderson dual to sphere spectrum
• Lattice Systems → Continuum field theory
•


MH⊥
Hn ↪ Hn+1 → ...
↓pn ↓pn+1
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H
↓
O
MTHn = Thom(−ξn ↓ BHn)
MTH = colim n→∞ Σ^n MTHn
IZ = Anderson dual to sphere spectrum
• Lattice Systems —low energy→ Continuum field theory
• Gapped → topological
• SRE → invertible
Intuition: Low energy physics determines def. class.

$$\begin{array}{ccccc} H_n & \hookrightarrow & H_{n+1} & \rightarrow & \cdots & & H \\ \downarrow \rho_n & & \downarrow \rho_{n+1} & & & & \downarrow \\ O_n & \hookrightarrow & O_{n+1} & \rightarrow & \cdots & & O \end{array}$$

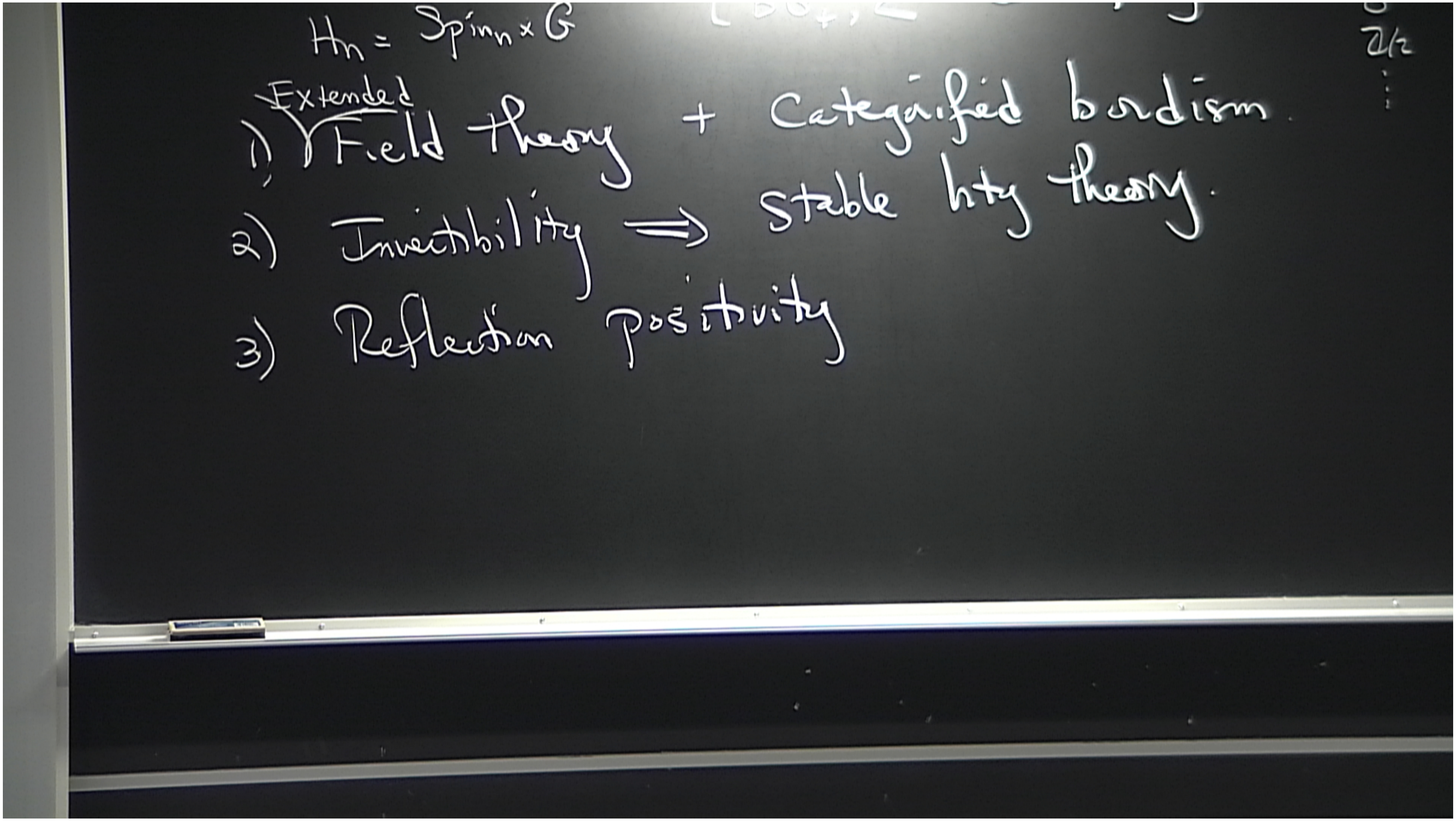
$$MTH_n = \mathrm{Thom}\begin{pmatrix} -\xi_n \\ \downarrow \\ BH_n \end{pmatrix}$$

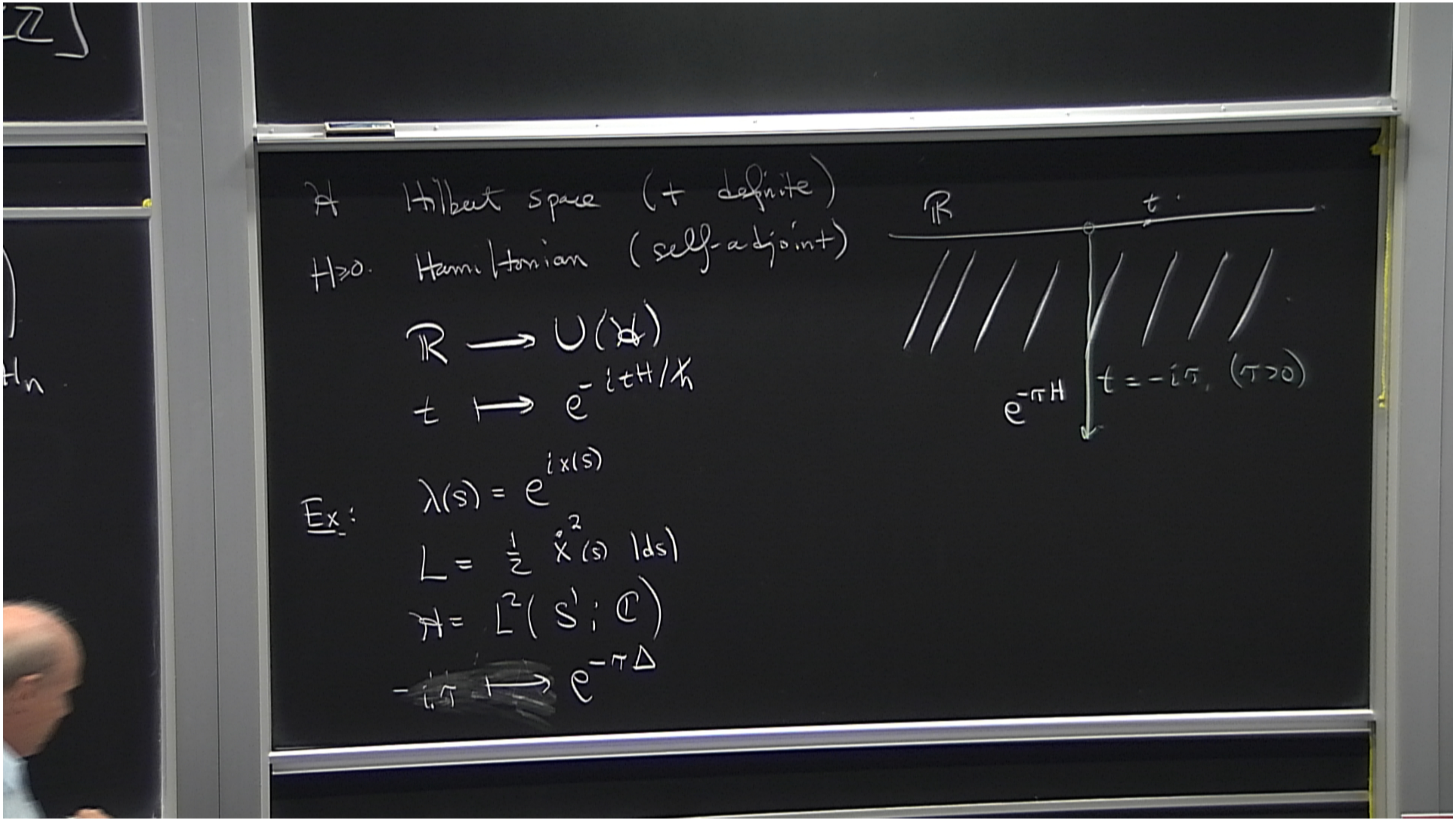
$$MTH = \operatorname*{colim}_{n \to \infty} \Sigma^n MTH_n$$

