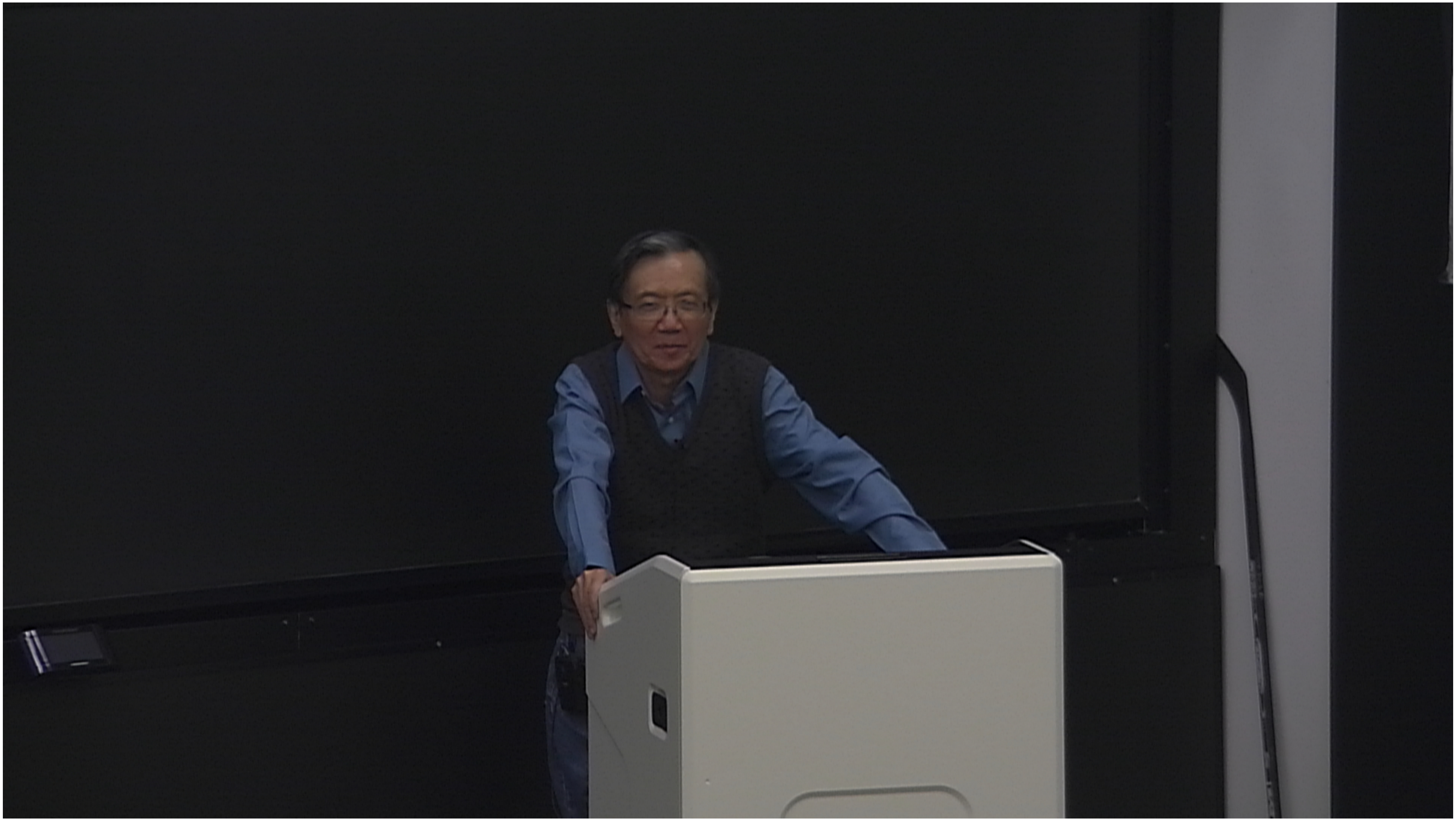
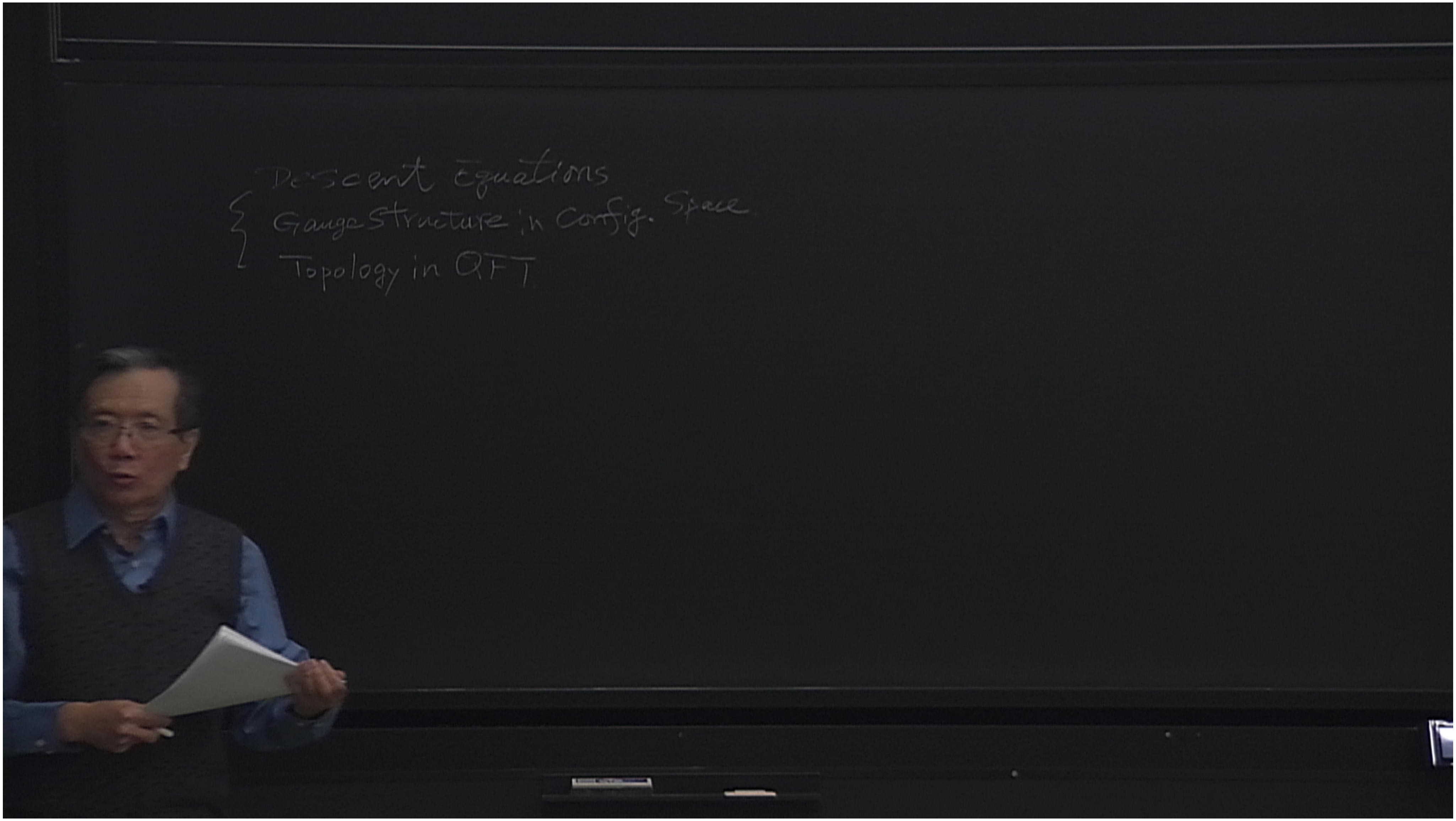
Title: Descent Equations, Gauge Structure in Configuration Space and Topology of Quantum Field Theory - Yongshi Wu

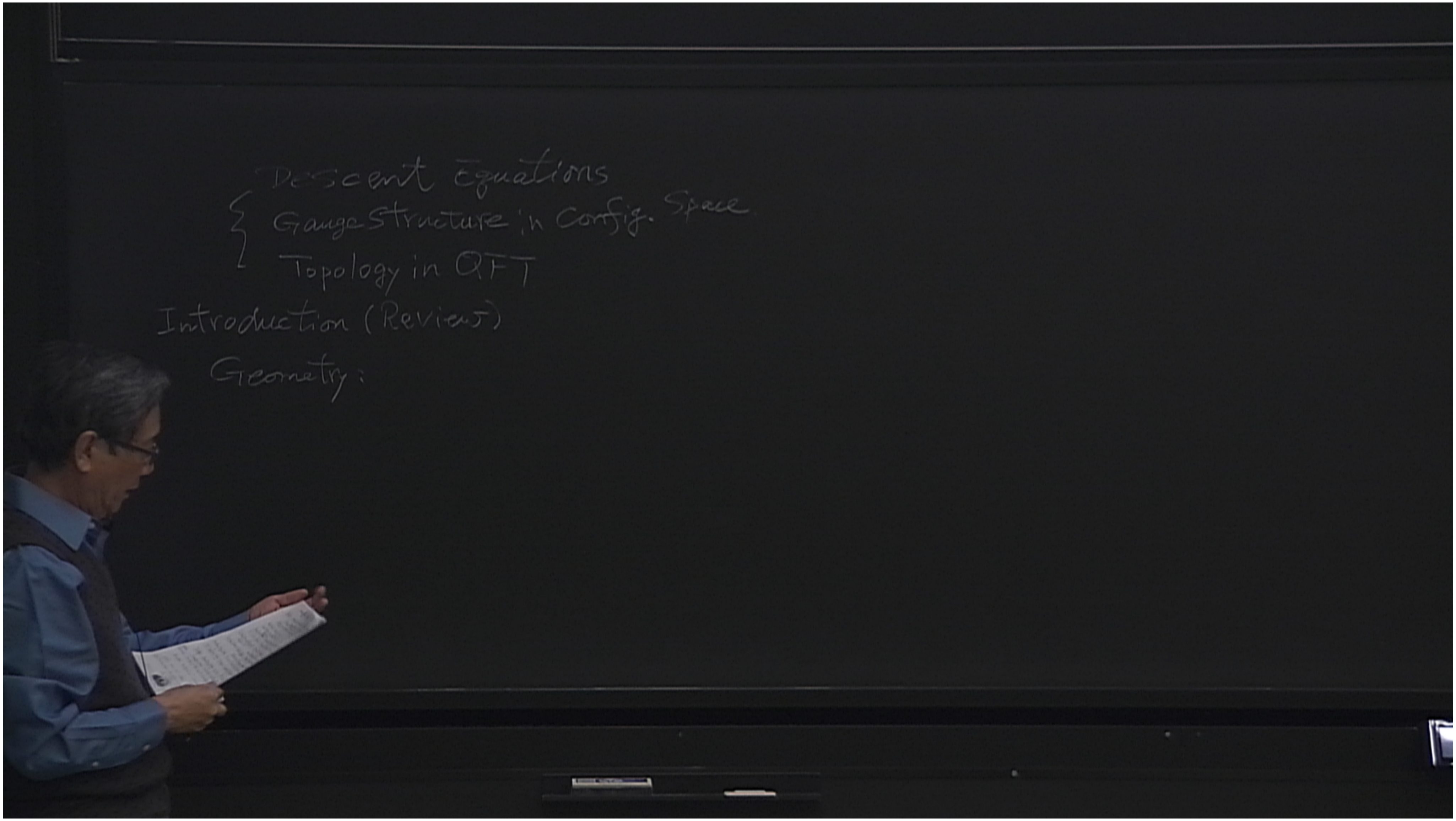
Date: Oct 14, 2015 02:00 PM

URL: http://pirsa.org/15100100

Abstract: <p>Topological aspects of physical systems, including the called topological states of matter, have become hot topics in the frontiers of physics in recent years. Here I would like to present a mathematically "popular" talk for professional physicists for a highlight or overview of how one can systematize knowledge of topological aspects of quantum field theories. Our starting points are Descent Equations and Gauge Structure in Configuration Space in Field Theory. (The audience needs only to know the meaning of "differential forms".)</p>




Descent Equations
Gauge Structure in Config. Space
Topology in QFT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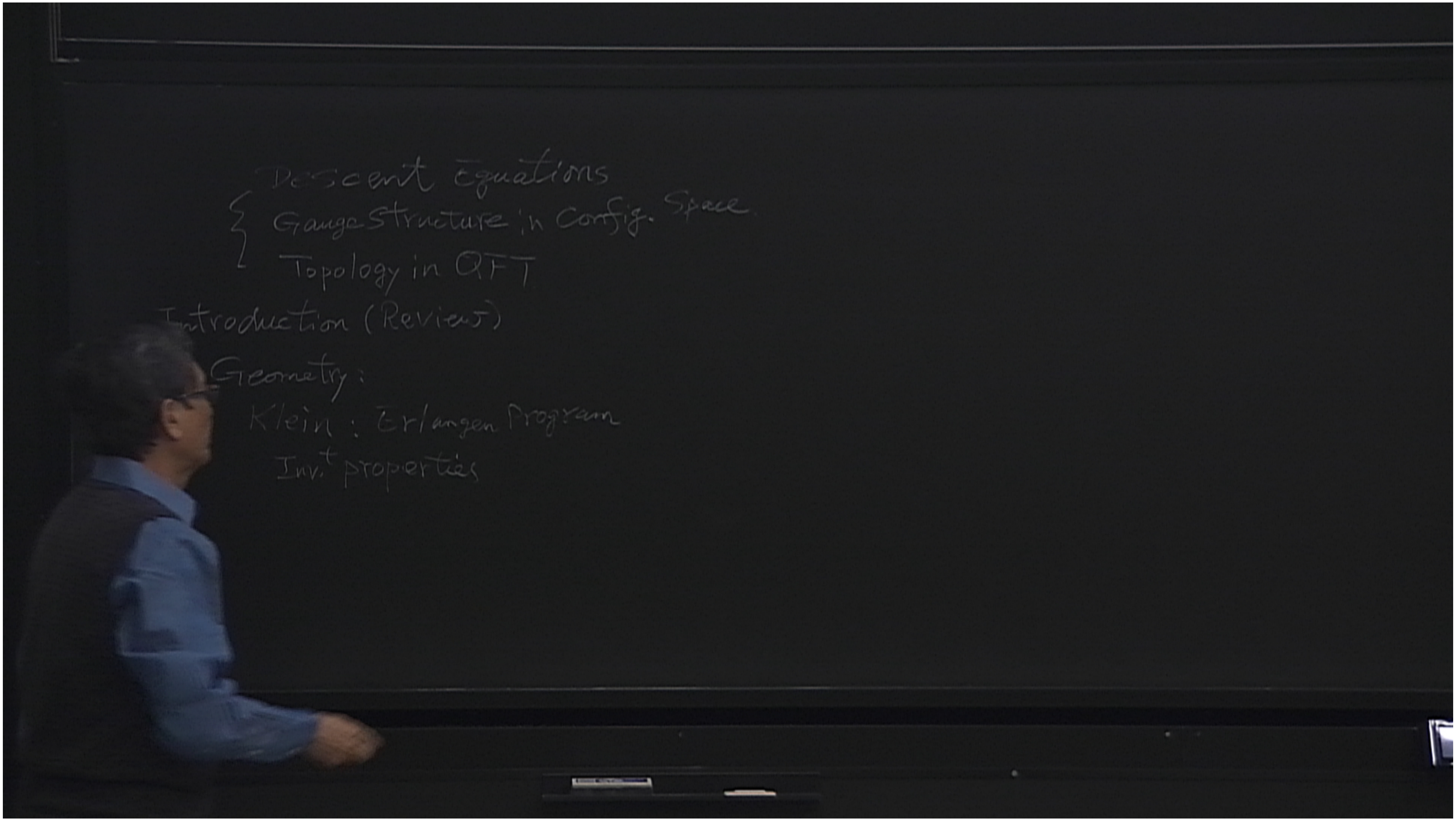
Introduction (Review)
Geometry: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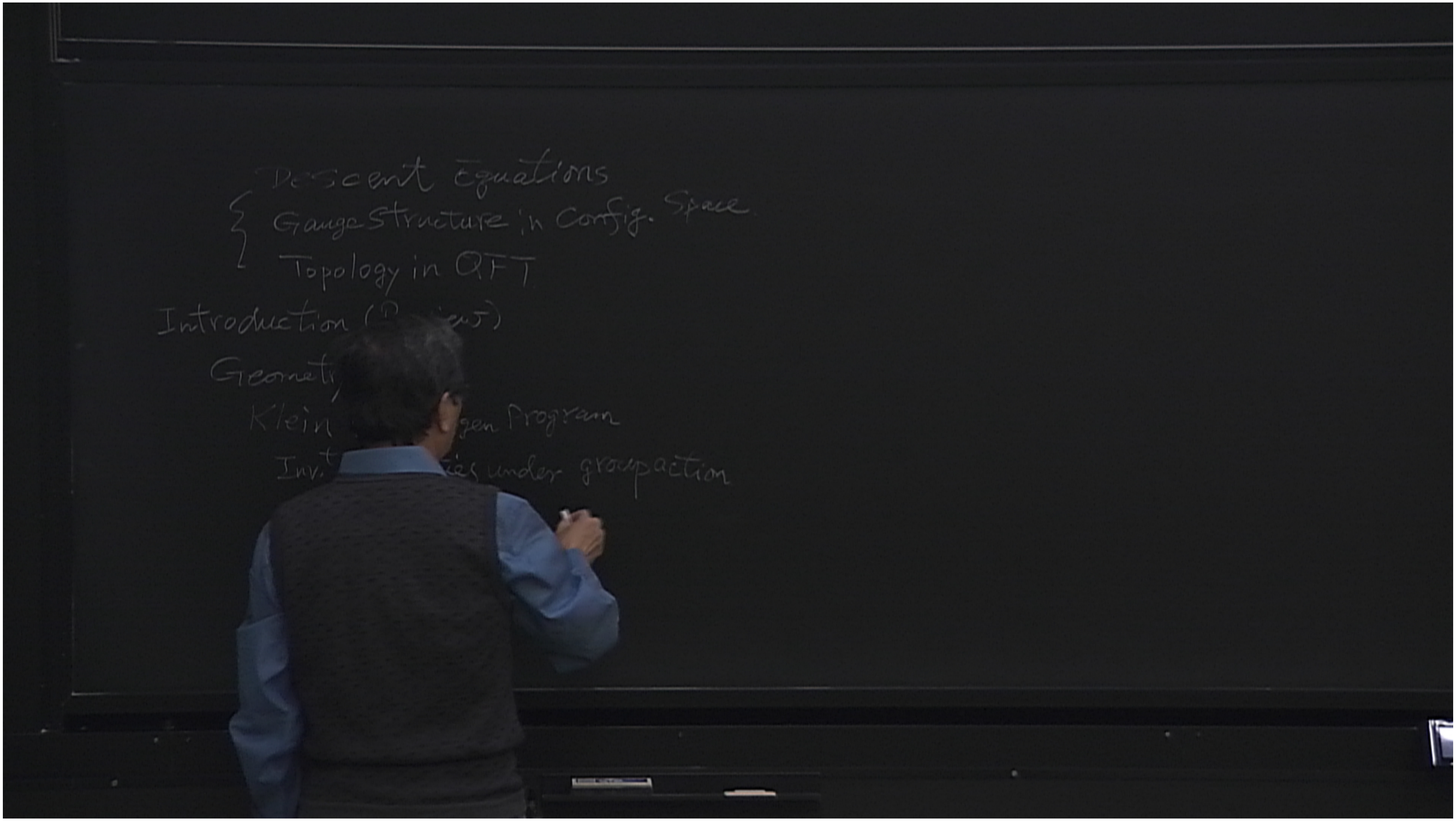
Klein: Erlangen Program
Inv.t properties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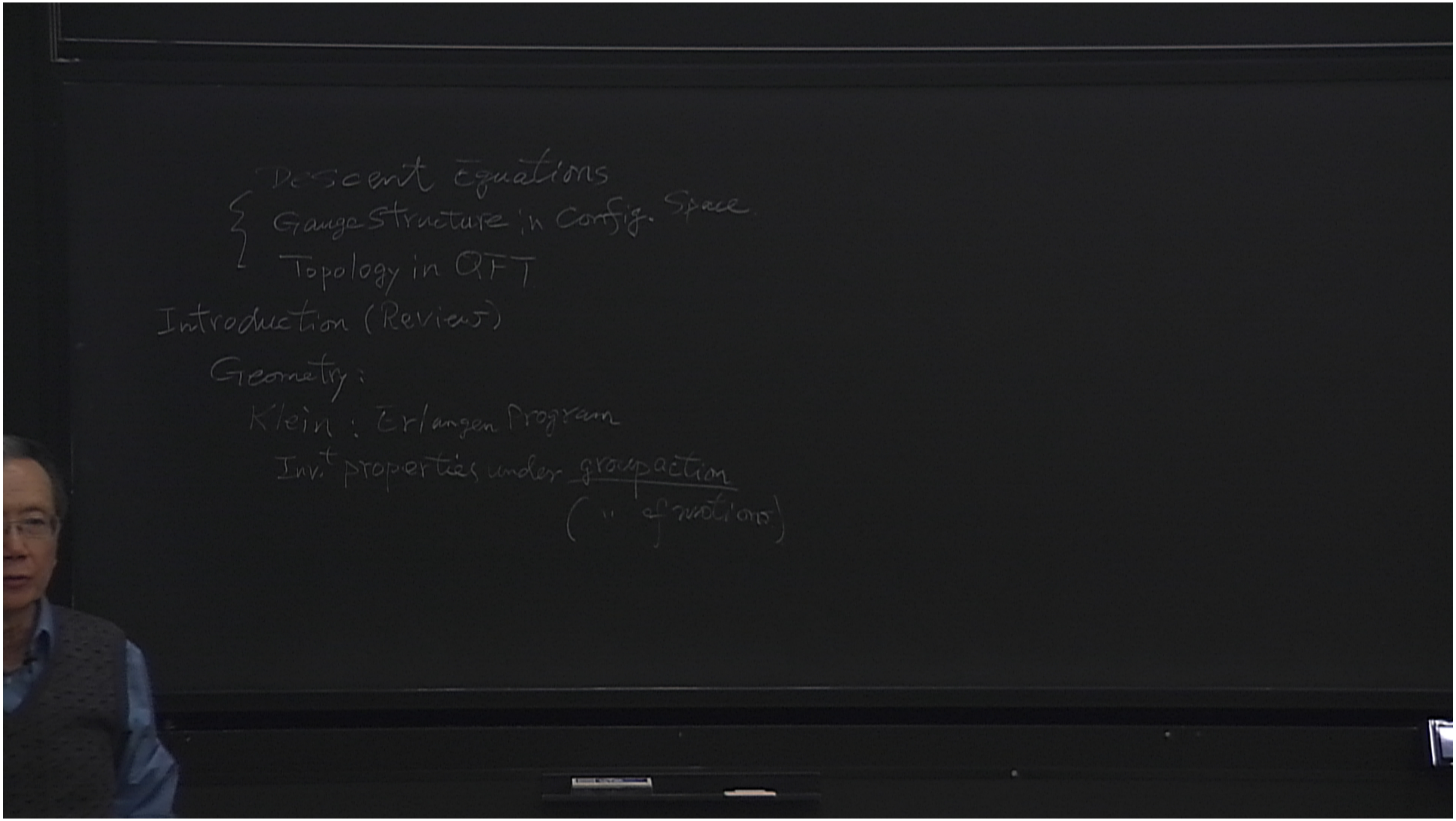
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Descent Equations
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Introduction (Review)

