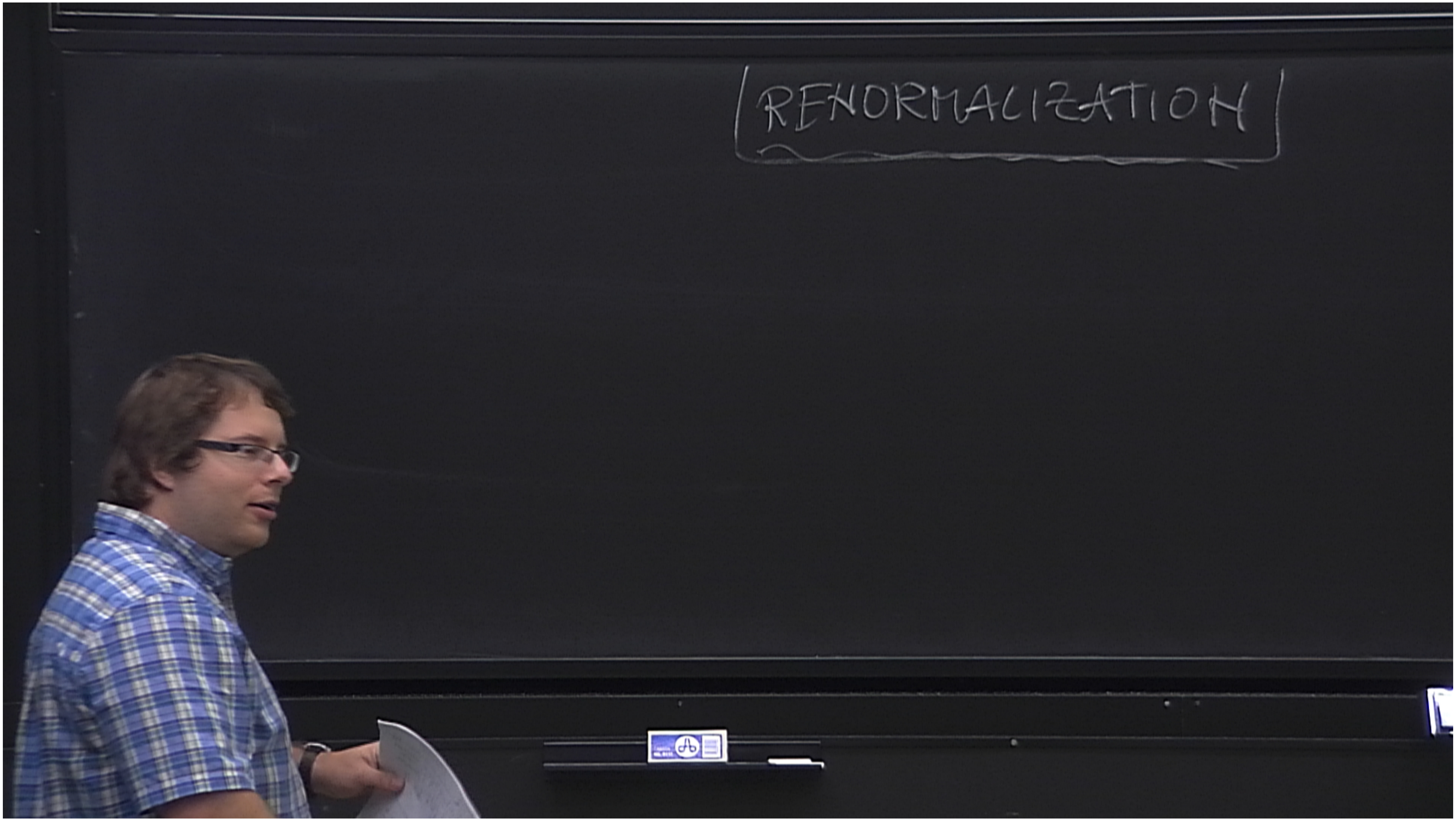
Title: PSI 2015/2016 Statistical Mechanics - G. Baskaran - Lecture 12