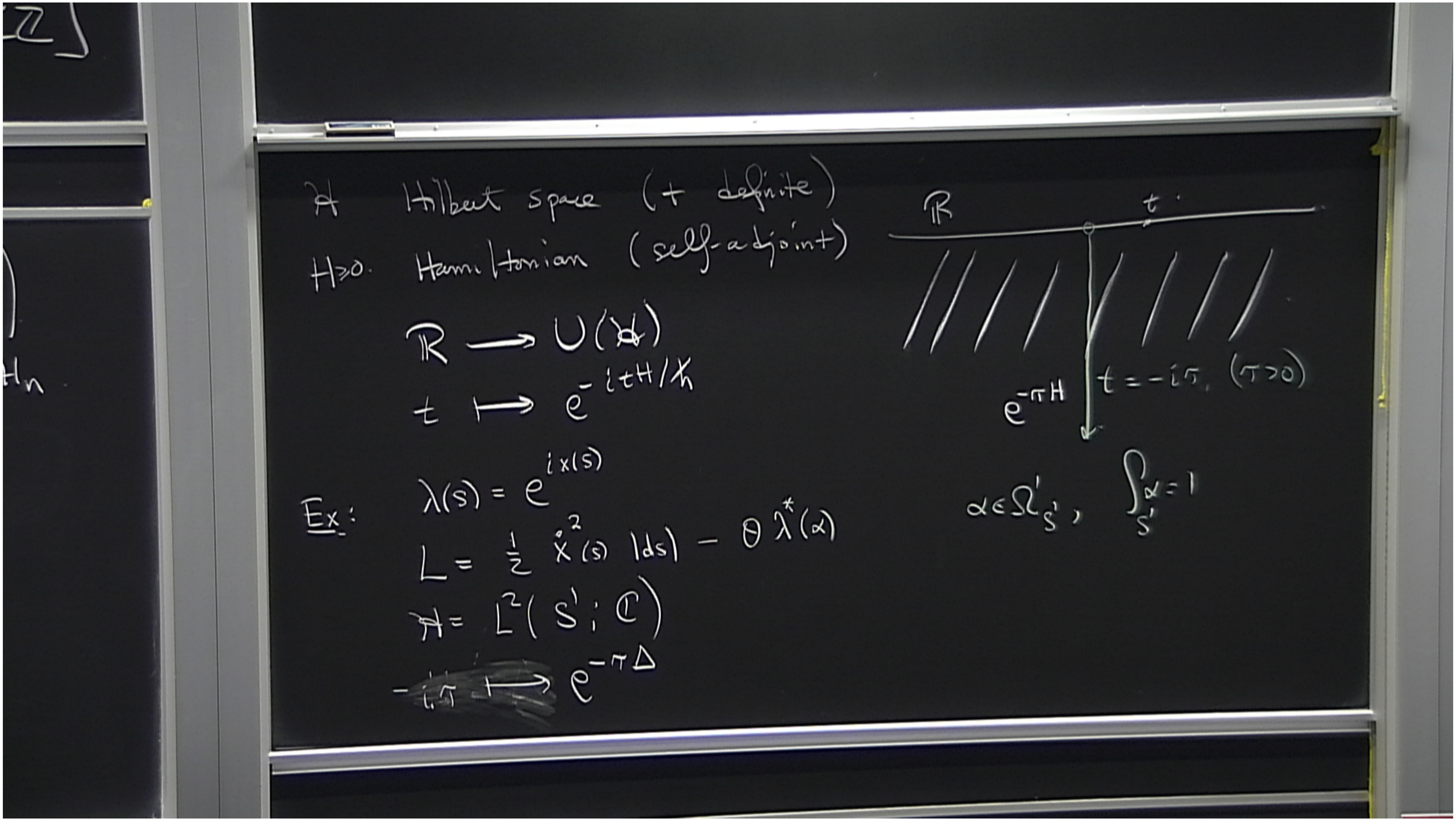
$I\mathbb{Z}$ = Anderson dual to sphere spectrum

- Lattice Systems $\xrightarrow{\text{low energy}}$ Continuum field theory
- Gapped $\longrightarrow$ topological
- SRE $\longrightarrow$ invertible

Intuition: Low energy physics determines def. class.


H_n = Spin_n × G
Z/2
1) Extended Field theory + Categorified bordism.
2) Invertibility ⟹ stable hty theory.
3) Reflection positivity