Geometry:

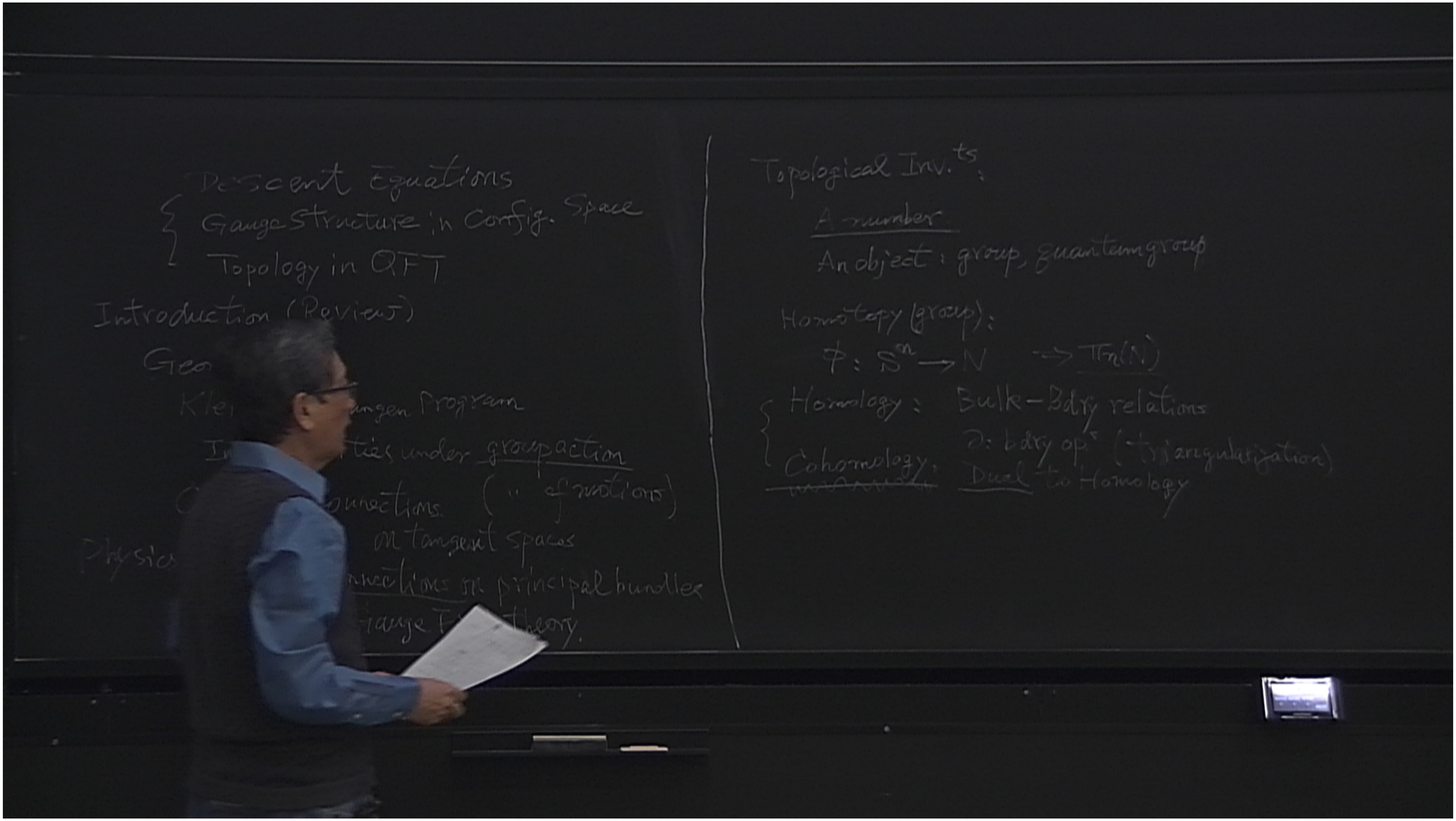
Klein: Erlangen Program

Inv.t properties under group action

Cartan: Connections (" of motions)
on tangent spaces

Physics:
Connections on principal bundles

Yang-Mills: Gauge Field theory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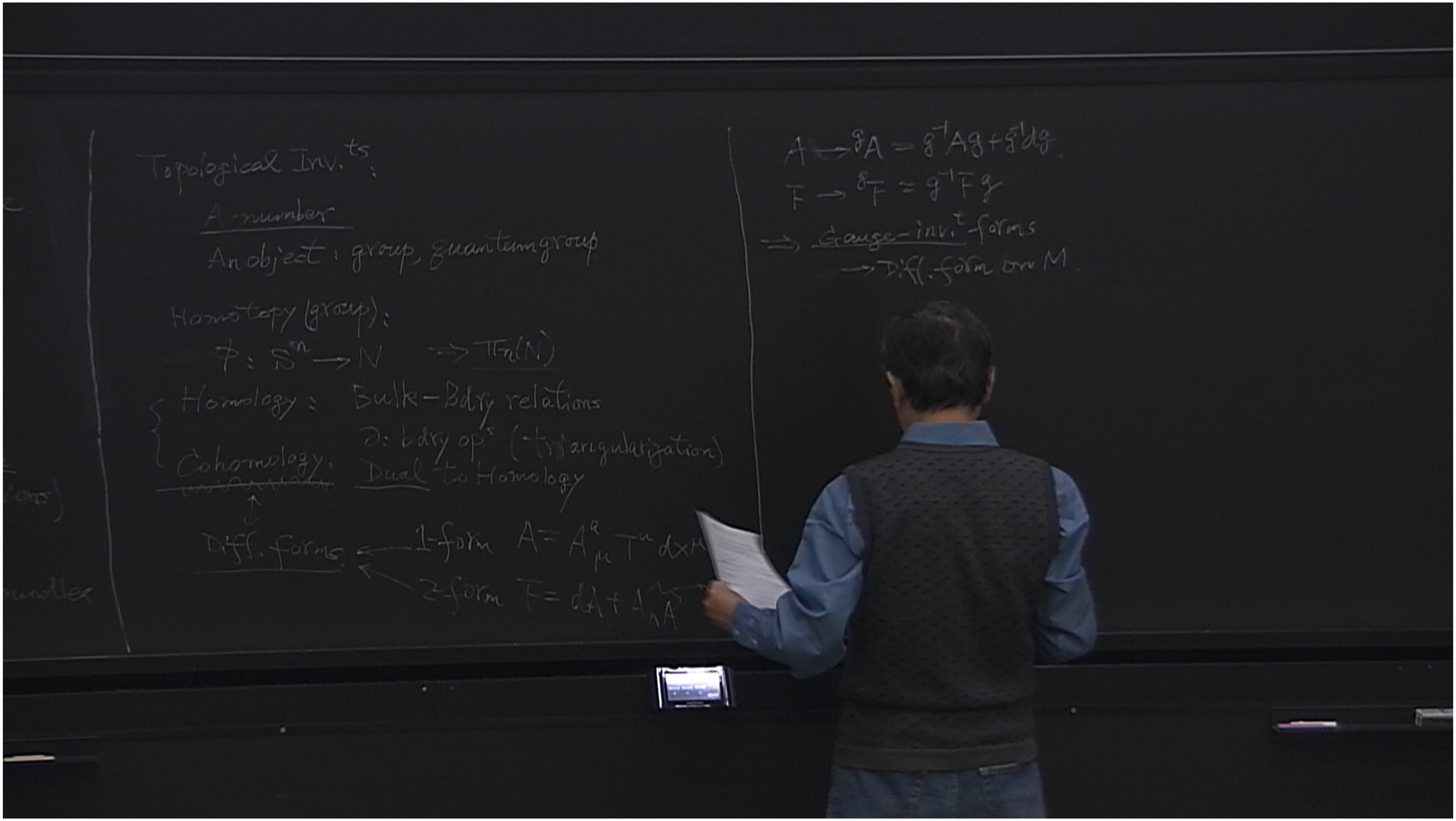
theory.
Topological Inv.ts:
A number
An object: group, quantum group
Homotopy (group):
Φ: S^n → N --> π_n(N)
Homology: Bulk–Bdry relations
∂: bdry op^r (triangularization)
Cohomology: Dual to Homology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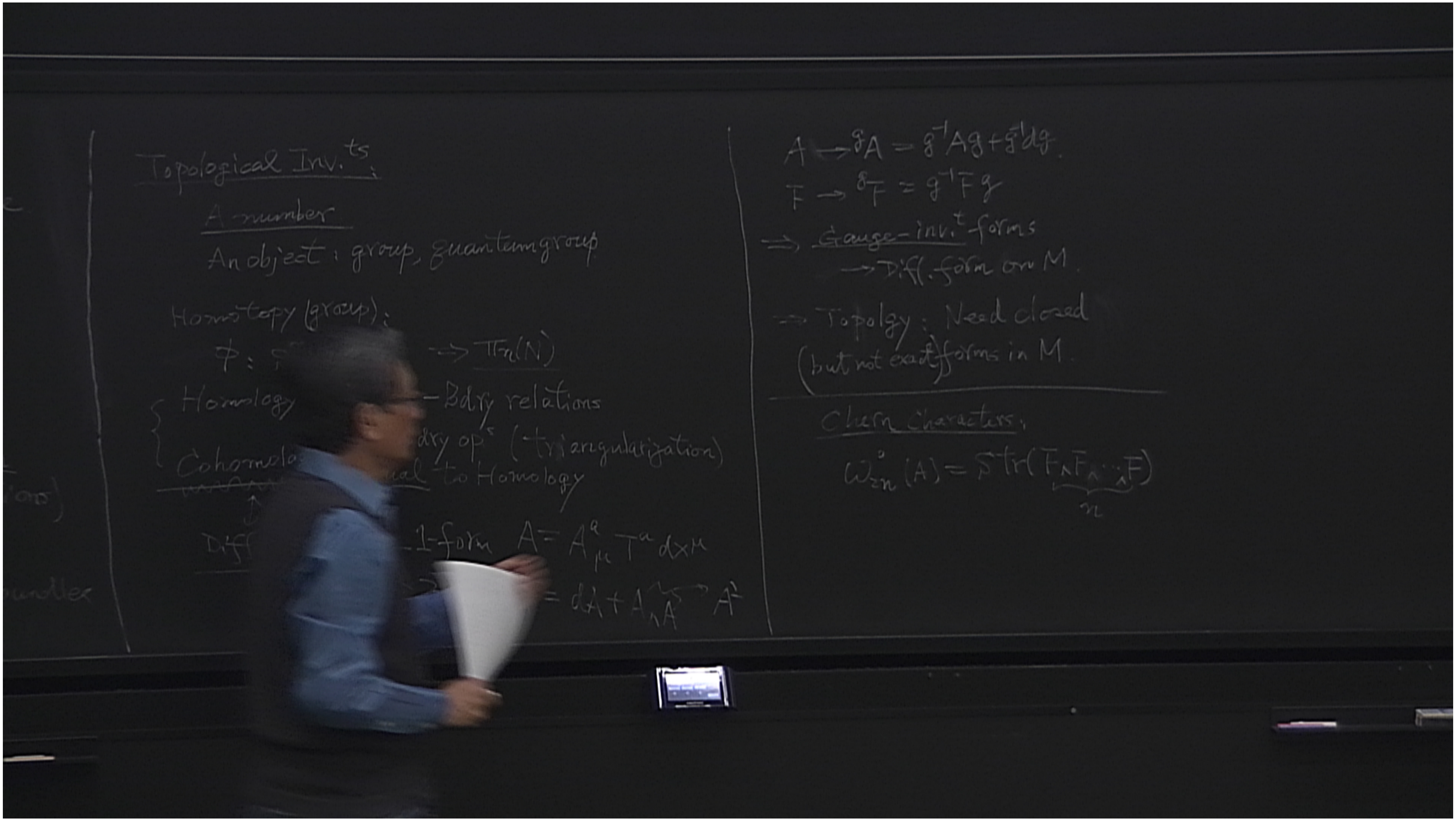
Diff. forms ← 1-form A = A^a_μ T^a dx^μ
← 2-form F = dA + A∧A
A → ^gA = g^{-1}Ag + g^{-1}dg
F → ^gF = g^{-1}Fg
⇒ Gauge-inv.t forms
→ Diff. form on M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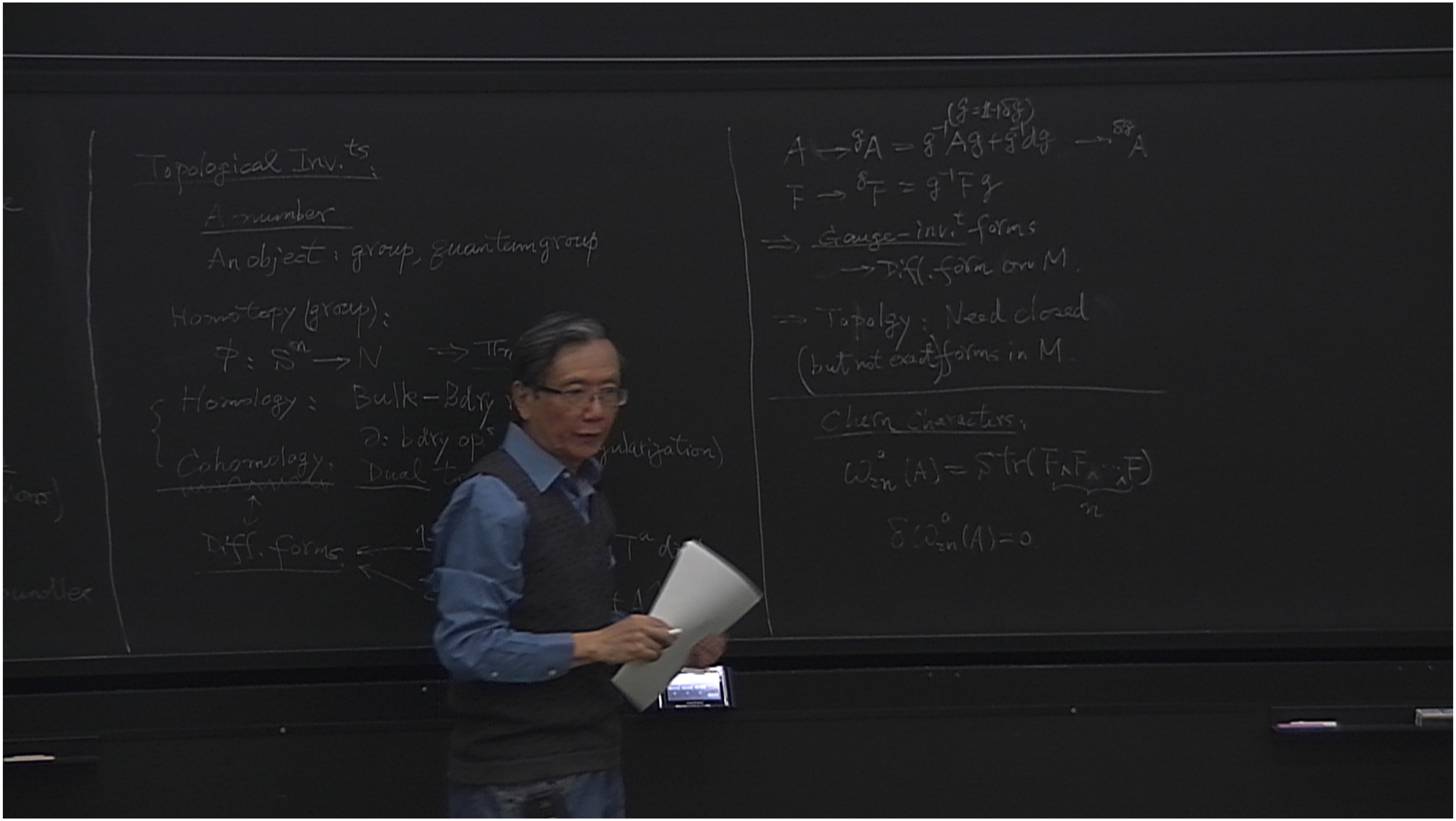
⇒ Topolgy: Need closed
(but not exact) forms in M.
Chern Characters:
ω^0_{2n}(A) = ∫ tr(F∧F∧…∧F)
n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Diff. forms
(g = 1 + δg)
A → ^gA = g^-1 A g + g^-1 dg → ^δg A
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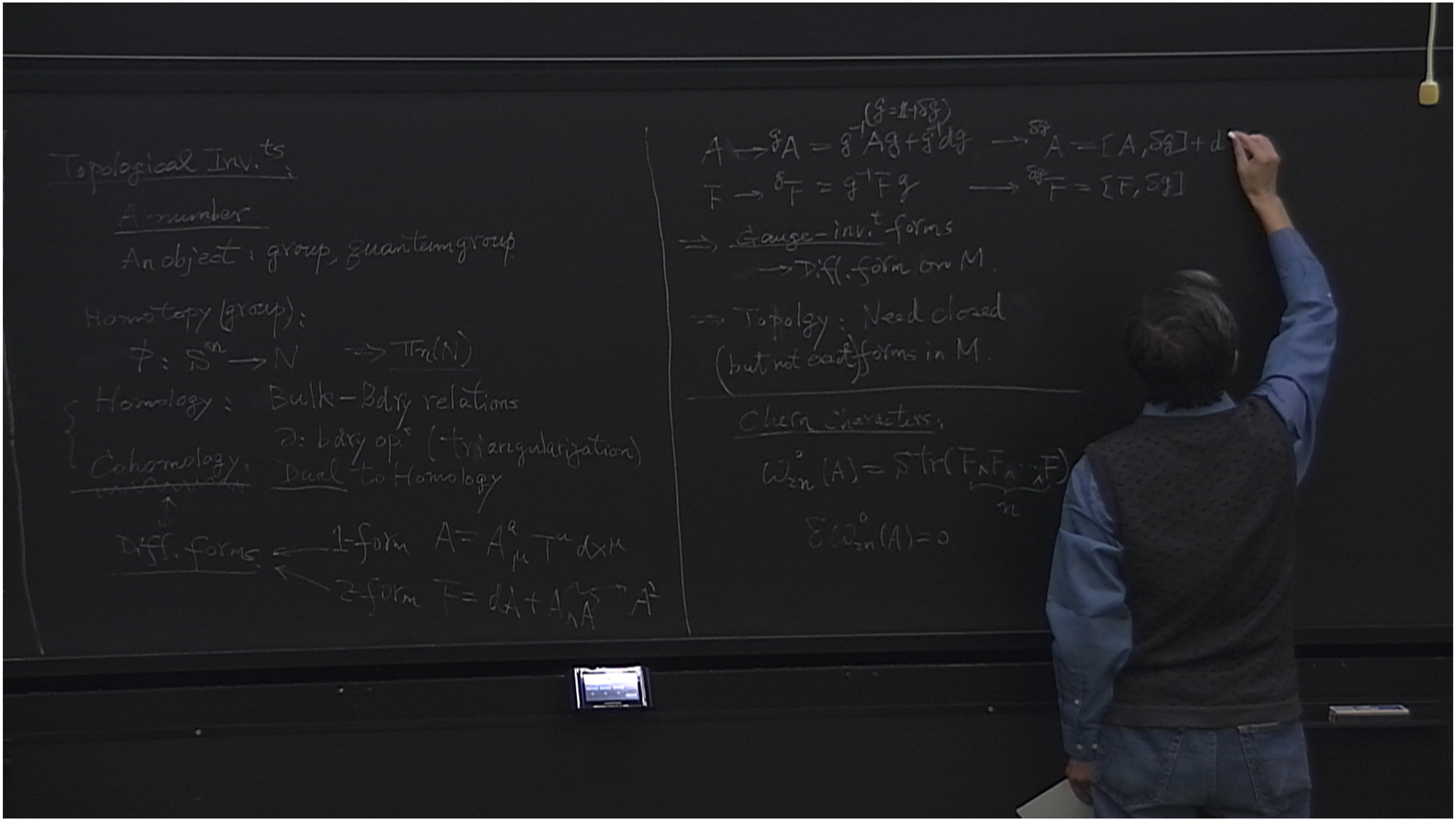
δω^0_2n (A) = 0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← 2-form F = dA + A∧A → A²
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F → ^gF = g⁻¹Fg ⟶ ^δgF = [F, δg]
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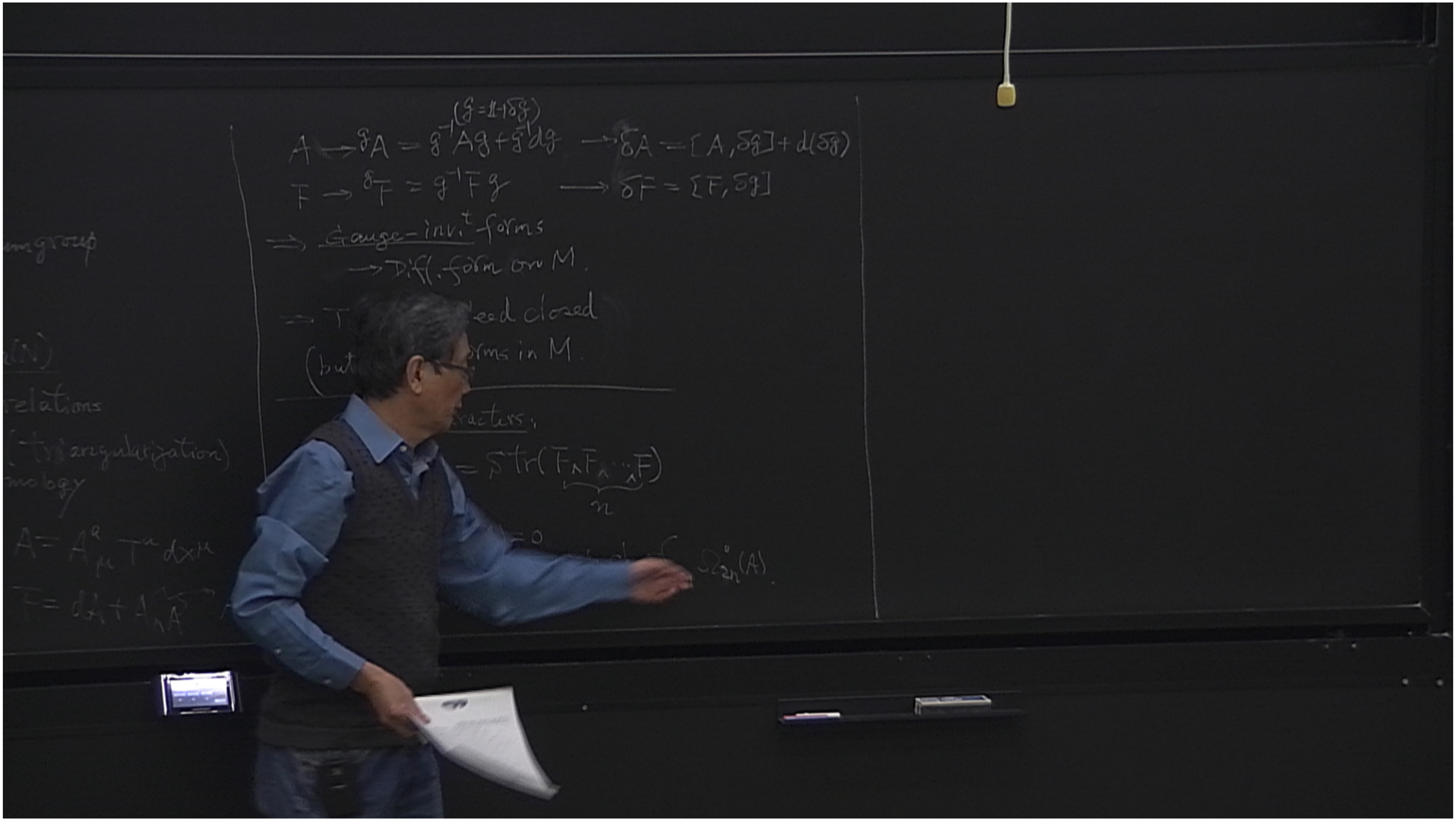
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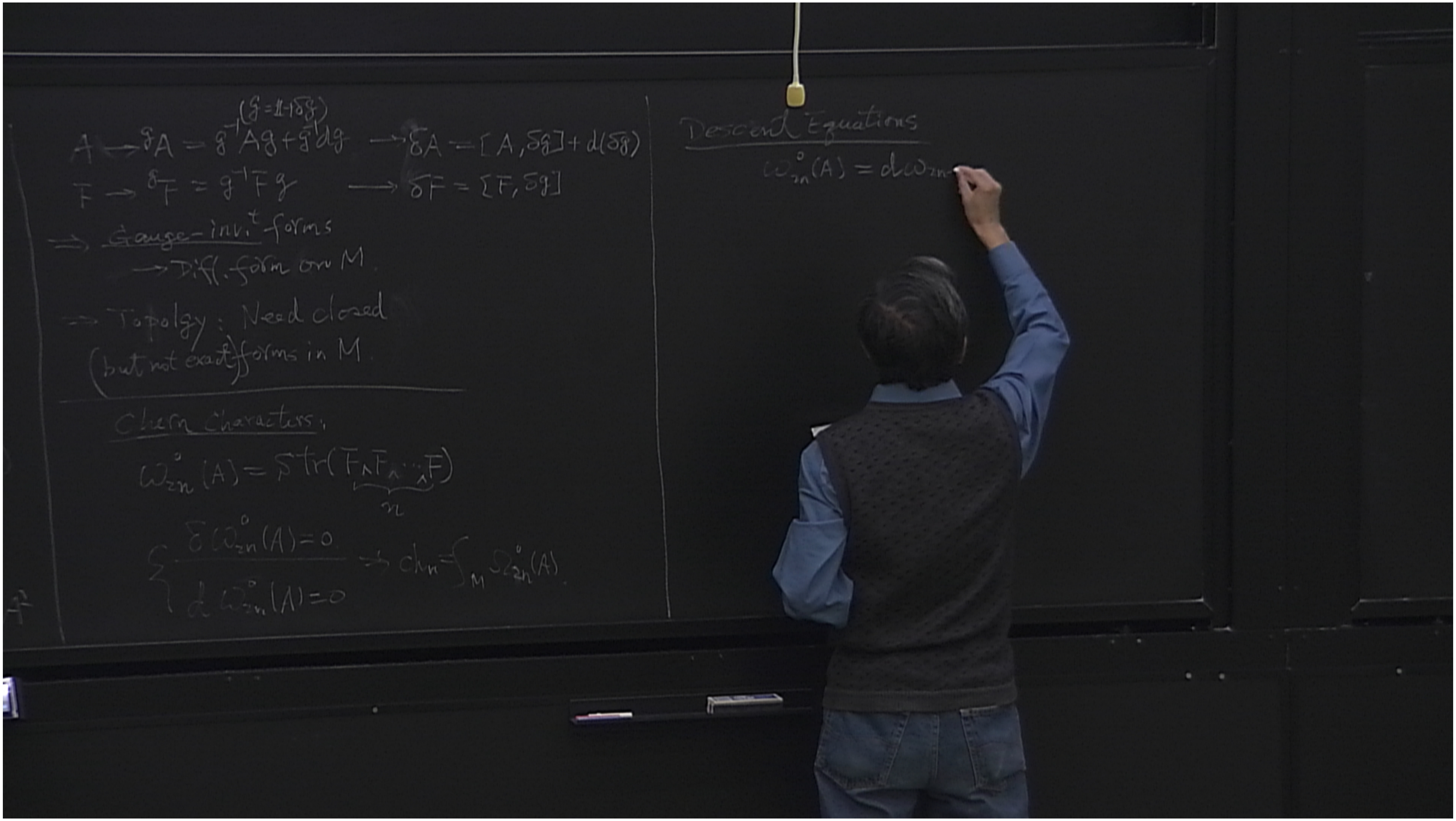
$(g = 1 + \delta g)$
$A \to {}^{g}A = g^{-1}Ag + g^{-1}dg \longrightarrow \delta A = [A, \delta g] + d(\delta g)$
$F \to {}^{g}F = g^{-1}Fg \longrightarrow \delta F = [F, \delta g]$
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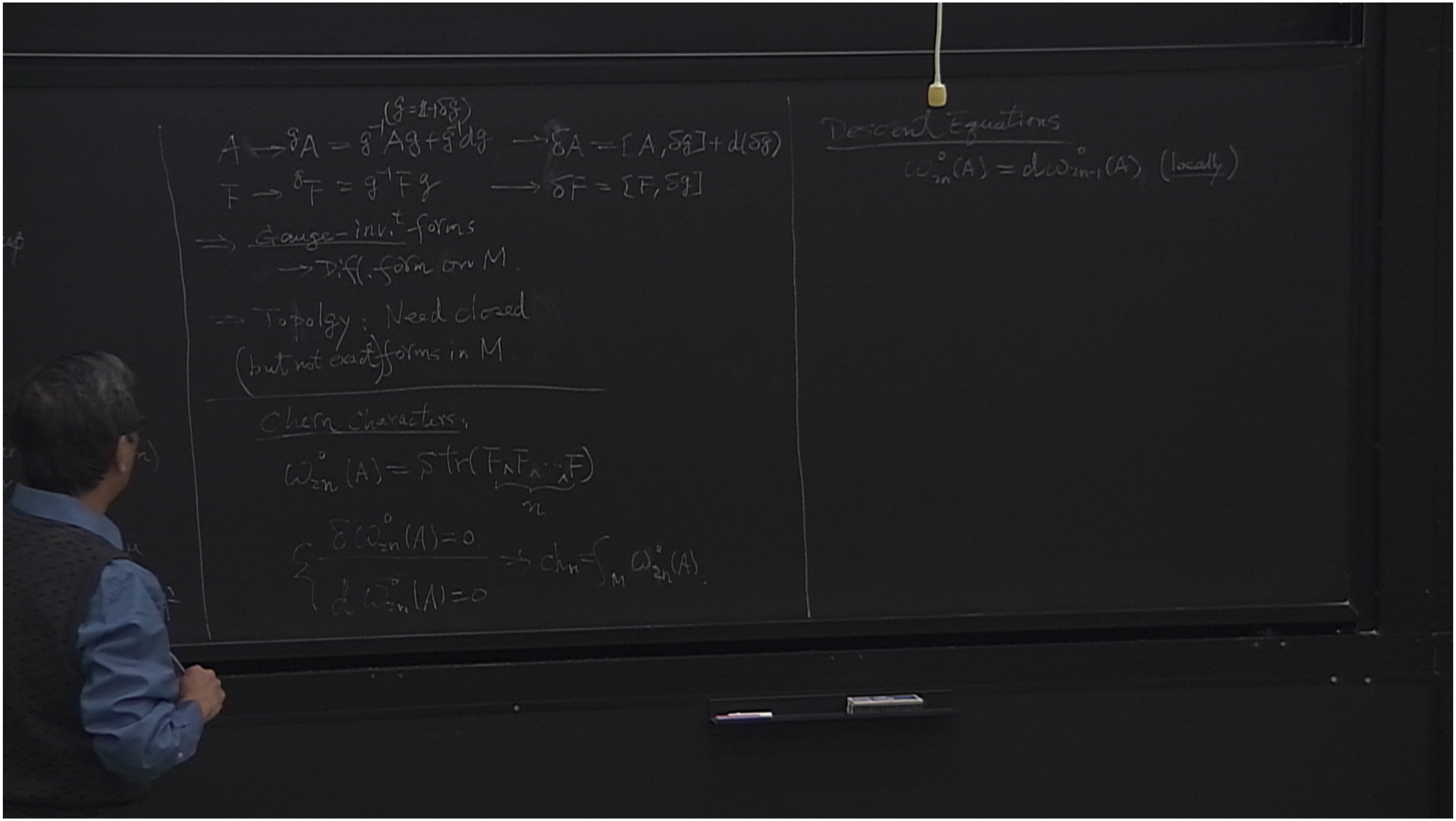
Chern Characters:
$\omega^{0}_{2n}(A) = \mathrm{Str}(\underbrace{F \wedge F \wedge \cdots \wedge F}_{n})$
$\delta\,\omega^{0}_{2n}(A) = 0$
$d\,\omega^{0}_{2n}(A) = 0$
$\Rightarrow ch_n = \int_M \omega^{0}_{2n}(A)$
Descent Equations
$\omega^{0}_{2n}(A) = d\,\omega_{2n}$