$\mathcal{H}$ Hilbert space (+ definite)
$H \geq 0$. Hamiltonian (self-adjoint)
$\mathbb{R} \longrightarrow U(\mathcal{H})$
$t \longmapsto e^{-itH/\hbar}$
Ex:
$\lambda(s) = e^{ix(s)}$
$L = \frac{1}{2} \dot{x}^2(s) \, |ds|$
$\mathcal{H} = L^2(S^1; \mathbb{C})$
$-i\tau \longmapsto e^{-\tau\Delta}$
$\mathbb{R}$
t
$e^{-\tau H}$
$t = -i\tau, \ (\tau > 0)$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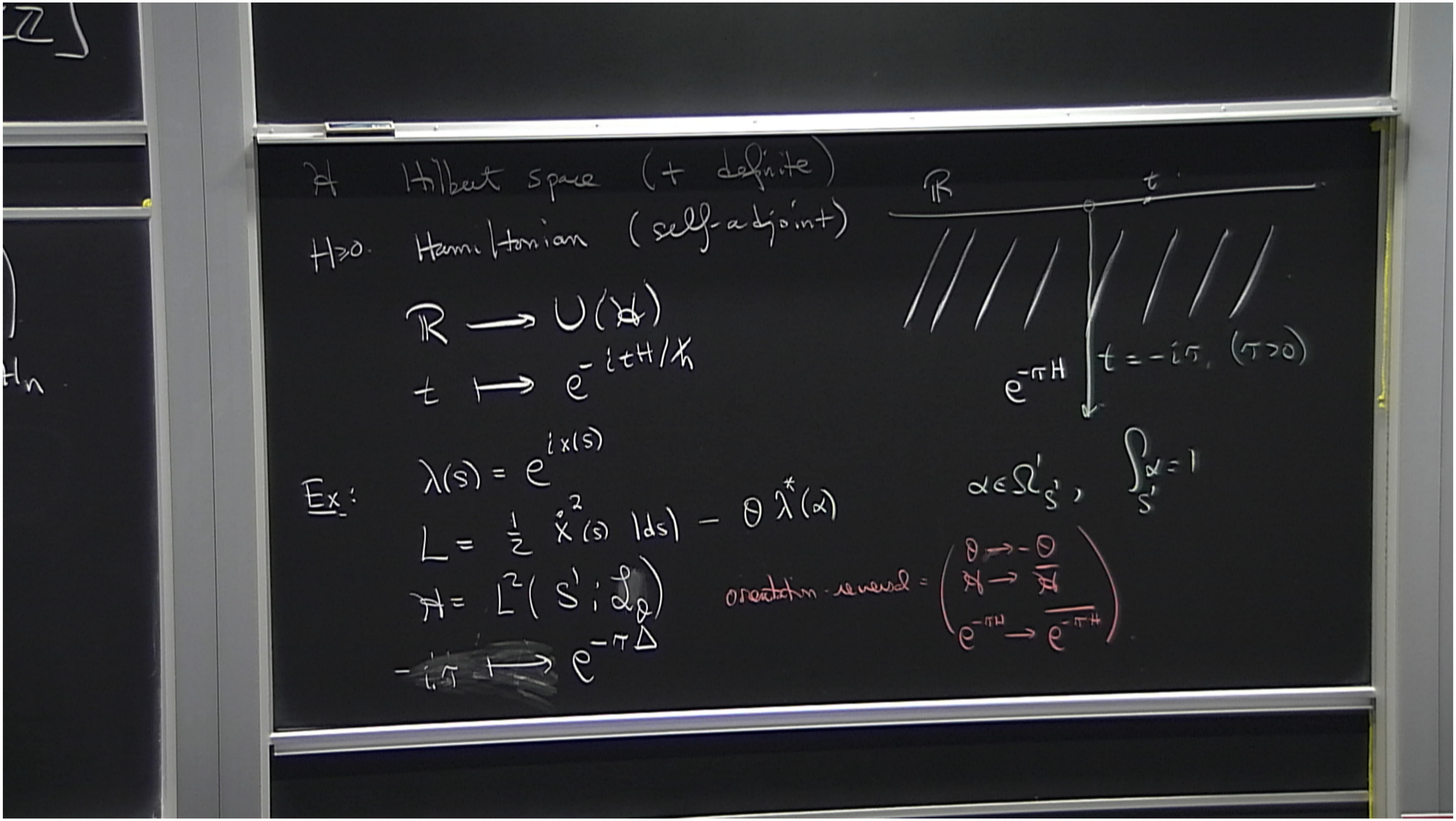
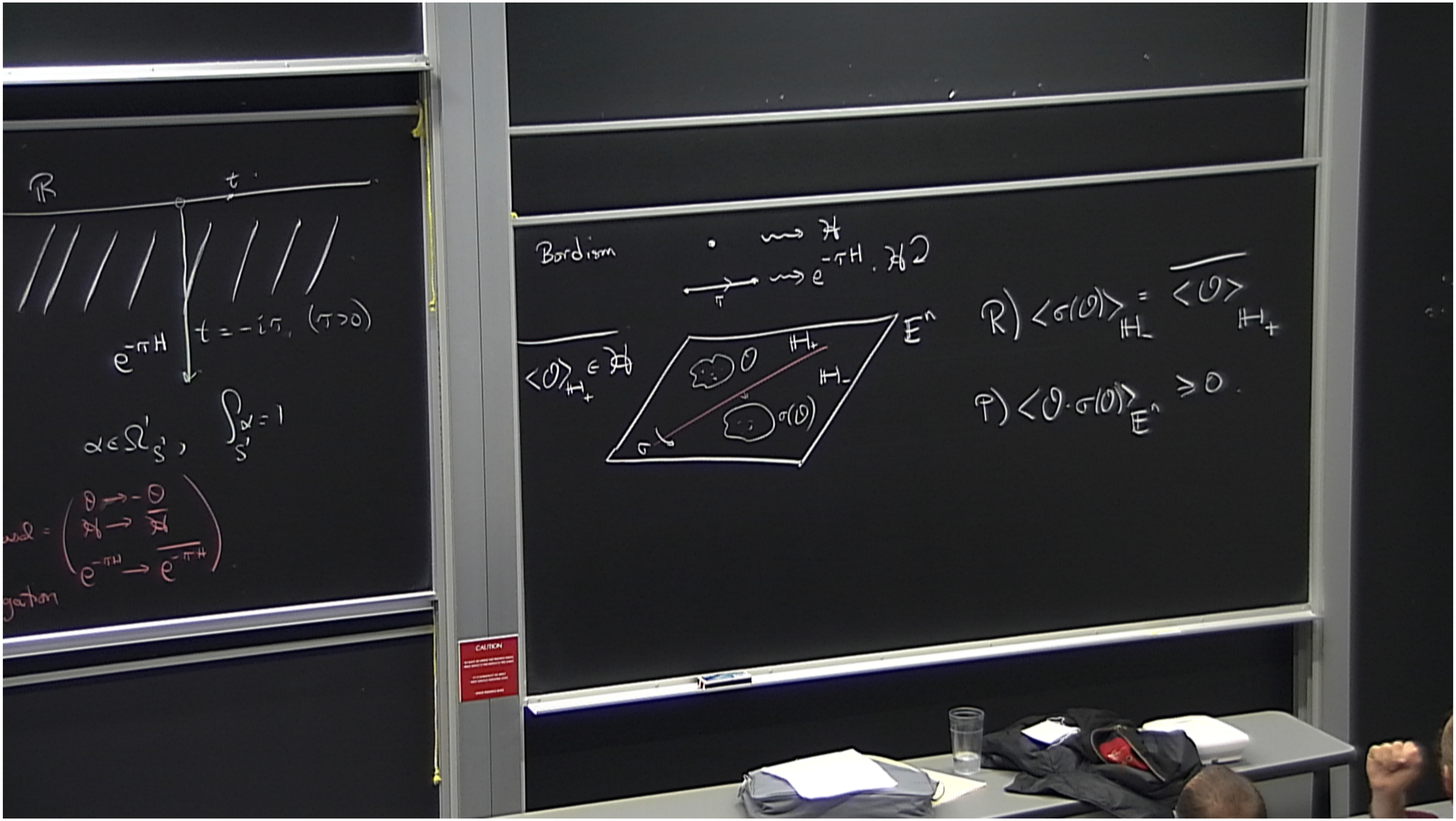
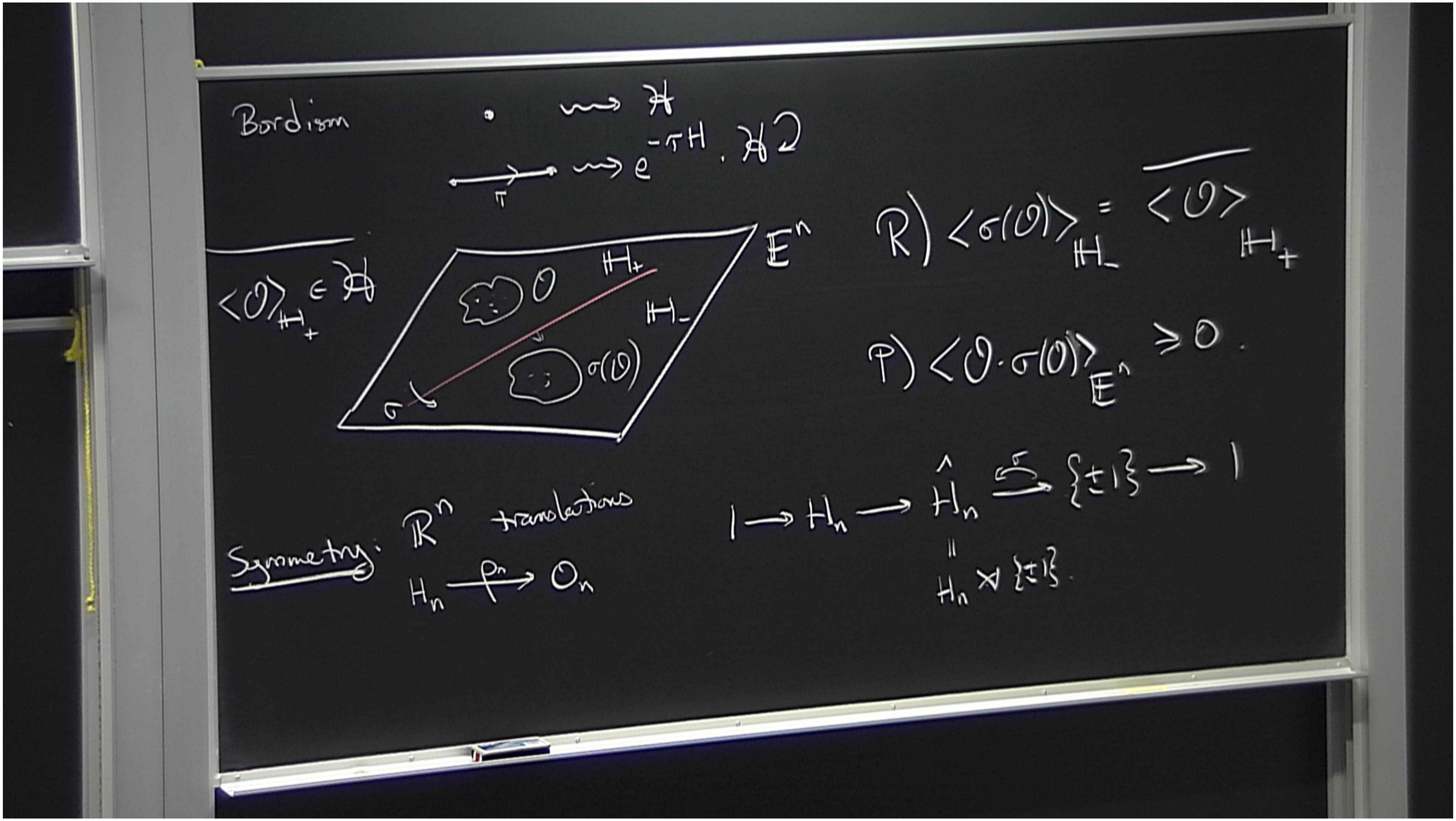
Ex:
$\lambda(s) = e^{ix(s)}$
$L = \frac{1}{2} \dot{x}^2(s) \, |ds| - \theta \lambda^*(\alpha)$
$\mathcal{H} = L^2(S^1; \mathbb{C})$
$-i\tau \longmapsto e^{-\tau\Delta}$
$\mathbb{R}$
t
$e^{-\tau H}$
$t = -i\tau, \ (\tau > 0)$
$\alpha \in \Omega^1_{S^1}, \ \int_{S^1} \alpha = 1$



$\mathcal{H}$ Hilbert space (+ definite)
$H \geq 0$. Hamiltonian (self-adjoint)
$\mathbb{R} \longrightarrow U(\mathcal{H})$
$t \longmapsto e^{-itH/\hbar}$
Ex: $\lambda(s) = e^{ix(s)}$
$L = \frac{1}{2} \dot{x}^2(s) \, |ds| - \theta \lambda^*(\alpha)$
$\mathcal{H} = L^2(S^1 ; \mathcal{L}_\theta)$
$-i\tau \longmapsto e^{-\tau\Delta}$
$\mathbb{R}$
t
$e^{-\tau H}$
$t = -i\tau \quad (\tau > 0)$
$\alpha \in \Omega^1_{S^1}, \quad \int_{S^1} \alpha = 1$
orientation-reversal =
$\theta \to -\theta$
$\mathcal{H} \to \overline{\mathcal{H}}$
$e^{-\tau H} \to \overline{e^{-\tau H}}$



t = −iτ, (τ>0)
Bordism
E^n
H+
H−
CAUTION


Bordism
E^n
H_+
H_-
σ(O)
R) ⟨σ(O)⟩_{H_-} = ⟨O⟩_{H_+}
P) ⟨O·σ(O)⟩_{E^n} ≥ 0.
Symmetry: R^n translations
H_n —ρ_n→ O_n
1 → H_n → Ĥ_n —σ→ {±1} → 1
Ĥ_n = H_n ⋊ {±1}

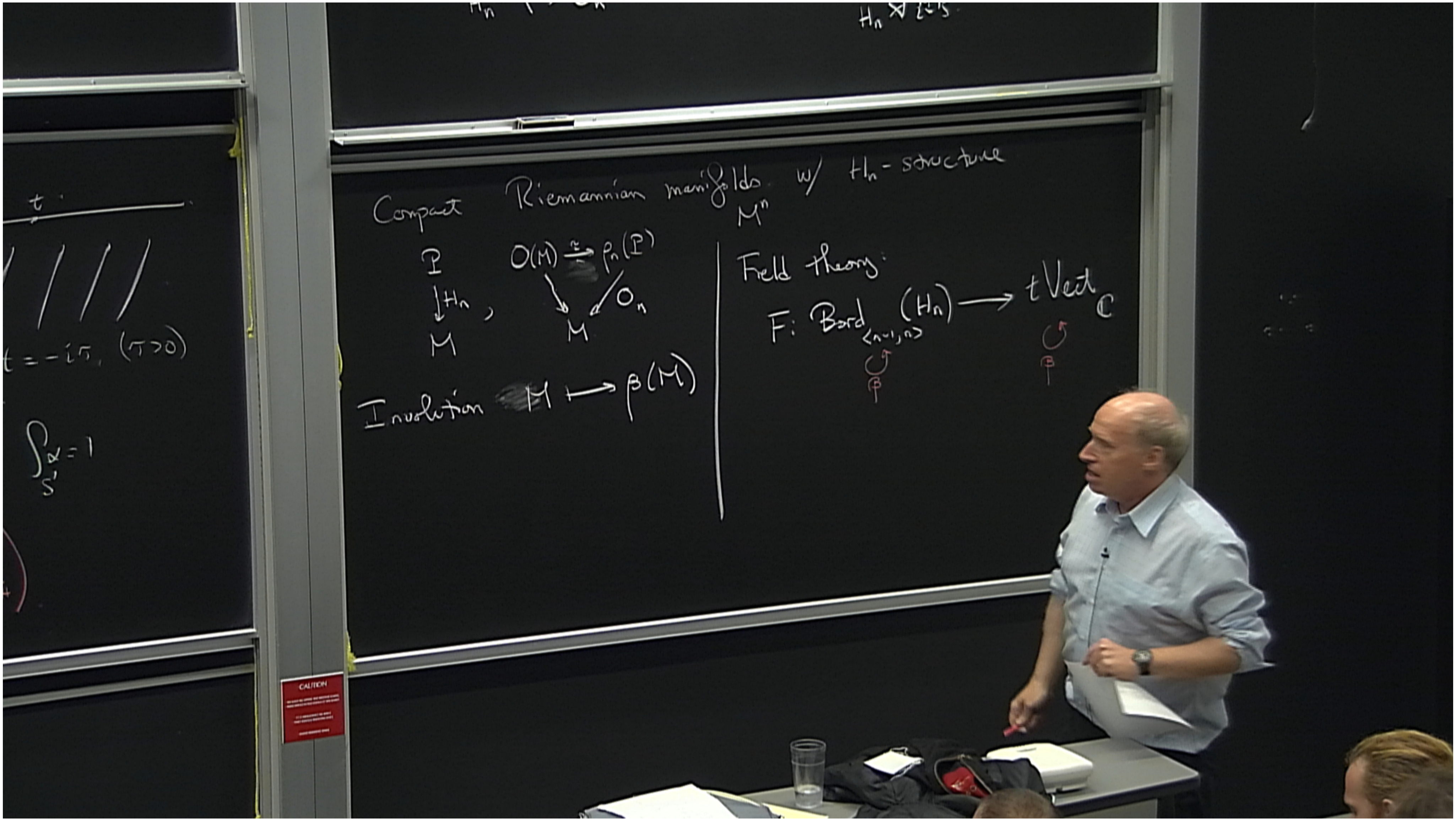
n = spacetime dim

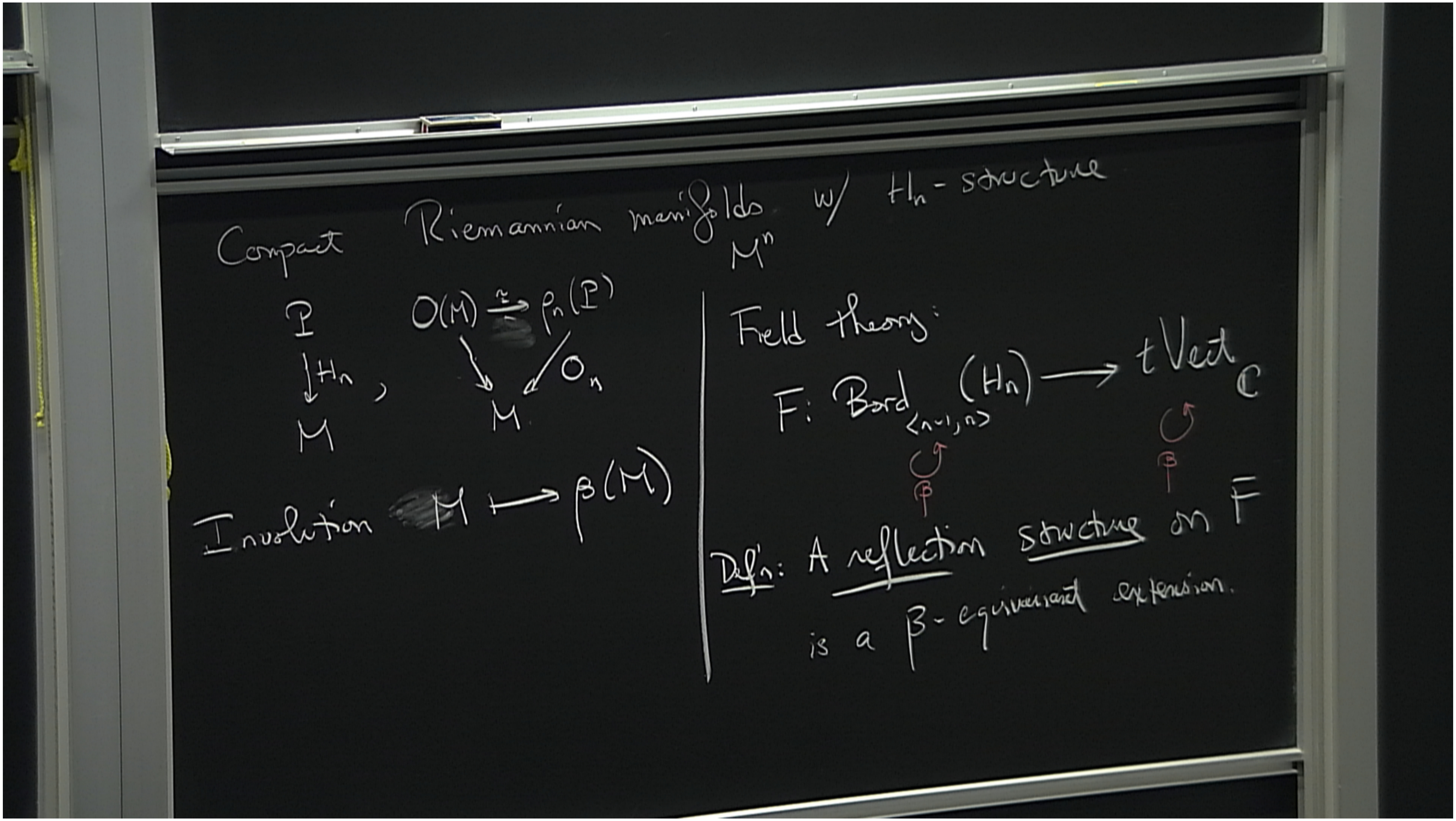
H_n = symmetry group

Fact: $\left\{ \begin{array}{l} \text{Deformation classes of} \\ n\text{-dim'l invertible TFT} \\ \text{w/ symmetry } H_n \end{array} \right\} \cong [\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}]$