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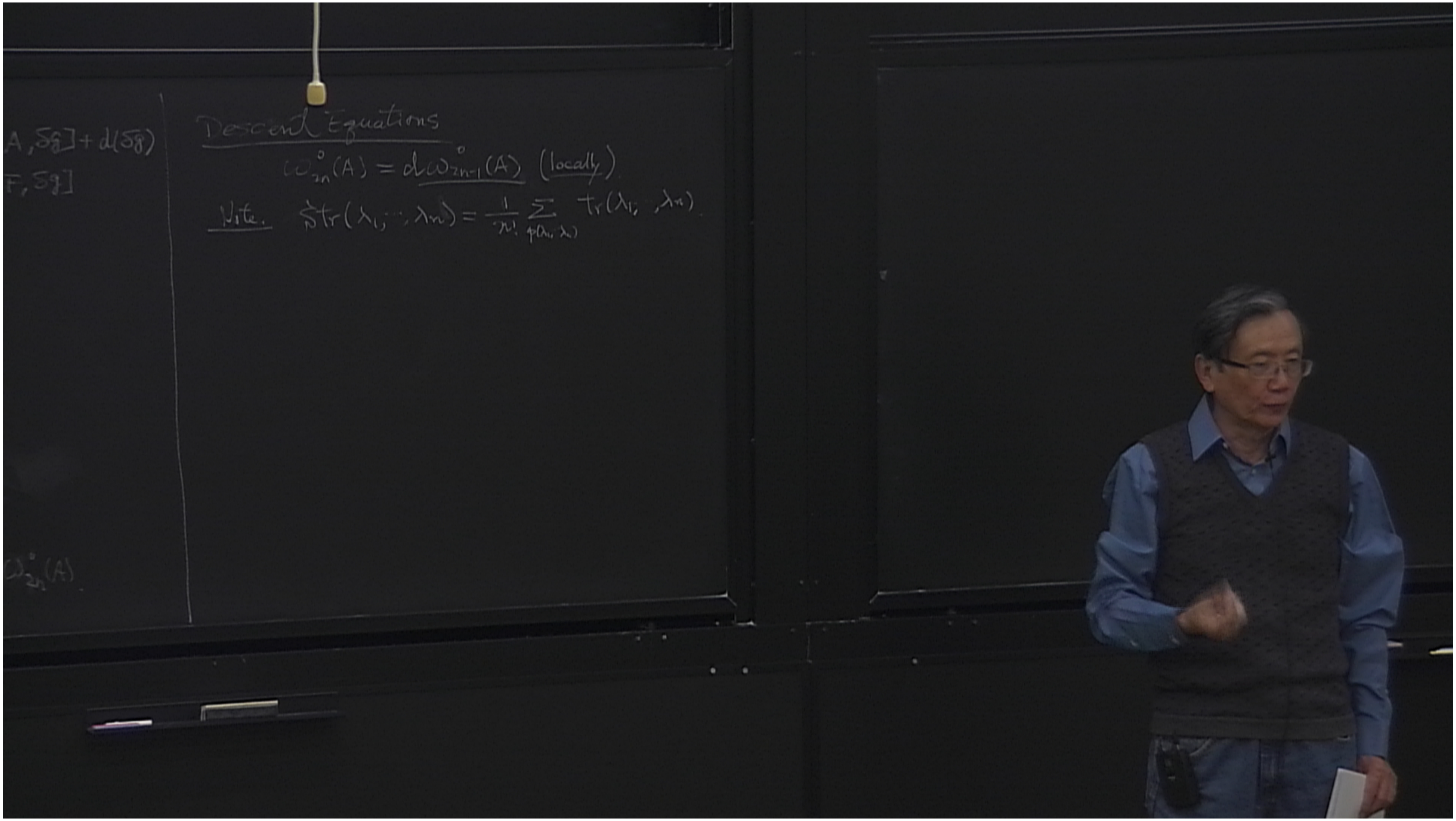
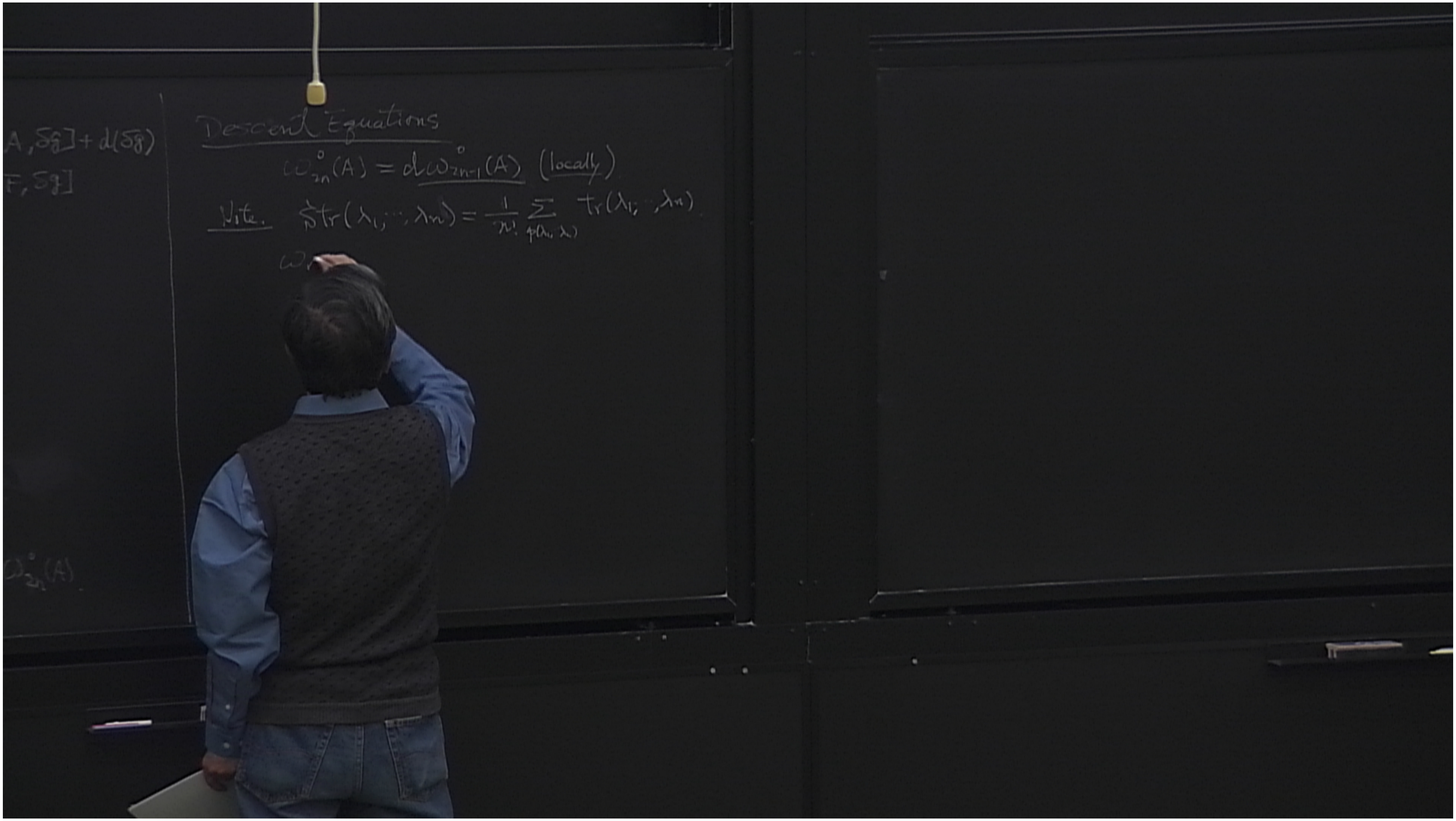
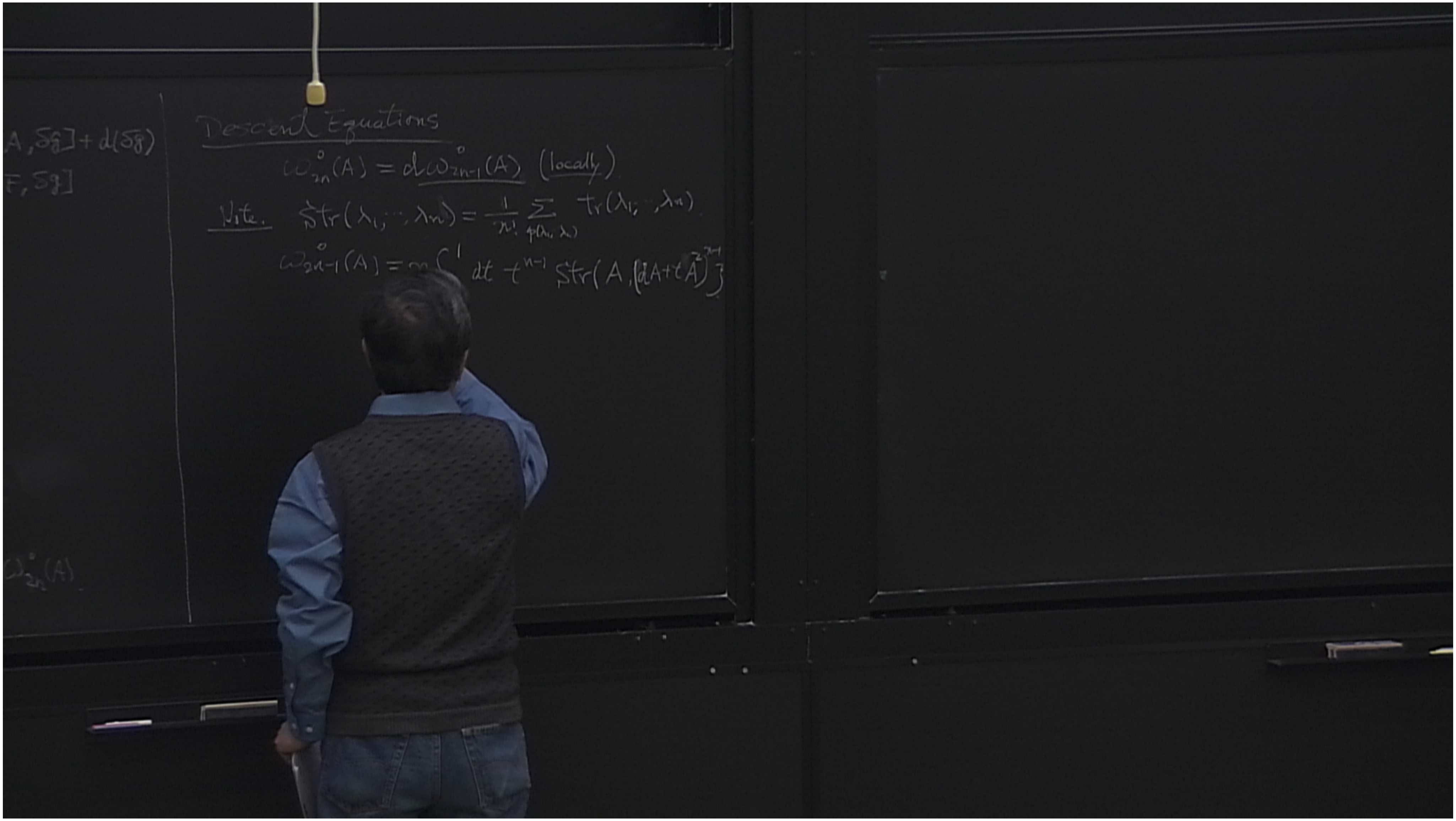
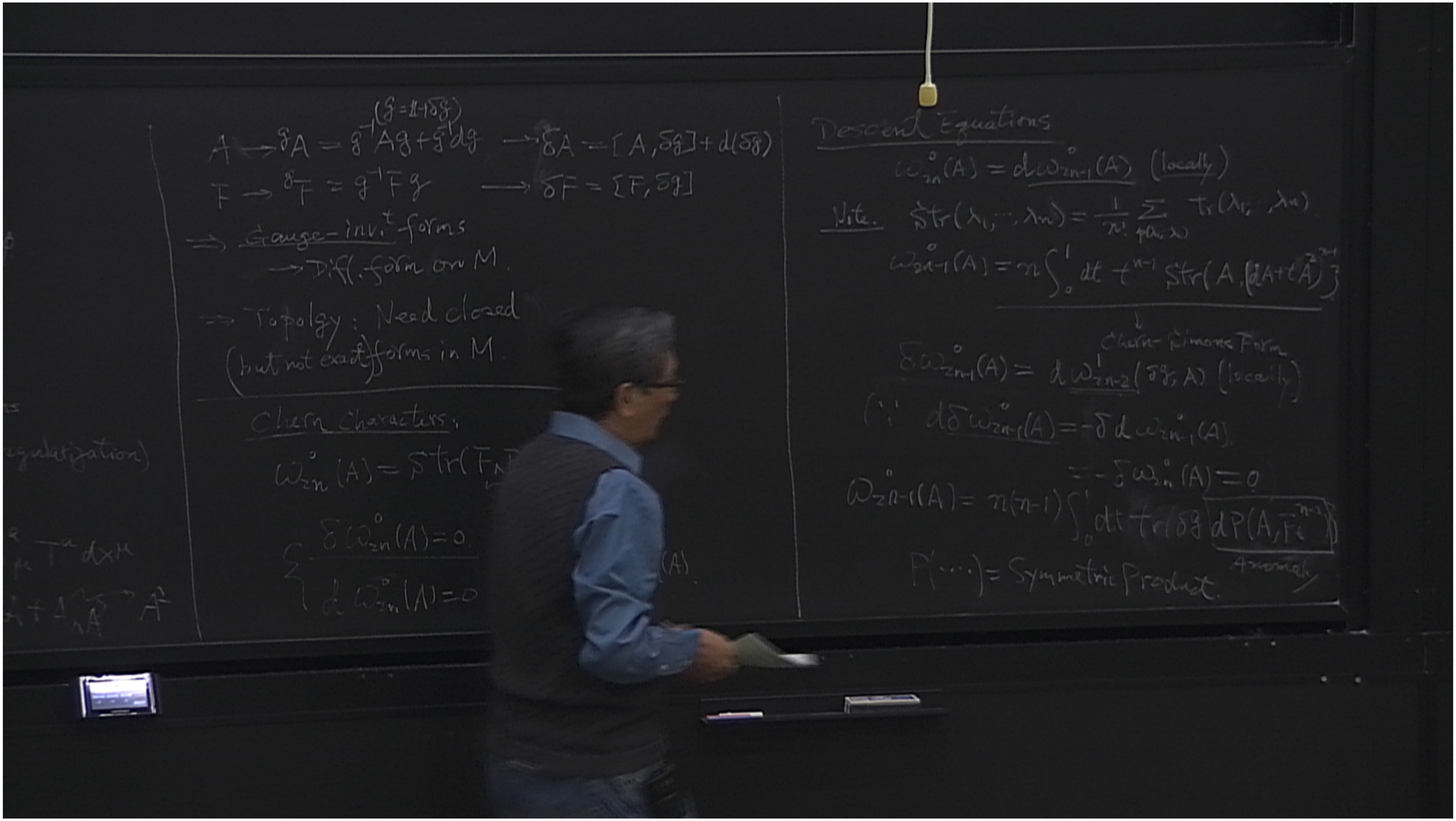
$\omega^0_{2n}(A) = d\omega^0_{2n-1}(A)$ (locally)


Descent Equations
$\omega^0_{2n}(A) = d\omega^0_{2n-1}(A)$ (locally)
Note. $\mathrm{Str}(\lambda_1,\cdots,\lambda_n) = \frac{1}{n!}\sum_{p(\lambda_1,\lambda_n)} \mathrm{tr}(\lambda_1,\cdots,\lambda_n)$
$A, \delta g] + d(\delta g)$
$F, \delta g]$
$\omega^0_{2n}(A)$


A, δg] + d(δg)
F, δg]
Descent Equations
ω⁰_2n(A) = d ω⁰_2n-1(A) (locally)
Note. Str(λ1, ..., λn) = 1/n! Σ_p(λ1, ..., λn) tr(λ1, ..., λn)
ω⁰_2n(A)


A, δg] + d(δg)
F, δg]
ω°2n(A)
Descent Equations
ω°2n(A) = dω°2n-1(A) (locally)
Note. Str(λ1, …, λn) = 1/n! Σ p(λ1, λn) tr(λ1, …, λn)
ω°2n-1(A) = n ∫ 1 dt t^(n-1) Str(A, (dA + tA²)^(n-1))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Chern Characters:
δ ω°2n (A) = 0
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Descent Equations
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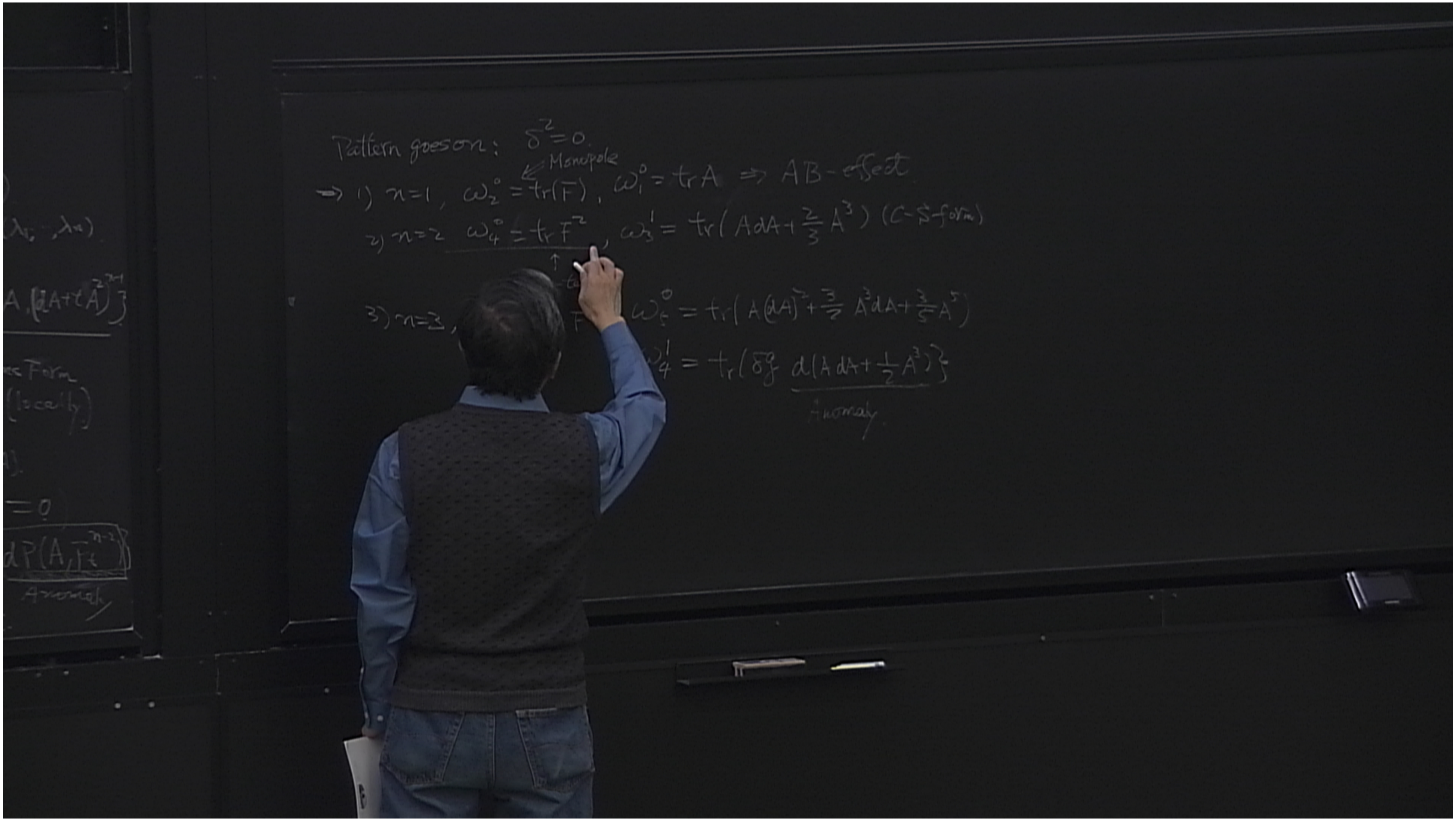
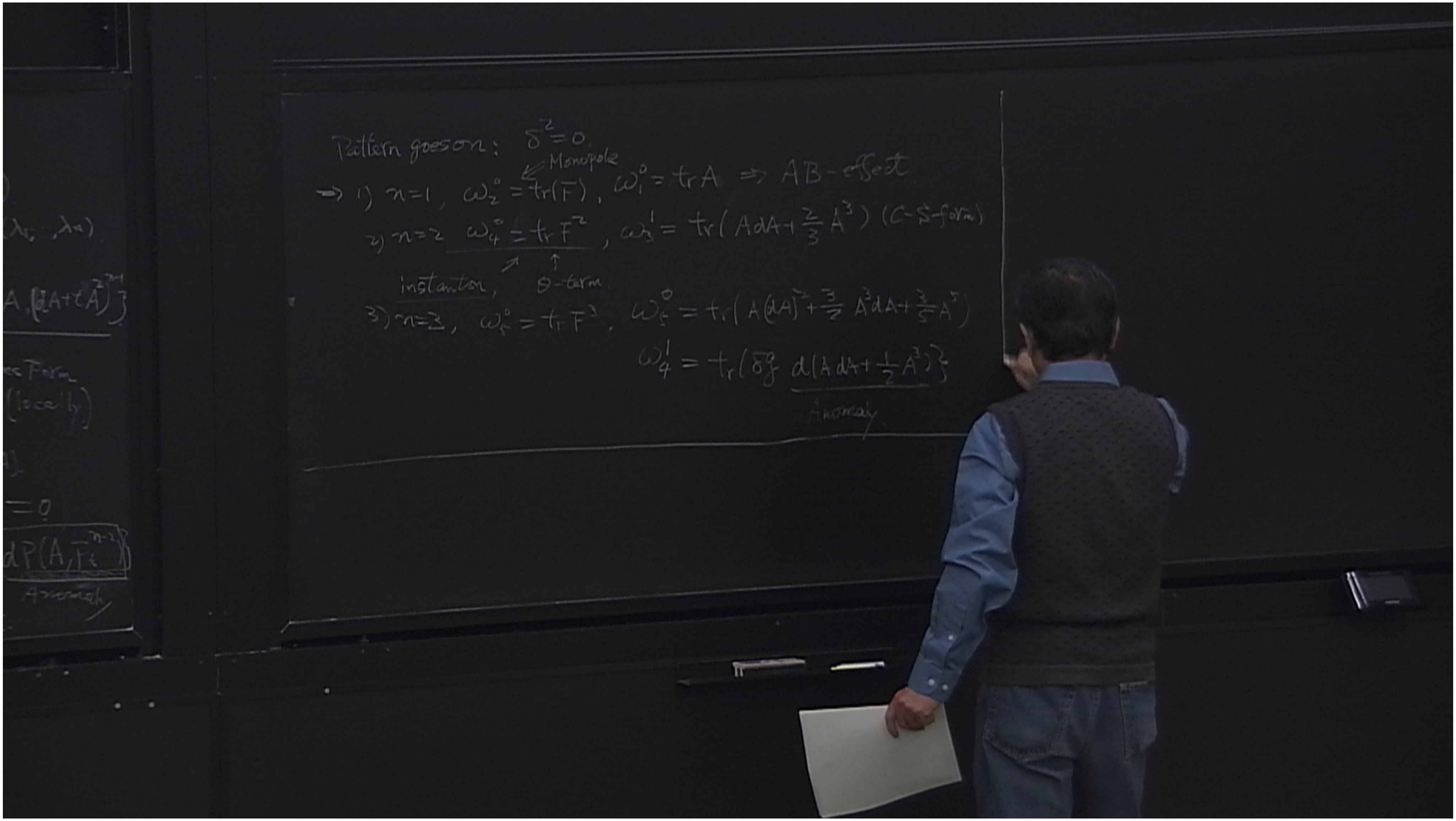
Note.
Chern-Simons Form
(locally)
P(...) = Symmetric Product.
Anomaly


Pattern goes on: $\delta^2 = 0$.
Monopole
$\Rightarrow$ 1) $n=1$, $\omega_2^0 = \mathrm{tr}(F)$, $\omega_1^0 = \mathrm{tr}A$ $\Rightarrow$ AB-effect
2) $n=2$ $\omega_4^0 = \mathrm{tr}F^2$, $\omega_3^1 = \mathrm{tr}(AdA + \frac{2}{3}A^3)$ (C-S-form)
3) $n=3$,
$\omega_5^0 = \mathrm{tr}(A(dA)^2 + \frac{3}{2}A^3dA + \frac{3}{5}A^5)$
$\omega_4^1 = \mathrm{tr}(\delta g\, d(AdA + \frac{1}{2}A^3))$
Anomaly



Pattern goes on: $\delta^2=0$
Monopole
$\Rightarrow$ 1) $n=1$, $\omega_2^0 = tr(F)$, $\omega_1^0 = tr A$ $\Rightarrow$ AB-effect
2) $n=2$ $\omega_4^0 = tr F^2$, $\omega_3^1 = tr(AdA + \frac{2}{3}A^3)$ (C-S-form)
instanton, θ-term
3) $n=3$, $\omega_6^0 = tr F^3$, $\omega_5^0 = tr(A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5)$
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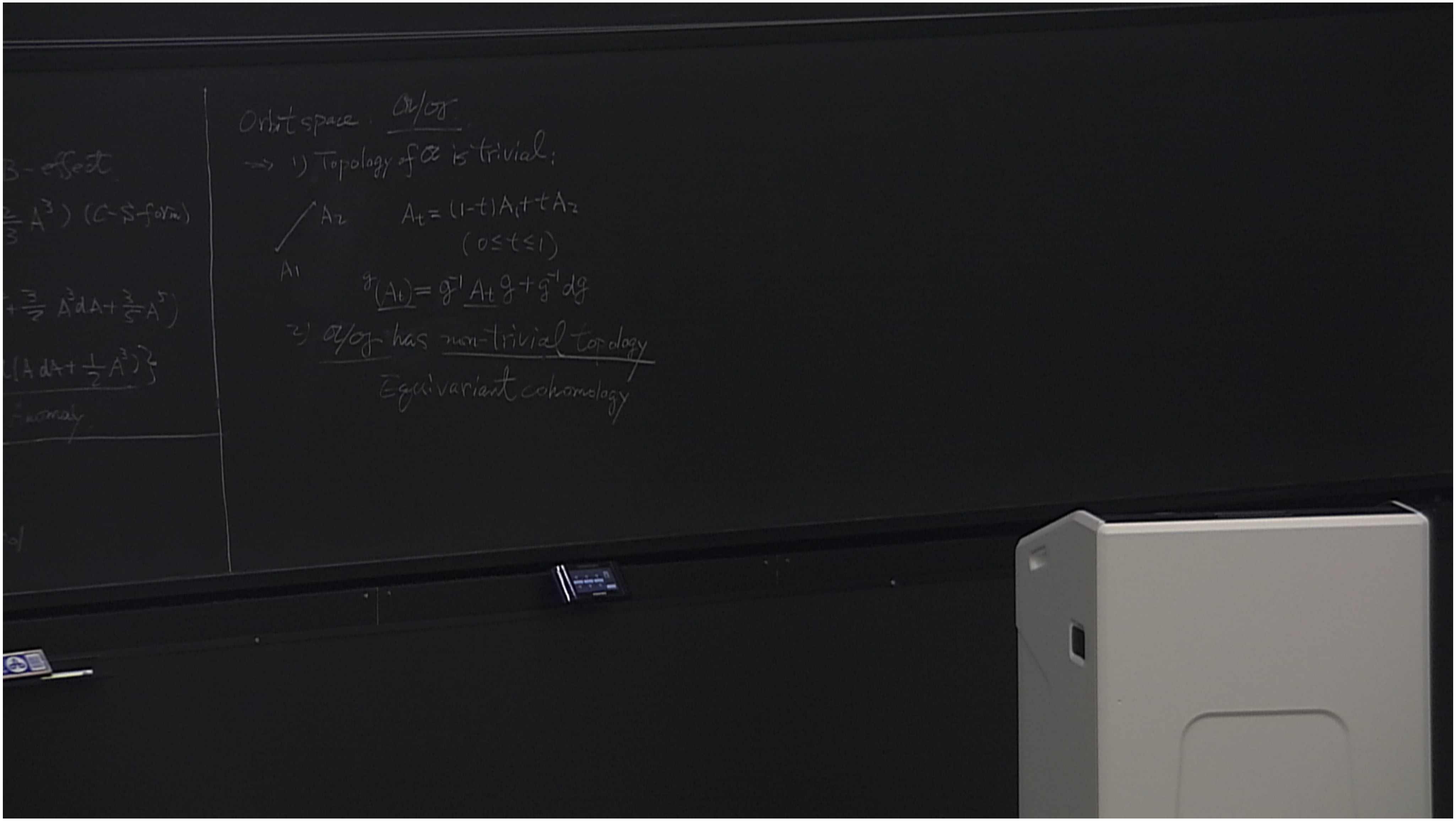
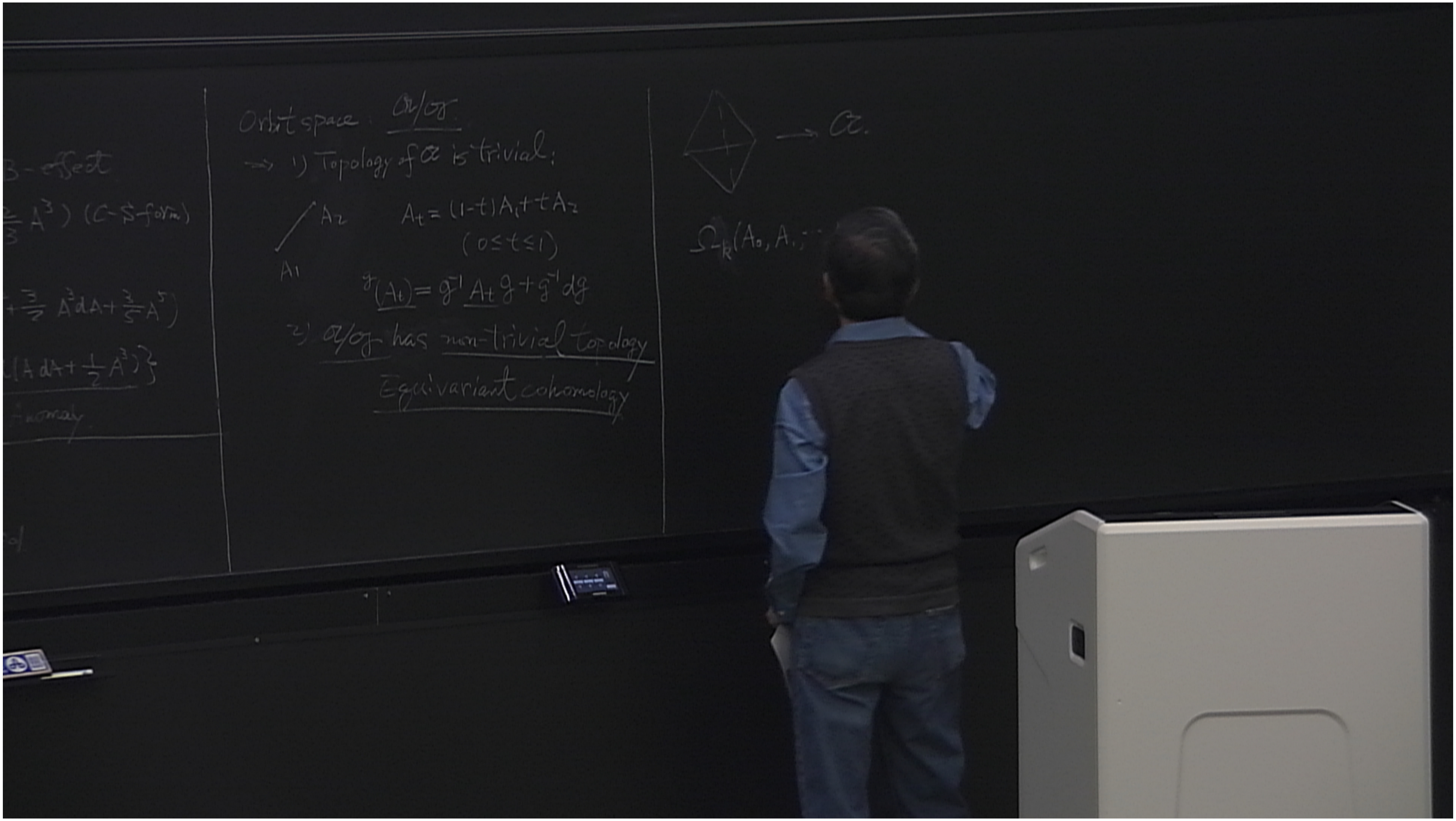
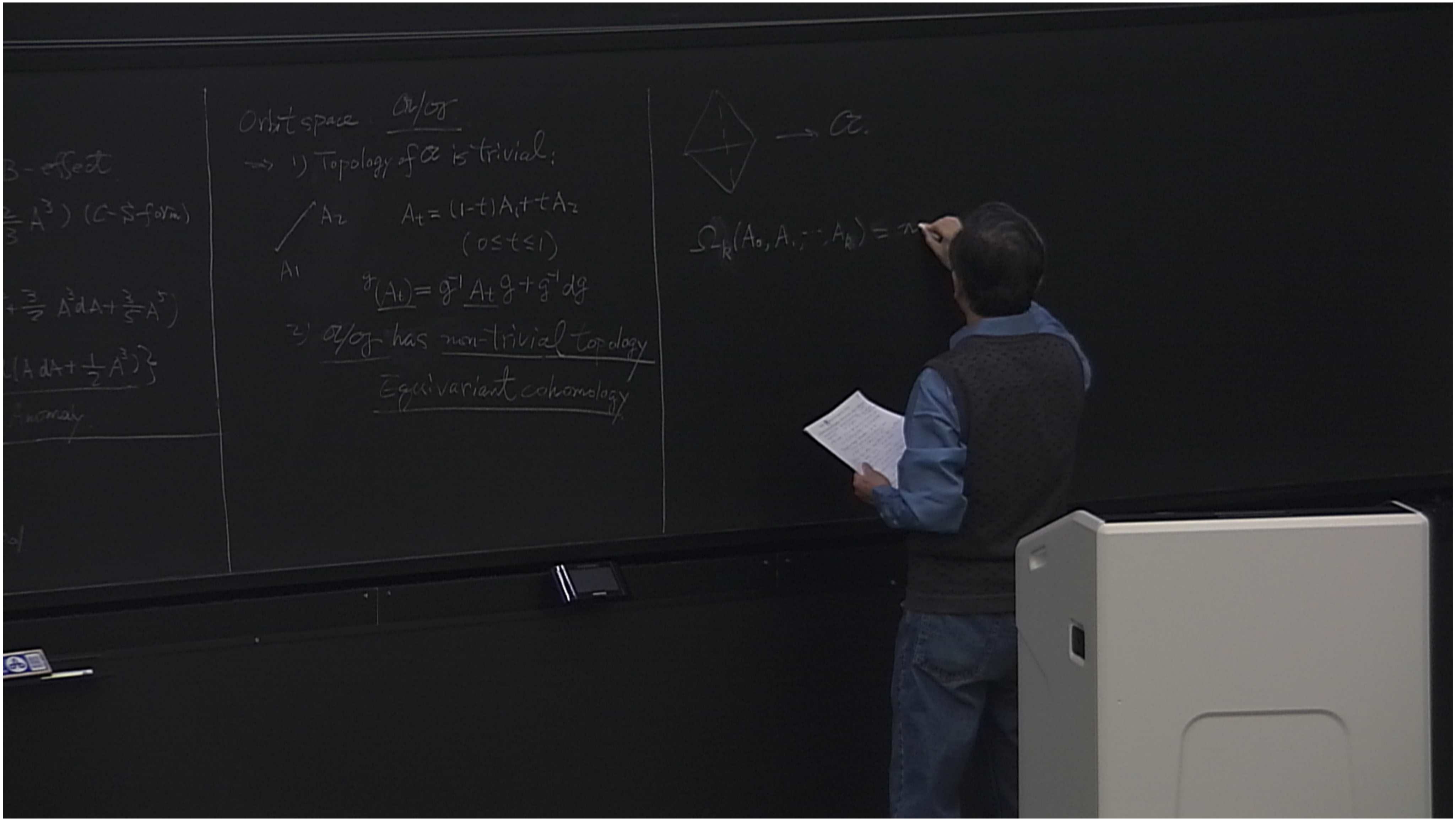
$\omega_4^1 = \mathrm{tr}(\delta g\ \underbrace{d(A dA + \frac{1}{2} A^3)}_{\text{Anomaly}})$

Configuration: $\mathcal{C} = \{A\}$

definite topol


Orbit space 𝒜/𝒢
→ 1) Topology of 𝒜 is trivial:
A2
A1
At = (1-t)A1 + tA2
(0 ≤ t ≤ 1)
g(At) = g⁻¹ At g + g⁻¹ dg
2) 𝒜/𝒢 has non-trivial topology
Equivariant cohomology
B-effect
(C-S form)