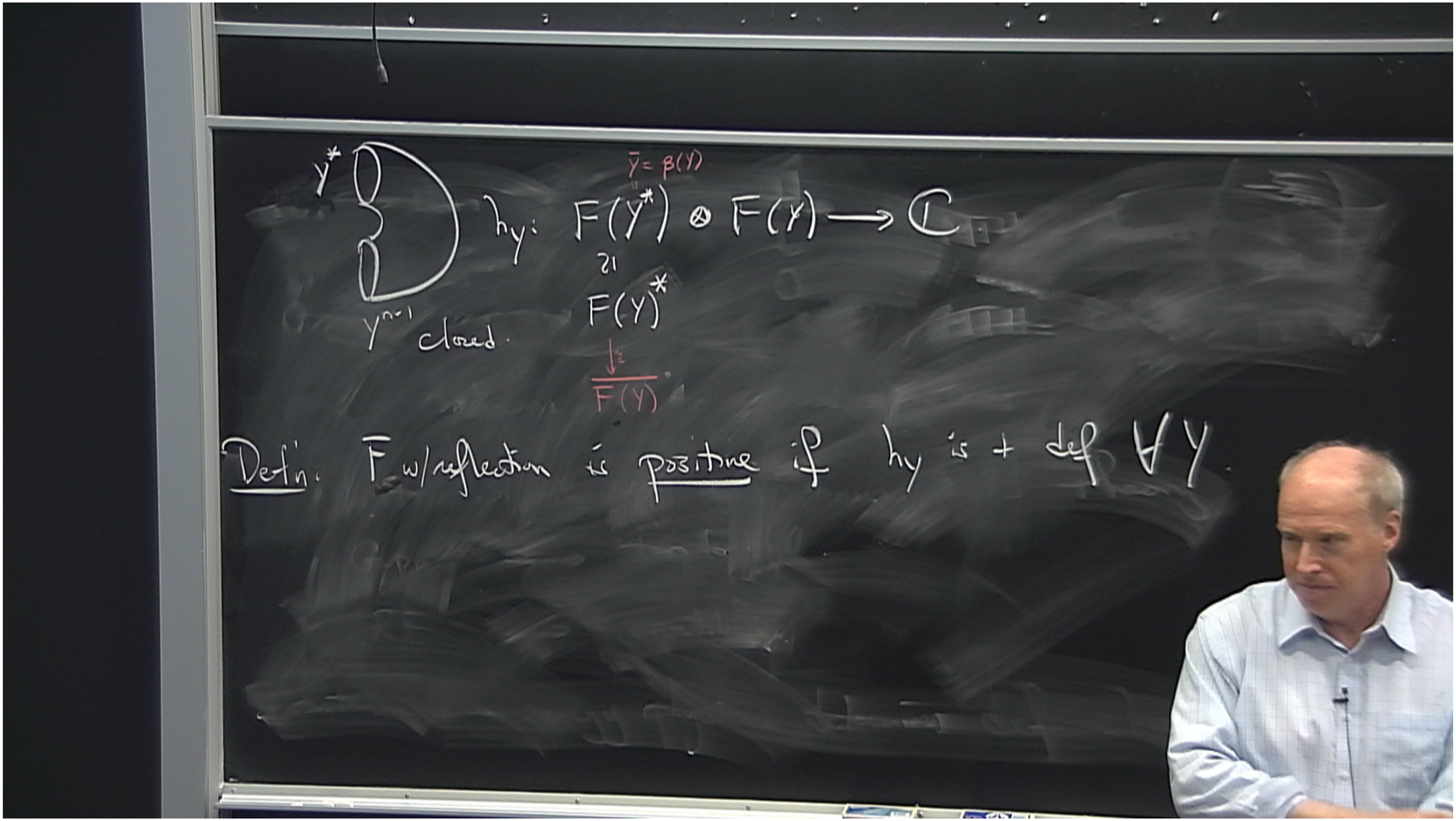
	H_n	$\hat{H}_n$
bosons only, no TR	SO_n	O_n
fermions allowed, no TR	$Spin_n$	Pin_n^+
bosons, TR	O_n	$O_n \times \{\pm 1\}$
fermions, $T^2 = (-1)^F$	Pin_n^+	
fermions, $T^2 = 1$	Pin_n^-	
— with global G	— $\times G$	

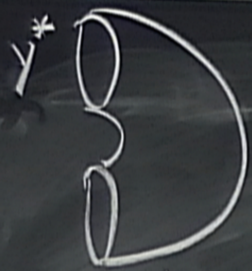
Thm (F-Hopkins): $\left\{ \begin{array}{l} \text{Def classes of} \\ n\text{-dim'l reflection positive} \\ \text{invertible TFT w/ symmetry } H_n \end{array} \right\} \cong [\underset{MH^\perp}{MTH}, \Sigma^{n+1} I\mathbb{Z}]$


Compact Riemannian manifolds M^n w/ H_n-structure
Involution $M \longmapsto \beta(M)$
Field theory:
$F: Bord_{\langle n-1,n \rangle}(H_n) \longrightarrow tVect_{\mathbb{C}}$
$t = -i\tau, \; (\tau > 0)$
CAUTION



Compact Riemannian manifolds w/ Hn-structure
M^n
P
↓Hn ,
M
O(M) ≅ ρn(P)
On
M
Involution M ⟼ β(M)
Field theory:
F: Bord<n-1,n>(Hn) ⟶ tVect C
β
β
Def'n: A reflection structure on F
is a β-equivariant extension.


Y*
Y^{n-1} closed.
h_Y: F(Y*) ⊗ F(Y) ⟶ ℂ
Ȳ = β(Y)
≅
F(Y)*
↓≅
F(Ȳ)
Def'n: F w/reflection is positive if h_Y is + def ∀ Y.



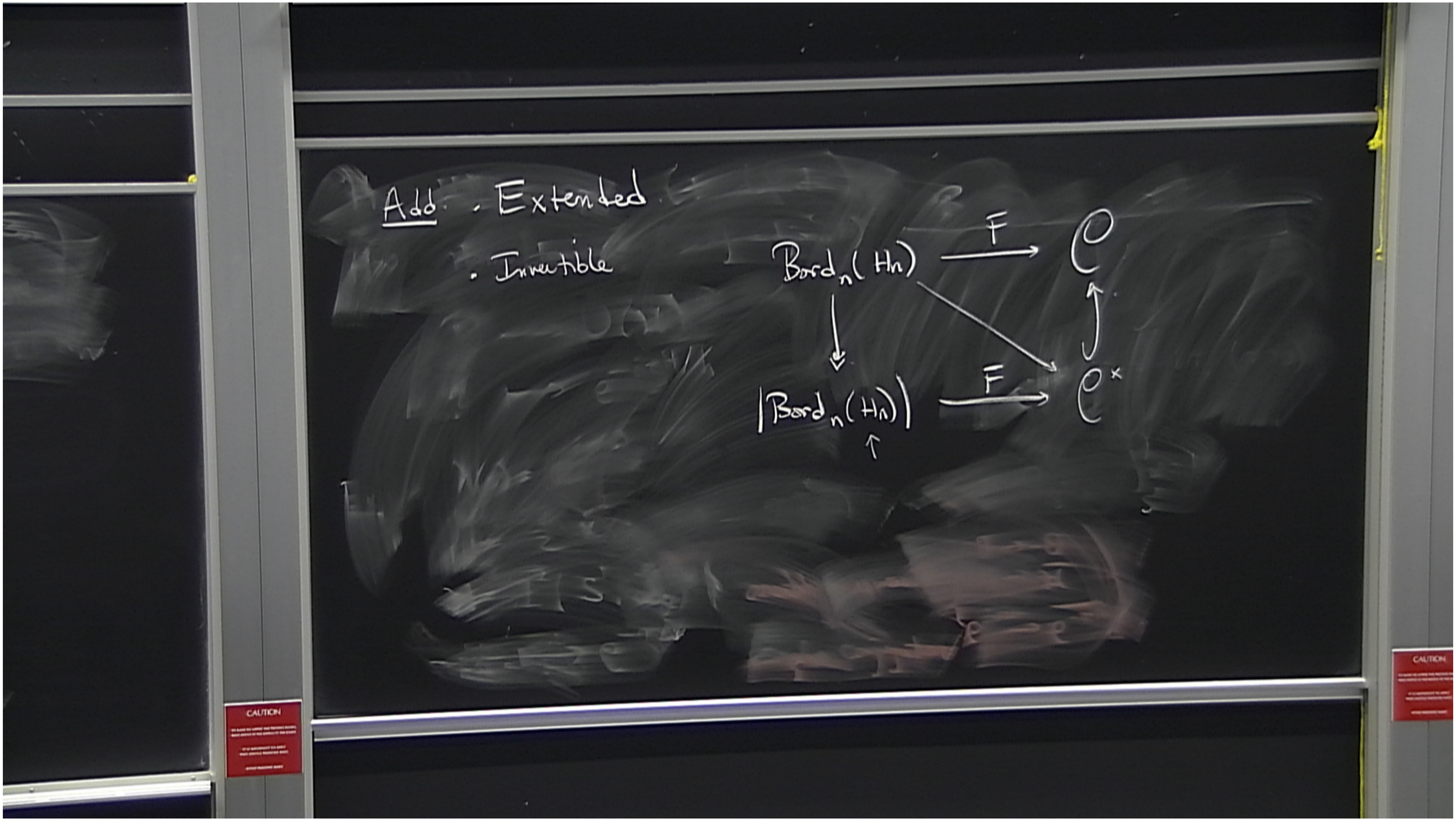
Y^{n-1} closed.