B-effect
(C-S form)
Anomaly
Orbit space $\mathcal{A}/\mathcal{G}$
1) Topology of $\mathcal{A}$ is trivial:
A_2
A_1
$A_t = (1-t)A_1 + tA_2$
$(0 \le t \le 1)$
$^g(A_t) = g^{-1}A_t g + g^{-1}dg$
2) $\mathcal{A}/\mathcal{G}$ has non-trivial topology
Equivariant cohomology
$\mathcal{A}$.
$\Omega_k(A_0, A_1, \cdots$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Equivariant cohomology
→ 𝔄.
Ωk(A0, A1, ⋯, Ak) =
B-effect
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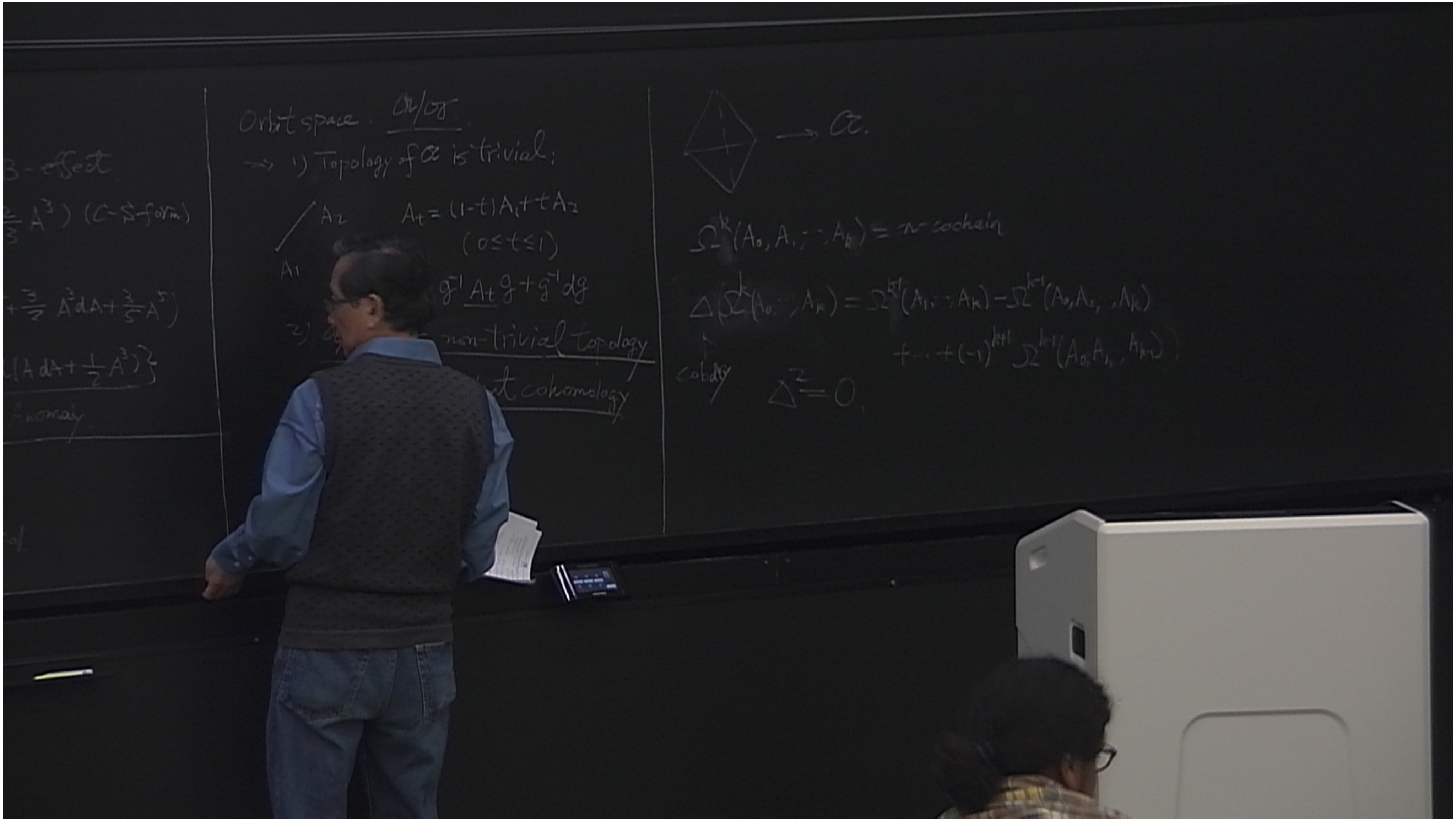
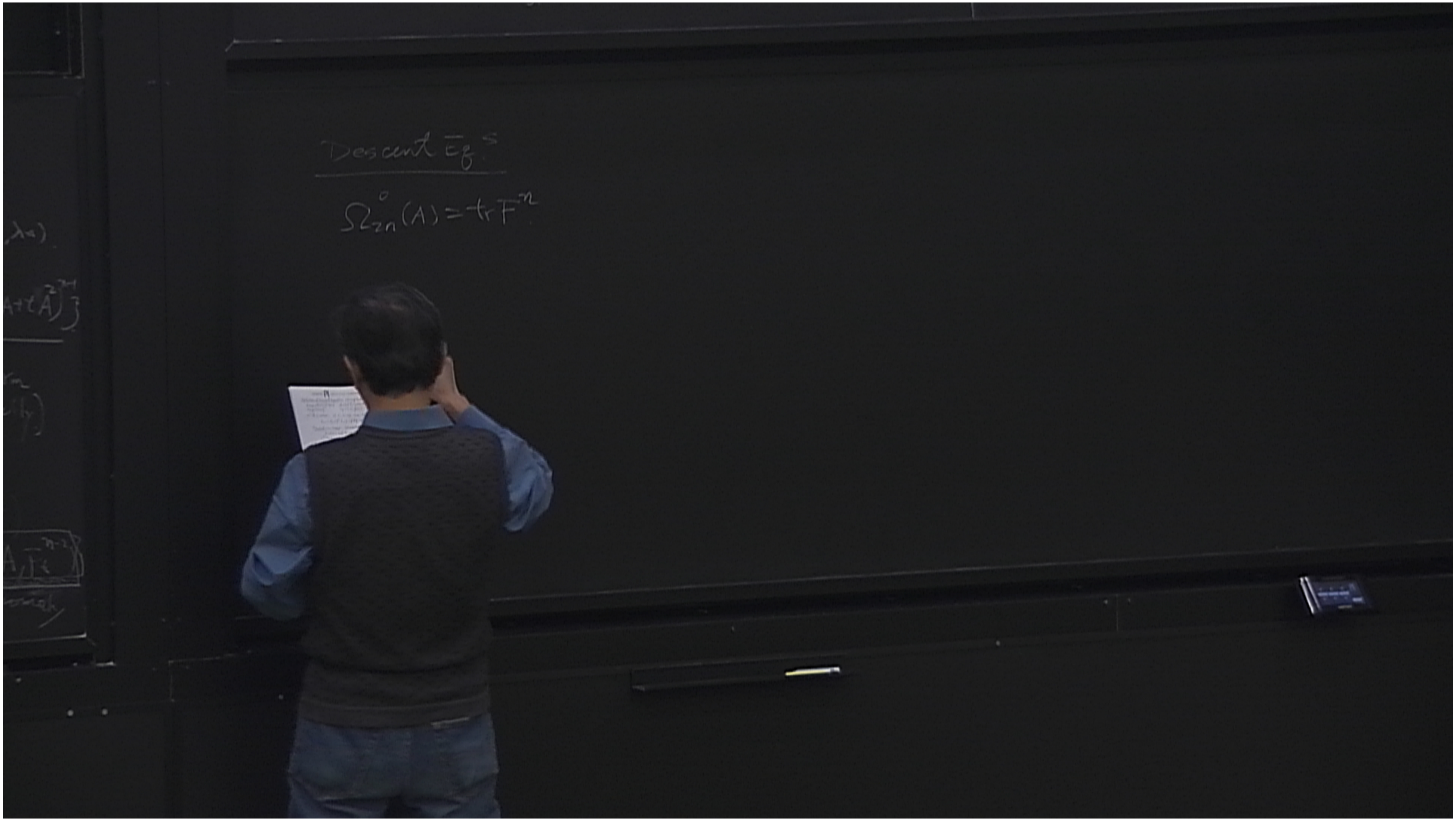
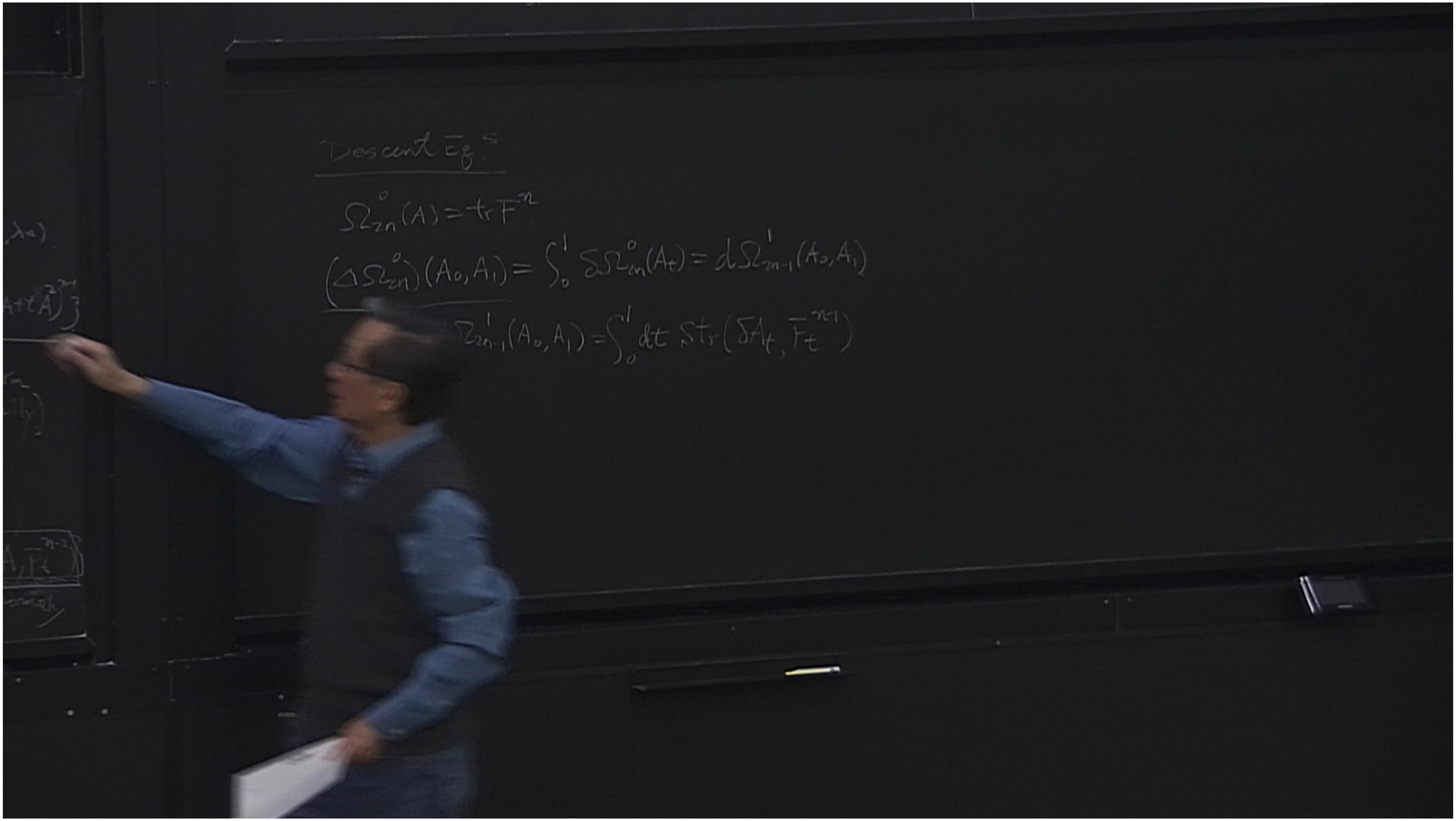
Anomaly



Orbit space : 𝔄/𝔊
1) Topology of 𝔄 is trivial:
A_t = (1-t)A_1 + tA_2
(0 ≤ t ≤ 1)
non-trivial topology
cohomology
Ω^k(A_0, A_1, ⋯, A_k) = n-cochain
cobdry
Δ^2 = 0