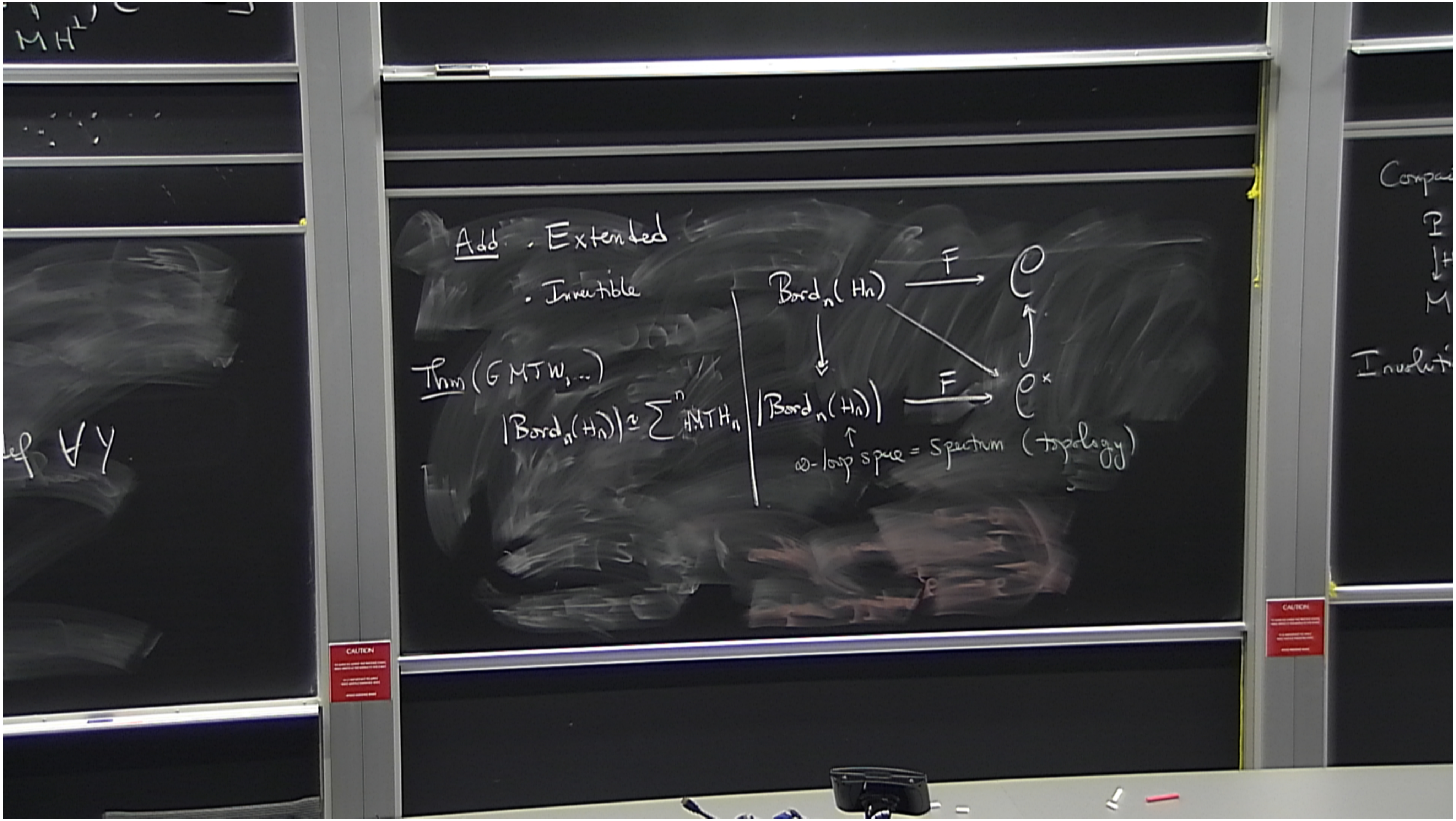
$$h_Y : F(Y^*) \otimes F(Y) \longrightarrow \mathbb{C}$$

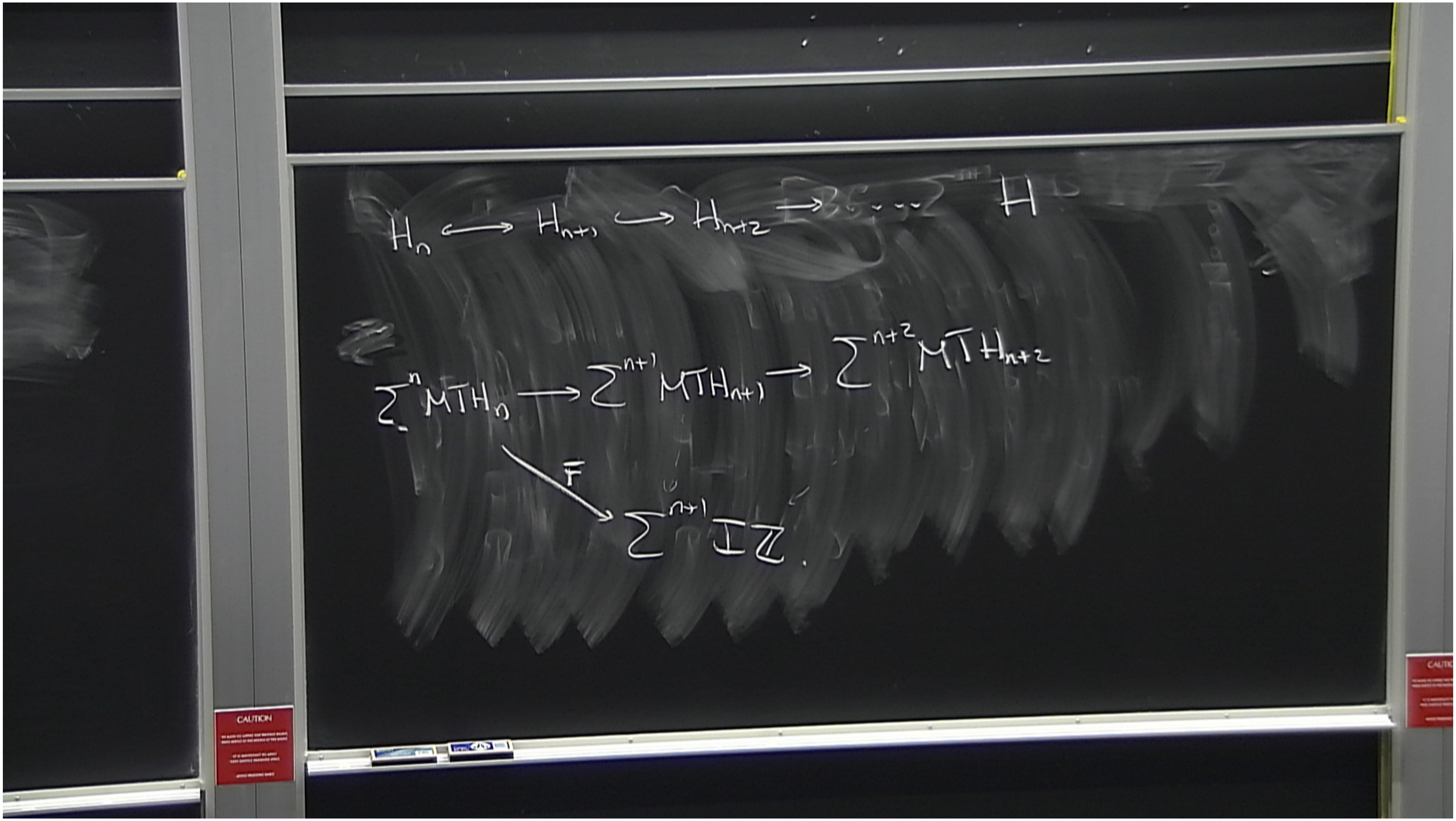
$$Y^* = \bar{Y} = \beta(Y)$$

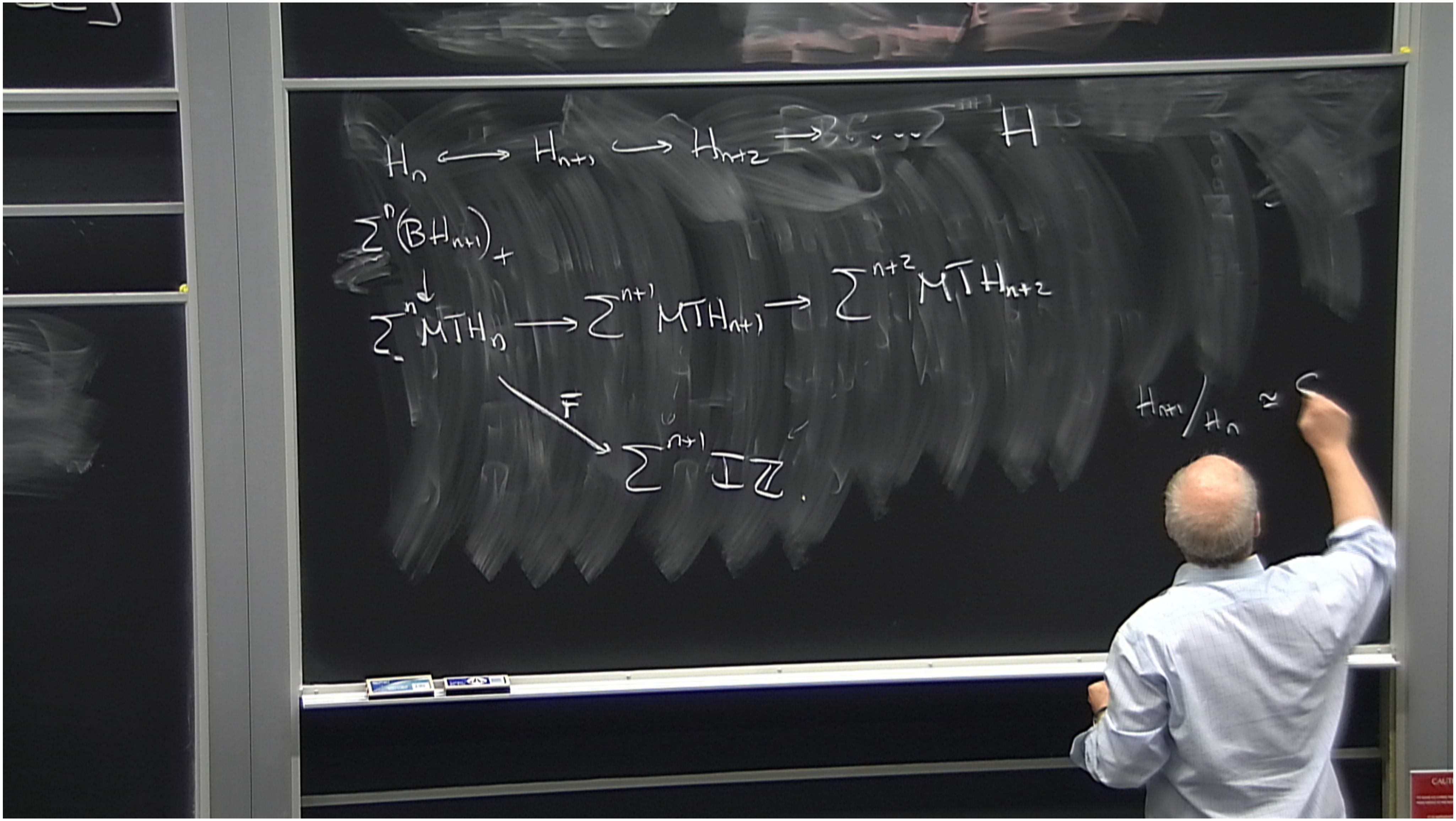
$$F(Y^*) \cong F(Y)^* \xrightarrow{\ \cong\ } \overline{F(Y)}$$

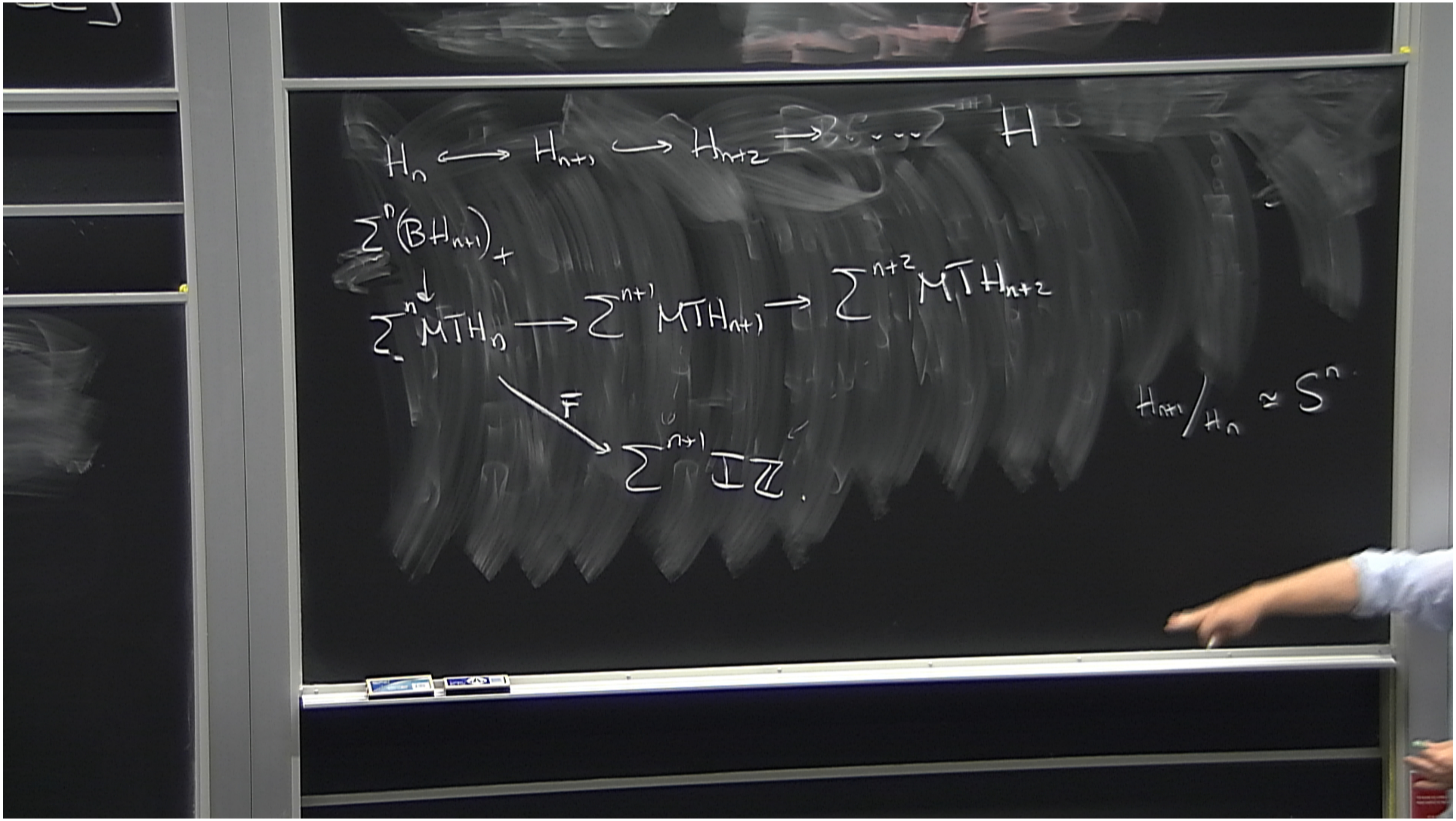
<u>Def'n</u>: F w/ reflection is <u>positive</u> if h_Y is + def $\forall\, Y$.