Descent Eqs
$\Omega^0_{2n}(A) = \mathrm{tr}\, F^n$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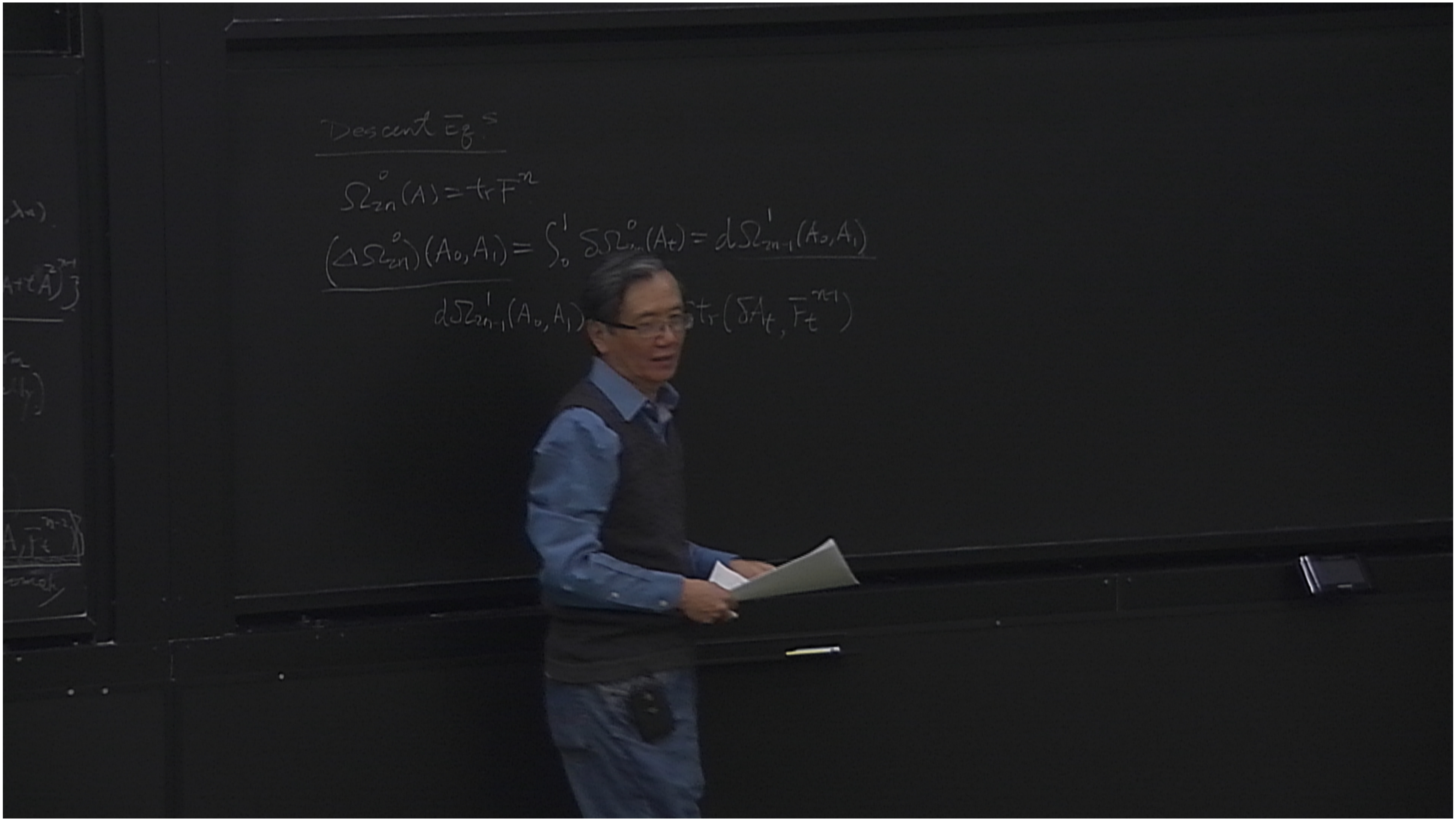
$\Omega^0_{2n}(A) = \mathrm{tr}\, F^n$
$(\Delta \Omega^0_{2n})(A_0, A_1) = \int_0^1 \delta \Omega^0_{2n}(A_t) = d\Omega^1_{2n-1}(A_0, A_1)$
$\Omega^1_{2n-1}(A_0, A_1) = \int_0^1 dt \, \mathrm{Str}(\delta A_t, F_t^{n-1})$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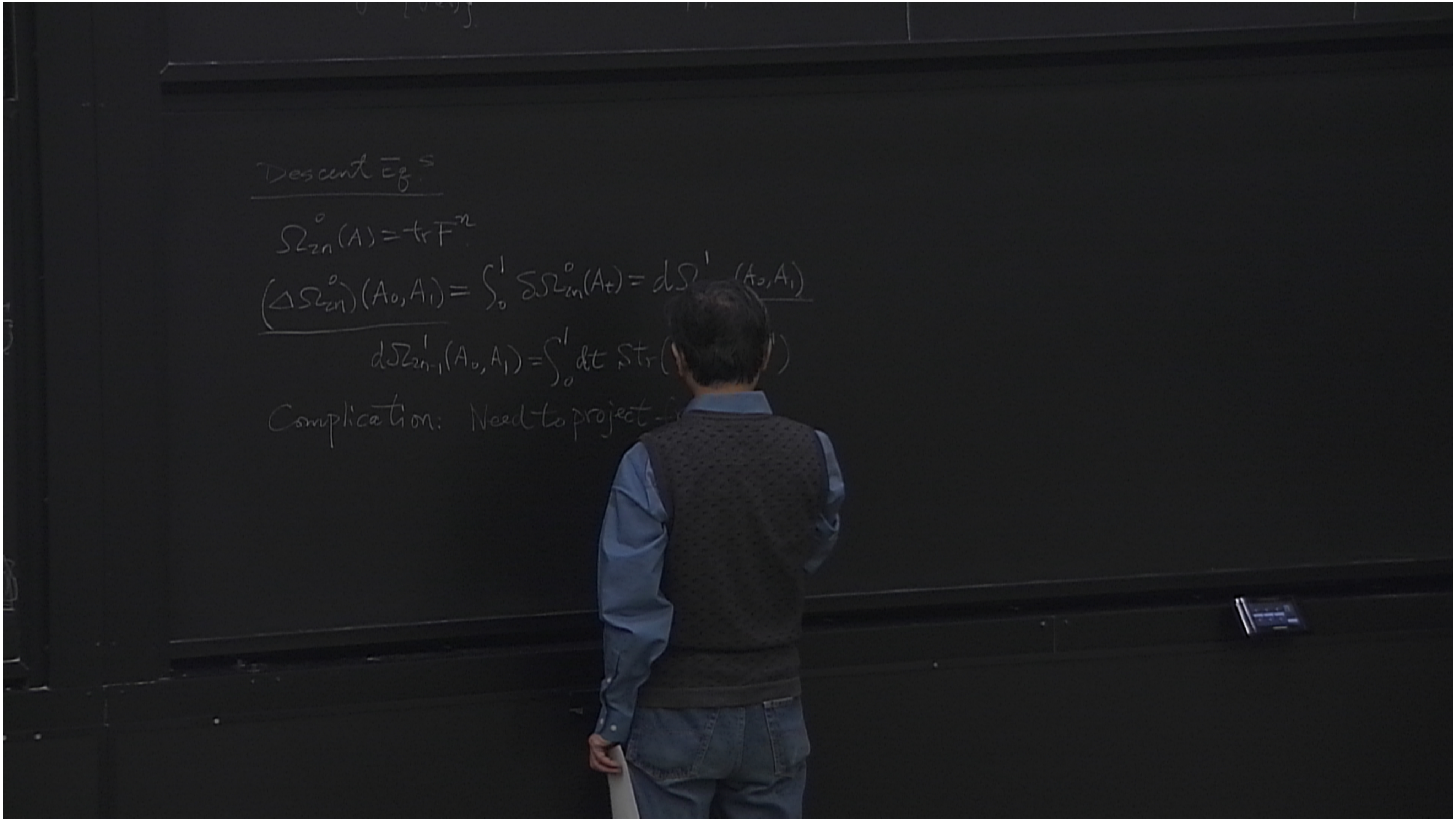
$d\Omega^{1}_{2n-1}(A_0, A_1)$
$\mathrm{tr}(\delta A_t, F_t^{n-1})$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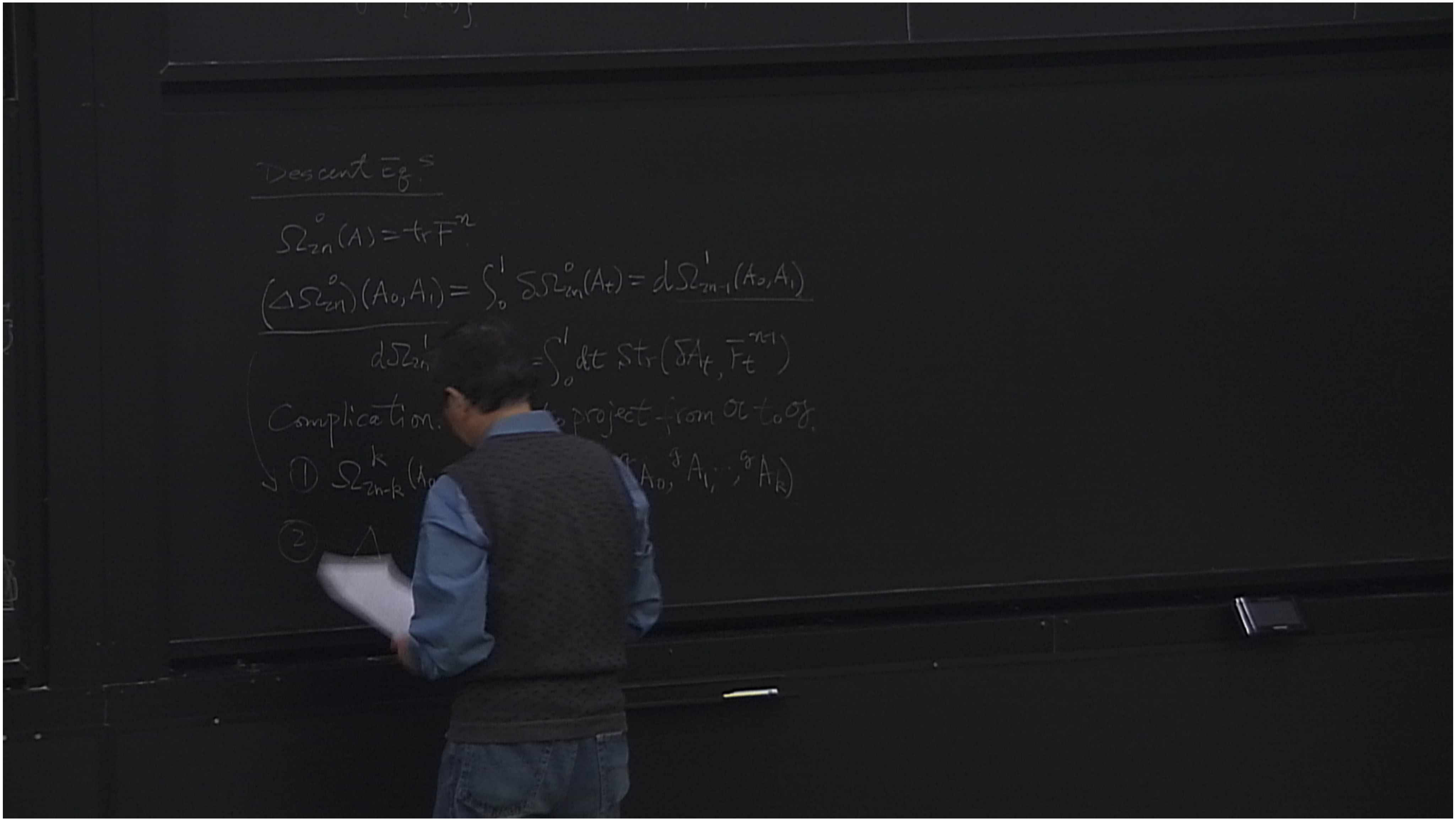
$d\Omega^1_{2n-1}(A_0, A_1) = \int_0^1 dt\ \mathrm{Str}($
Complication: Need to project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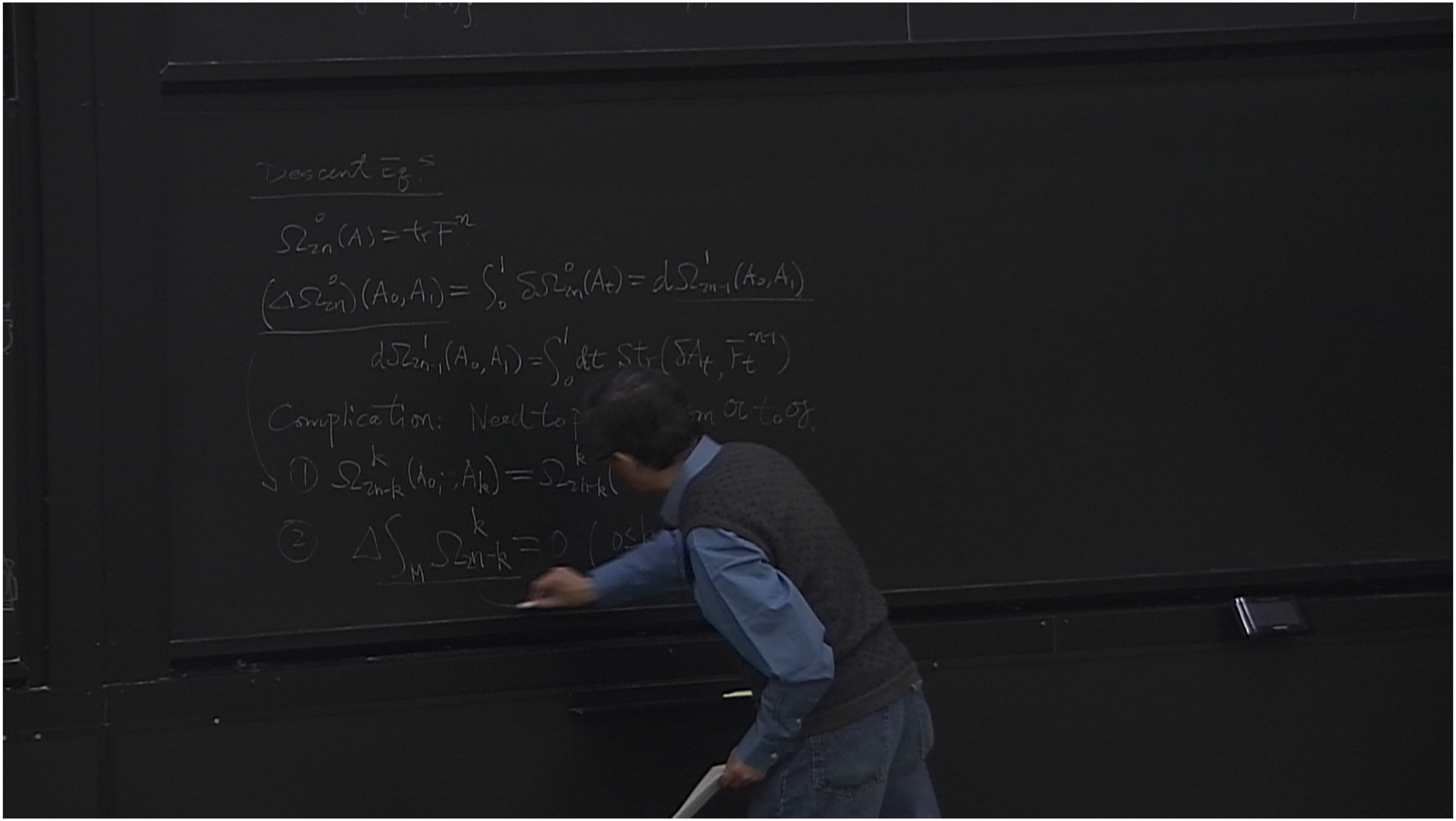
$d\Omega^1_{2n}$
$= \int_0^1 dt\, \mathrm{Str}(\delta A_t, F_t^{n-1})$
Complication:
project from
① $\Omega^k_{2n-k}(A_0$
$A_0, {}^gA_1, \dots, {}^gA_k)$
② Δ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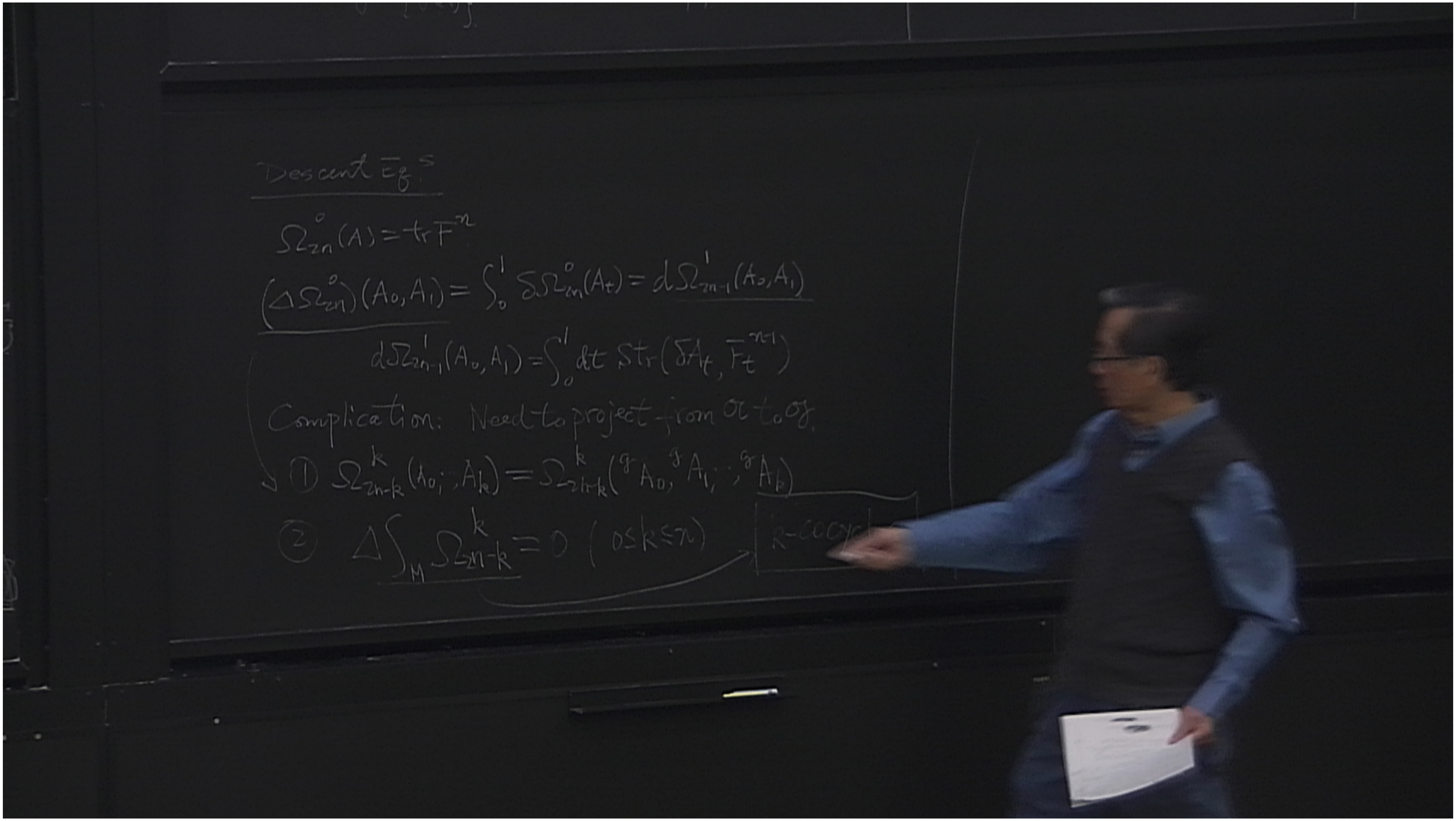
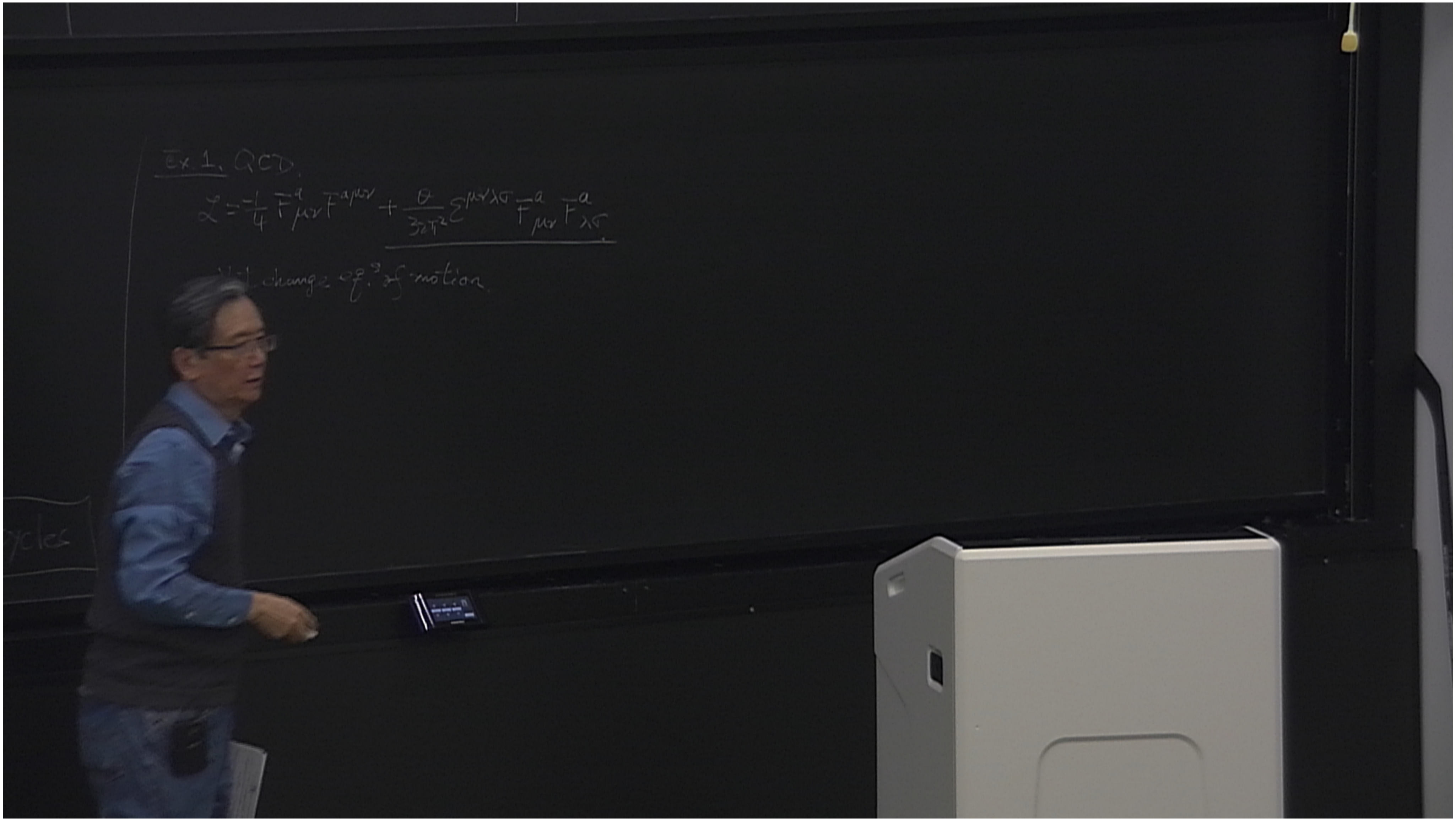
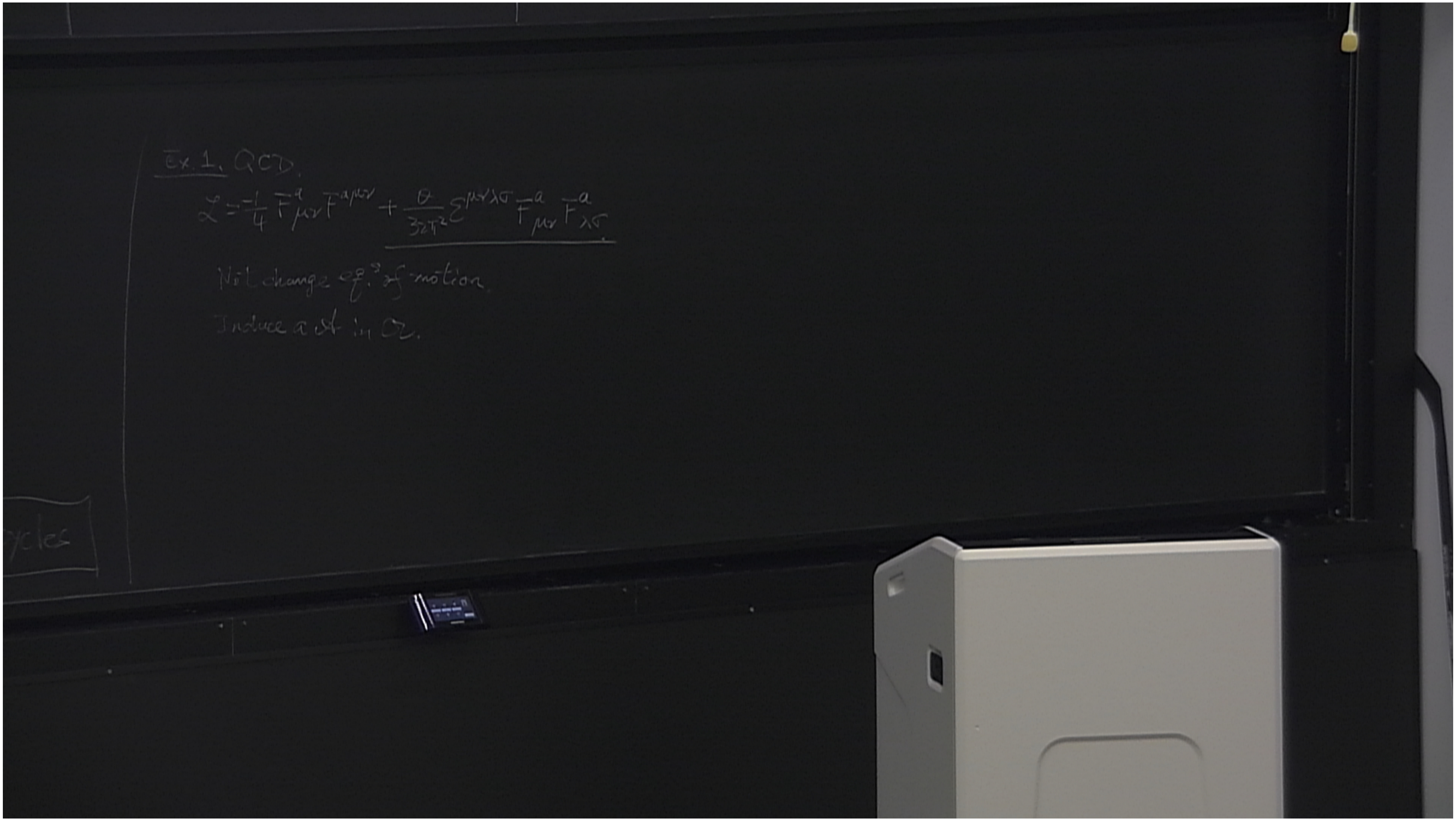
$d\Omega^1_{2n-1}(A_0, A_1) = \int_0^1 dt\, \mathrm{Str}(\delta A_t, F_t^{n-1})$
Complication: Need to
① $\Omega^k_{2n-k}(A_0, \dots, A_k) = \Omega^k_{2n-k}($
② $\Delta \int_M \Omega^k_{2n-k} = 0$


Descent Eq^s
$\Omega^0_{2n}(A) = \mathrm{tr}\, F^n$
$(\Delta\Omega^0_{2n})(A_0,A_1) = \int_0^1 \delta\Omega^0_{2n}(A_t) = d\Omega^1_{2n-1}(A_0,A_1)$
$d\Omega^1_{2n-1}(A_0,A_1) = \int_0^1 dt\, \mathrm{Str}(\delta A_t, F_t^{n-1})$
Complication: Need to project from $\mathfrak{a}$ to $\mathfrak{g}$.
① $\Omega^k_{2n-k}(A_0,\ldots,A_k) = \Omega^k_{2n-k}({}^gA_0, {}^gA_1, \ldots, {}^gA_k)$
② $\Delta\int_M \Omega^k_{2n-k} = 0 \quad (0 \le k \le n)$
k-cocycle