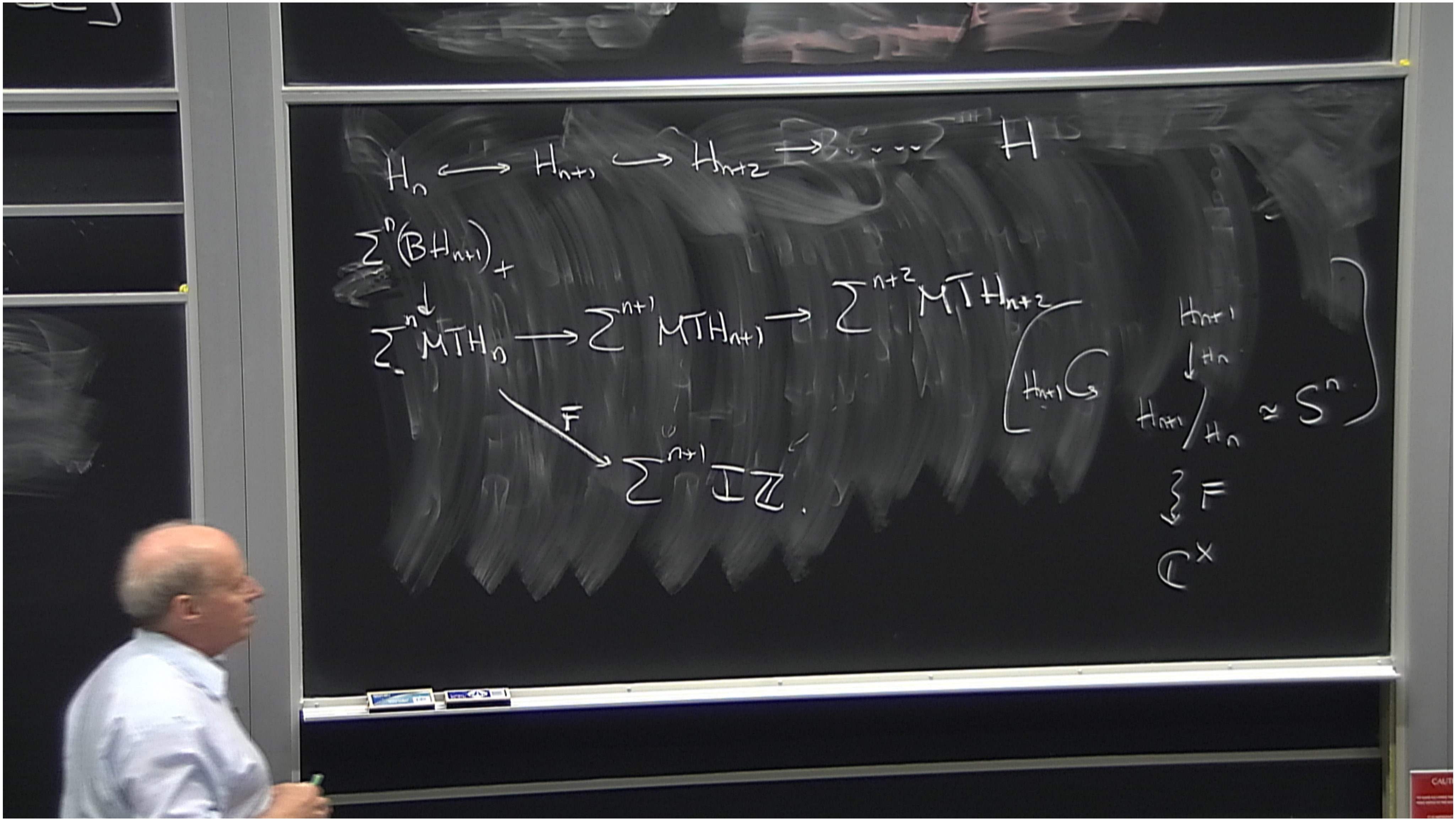

Add · Extended
· Invertible
Bord_n(Hn)
F
C
|Bord_n(Hn)|
F
C^×

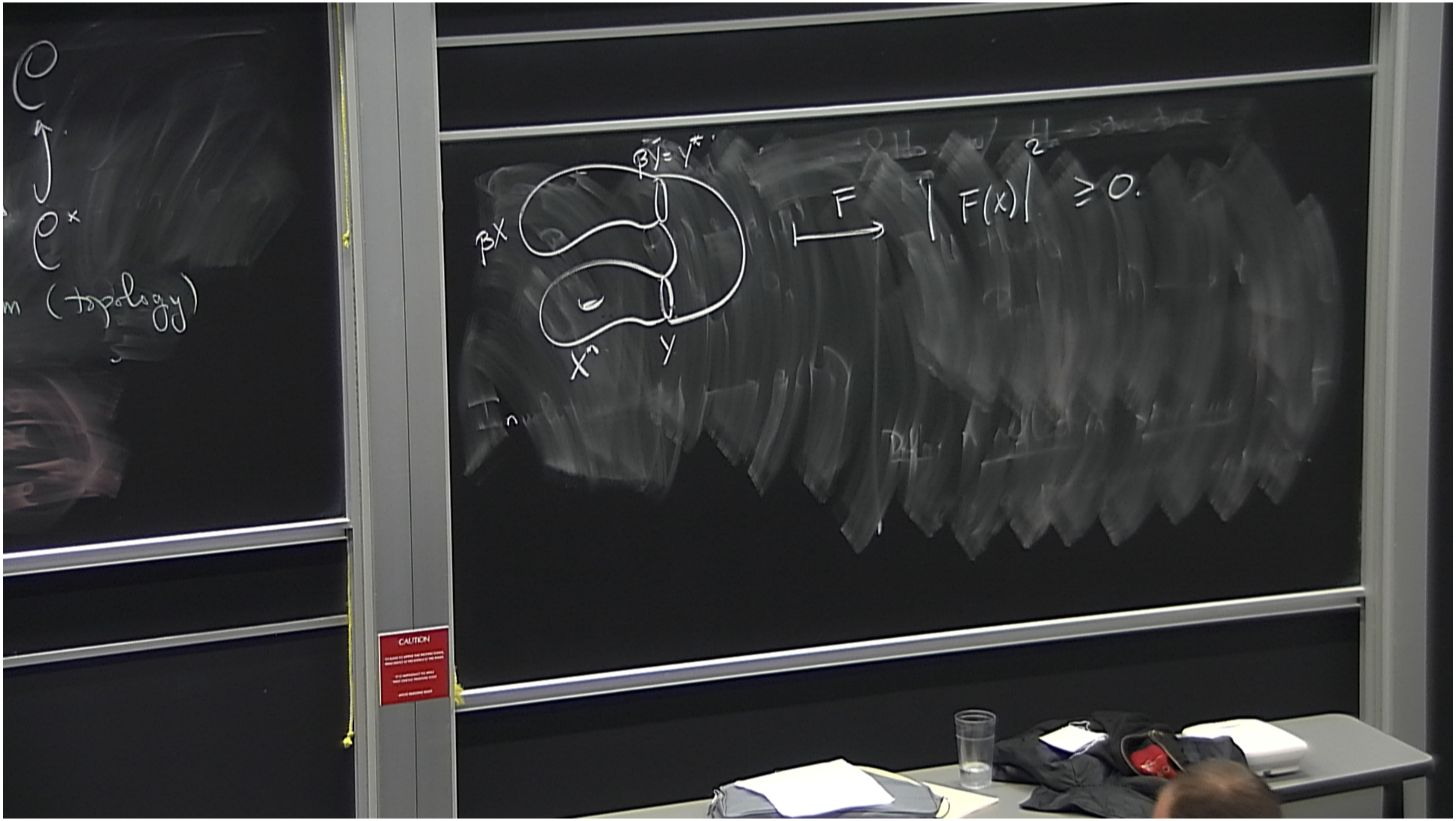

Add · Extended
· Invertible
Thm (GMTW, ..)
|Bord_n(H_n)| ≃ Σ^n MTH_n
Bord_n(H_n) F → C
|Bord_n(H_n)| F → C^×
∞-loop space = Spectrum (topology)


$H_n \hookrightarrow H_{n+1} \hookrightarrow H_{n+2} \to \cdots \quad H$
$\Sigma^n MTH_n \to \Sigma^{n+1} MTH_{n+1} \to \Sigma^{n+2} MTH_{n+2}$
F
$\Sigma^{n+1} I\mathbb{Z}$
CAUTION


$H_n \hookrightarrow H_{n+1} \hookrightarrow H_{n+2} \to \cdots\ H$
$\Sigma^n (BH_{n+1})_+$
$\Sigma^n MTH_n \to \Sigma^{n+1} MTH_{n+1} \to \Sigma^{n+2} MTH_{n+2}$
F
$\Sigma^{n+1} I\mathbb{Z}$
$H_{n+1}/H_n \simeq$


$H_n \hookrightarrow H_{n+1} \hookrightarrow H_{n+2} \to \cdots\ H$
$\Sigma^n (BH_{n+1})_+$
$\Sigma^n MTH_n \to \Sigma^{n+1} MTH_{n+1} \to \Sigma^{n+2} MTH_{n+2}$
F
$\Sigma^{n+1} I\mathbb{Z}$
$H_{n+1}/H_n \simeq S^n$


$H_n \hookrightarrow H_{n+1} \hookrightarrow H_{n+2} \to \cdots \quad H$
$\Sigma^n (BH_{n+1})_+$
$\Sigma^n MTH_n \to \Sigma^{n+1} MTH_{n+1} \to \Sigma^{n+2} MTH_{n+2}$
F
$\Sigma^{n+1} I\mathbb{Z}$
H_{n+1}
$H_{n+1} / H_n \simeq S^n$
F
$\mathbb{C}^\times$


(topology)
βX
βY = Y*
Y
F
|F(x)|² ≥ 0.