Ex 1. QCD.
$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} \varepsilon^{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma}$
change eq.^{s} of motion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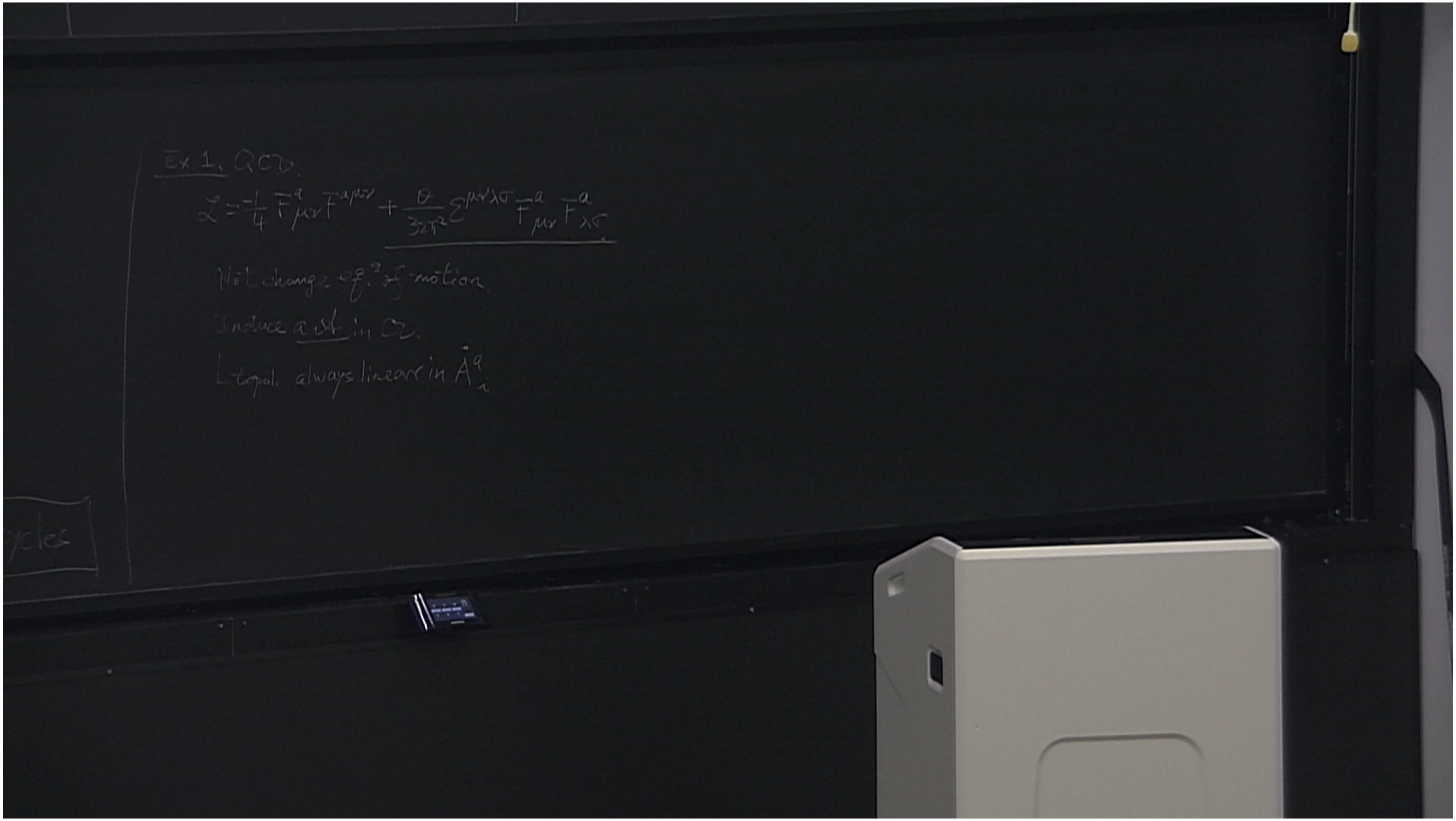
Not change eq.^{s} of motion.
Induce a $\mathcal{A}$ in Ω.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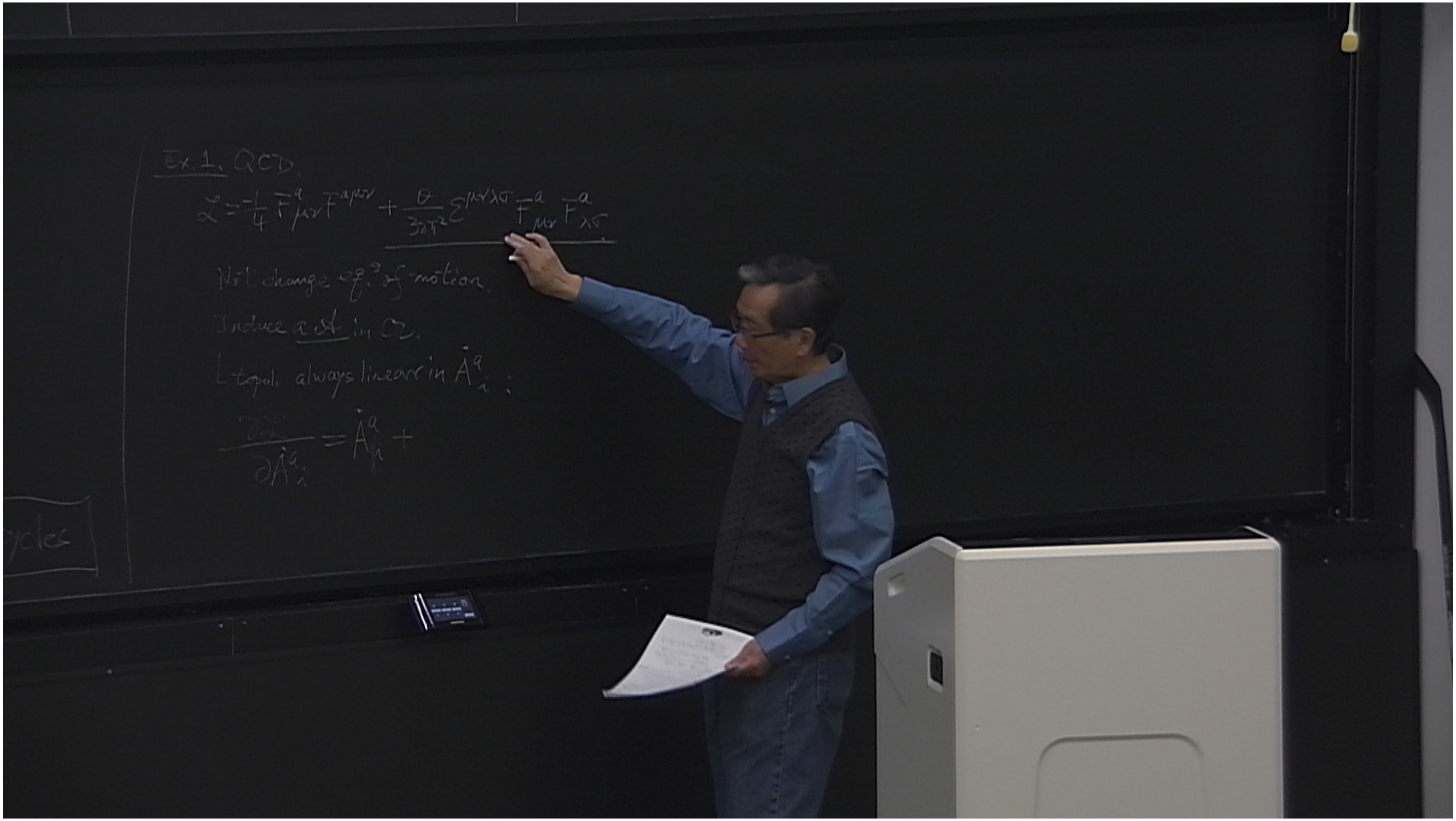
Not change eq.s of motion.
Induce a $\mathcal{A}$ in Ω.
$\mathcal{L}$-topol. always linear in $\dot{A}^a_i$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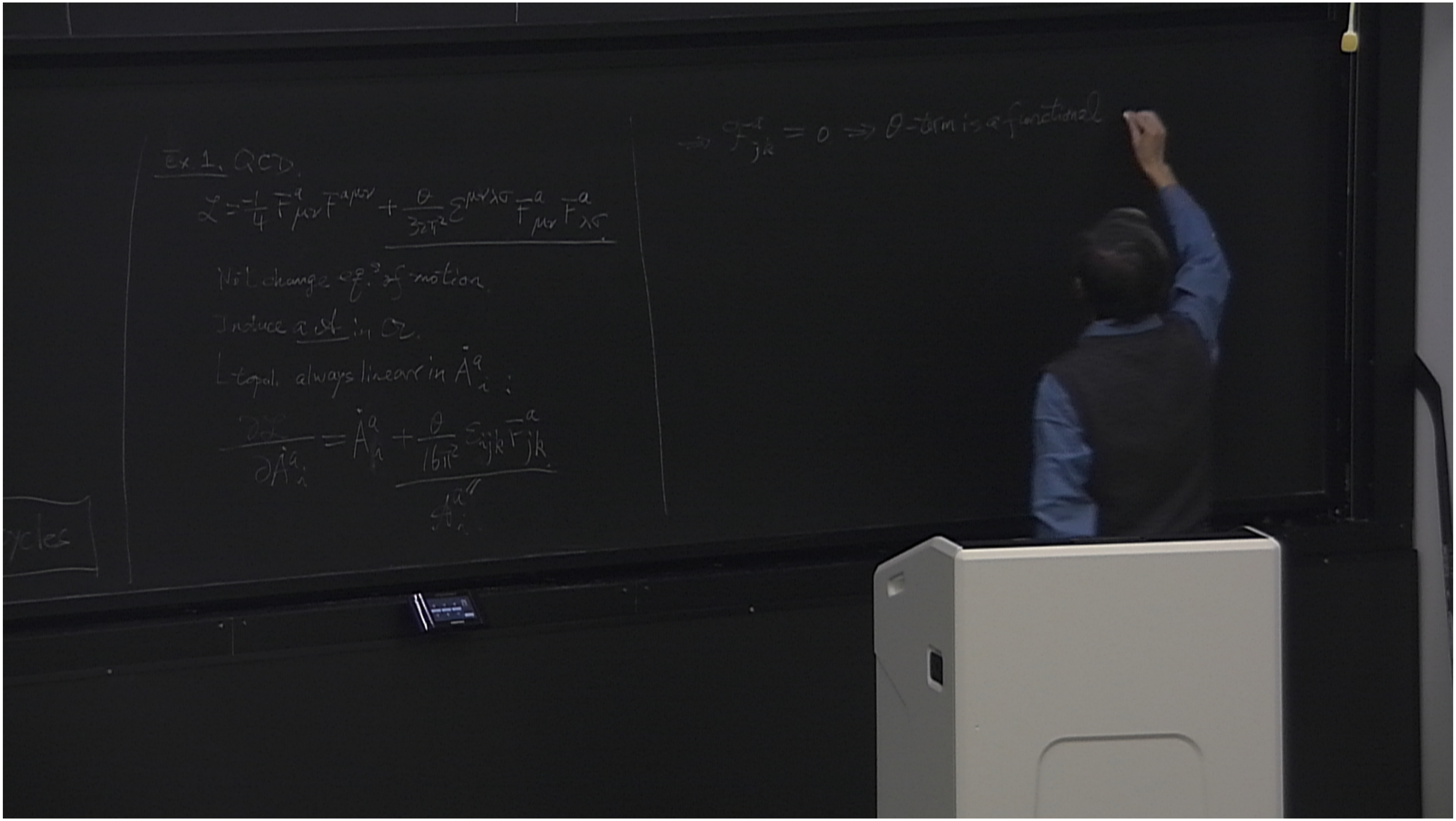
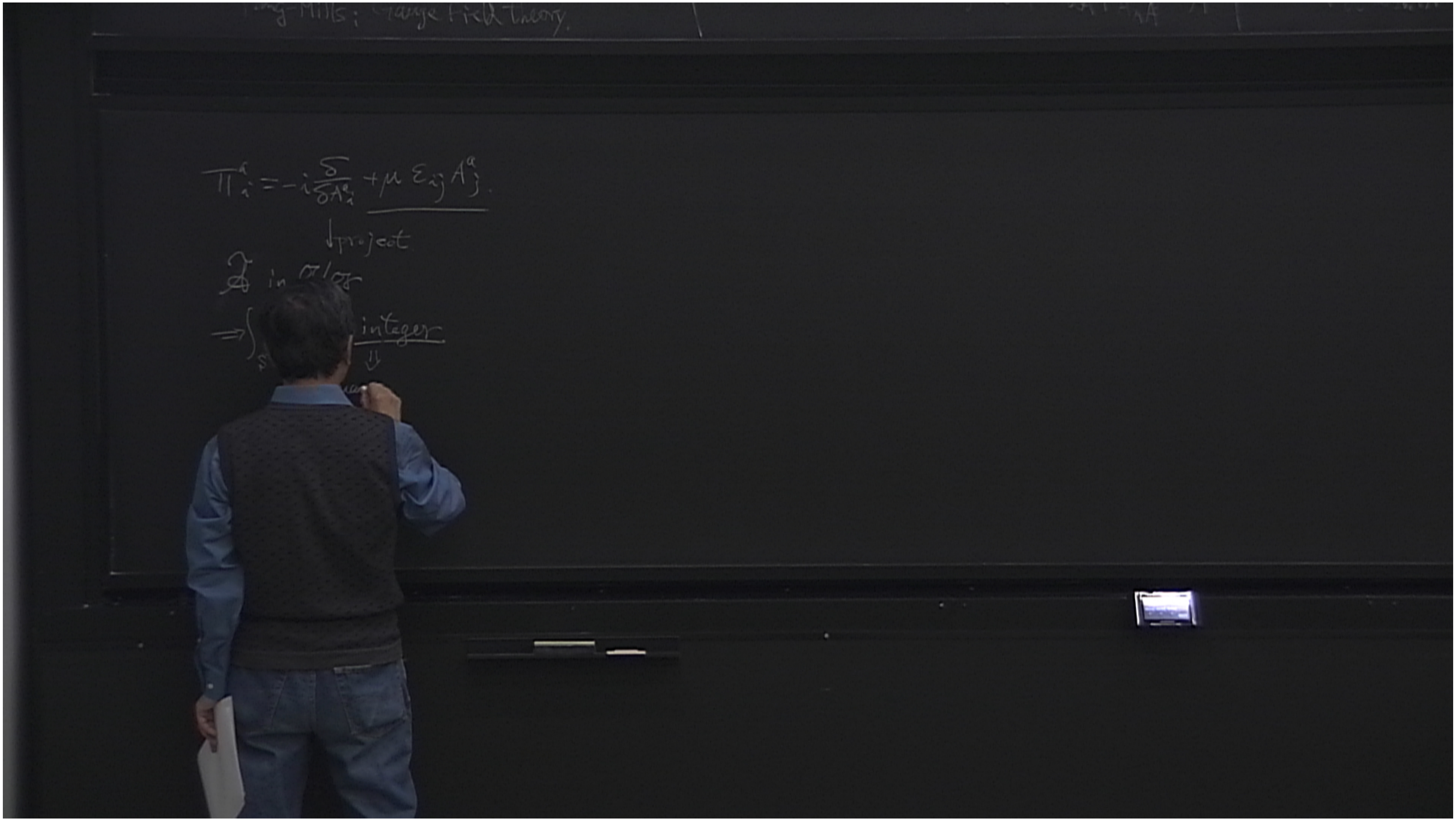
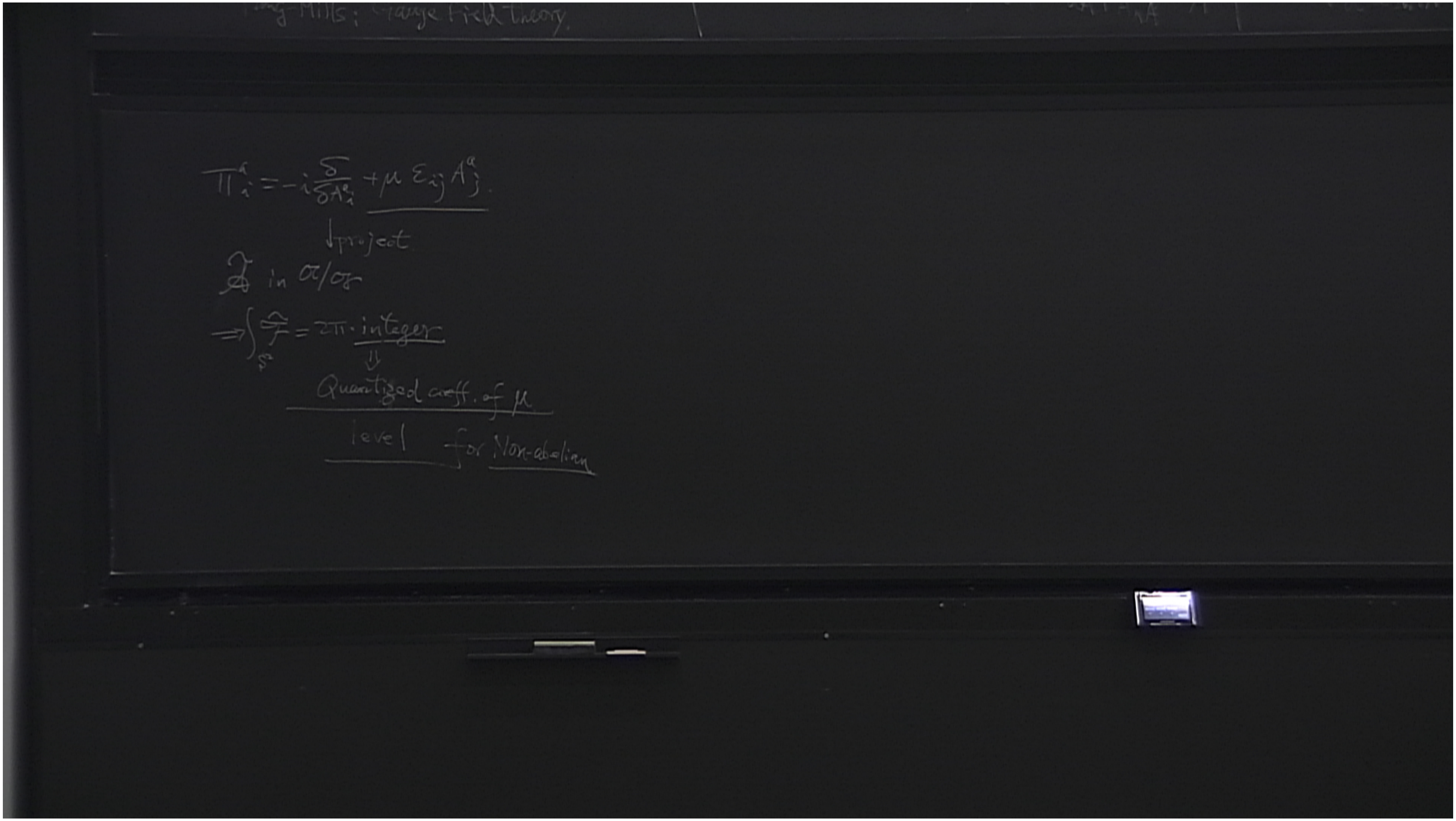
$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} \varepsilon^{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma}$
Not change eq.s of motion.
L-topol. always linear in $\dot{A}^a_i$:
$\frac{\partial \mathcal{L}}{\partial \dot{A}^a_i} = \dot{A}^a_i +$
ycles


Ex. 1. QCD.
$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} \varepsilon^{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma}$
Not change eq. of motion.
Induce a $\mathcal{A}$ in Ω.
L-topol. always linear in $\dot{A}^a_i$:
$\frac{\partial \mathcal{L}}{\partial \dot{A}^a_i} = \dot{A}^a_i + \frac{\theta}{16\pi^2} \varepsilon_{ijk} F^a_{jk}$
$\Rightarrow F^a_{jk} = 0 \Rightarrow \theta$-term is a functional


Gauge Field theory.
project
integer


Mills; Gauge Field theory.
$\Pi^a_i = -i\frac{\delta}{\delta A^a_i} + \mu\, \varepsilon_{ij} A^a_j$.
↓project
$\tilde{A}$ in $\mathfrak{a}/\mathfrak{g}$
$\Rightarrow \int_{S^2} \tilde{F} = 2\pi \cdot$ integer
⇓
Quantized coeff. of μ
level for Non-abelian