Date: Oct 28, 2015 10:45 AM

URL: http://pirsa.org/15100097

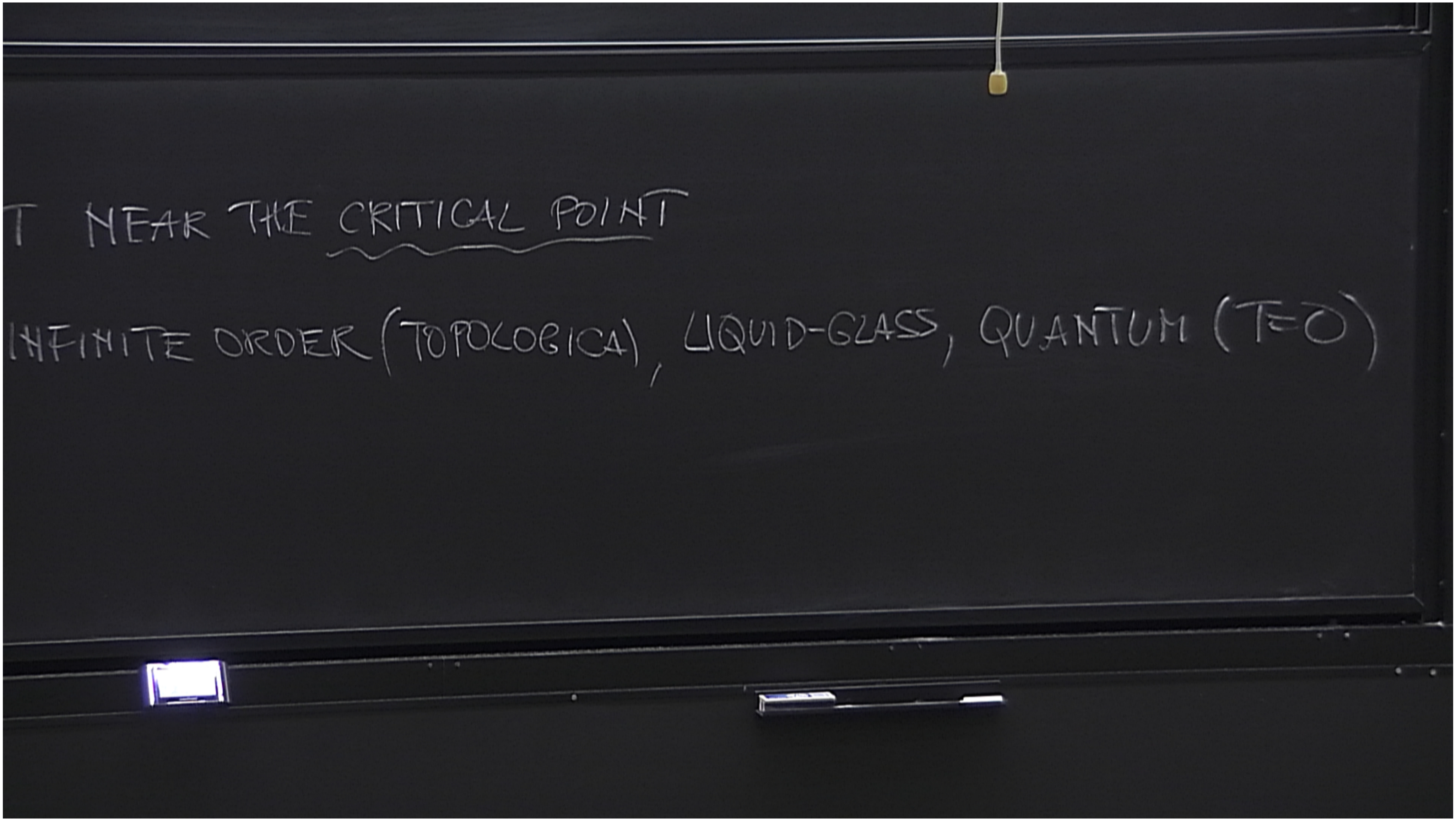
Abstract:

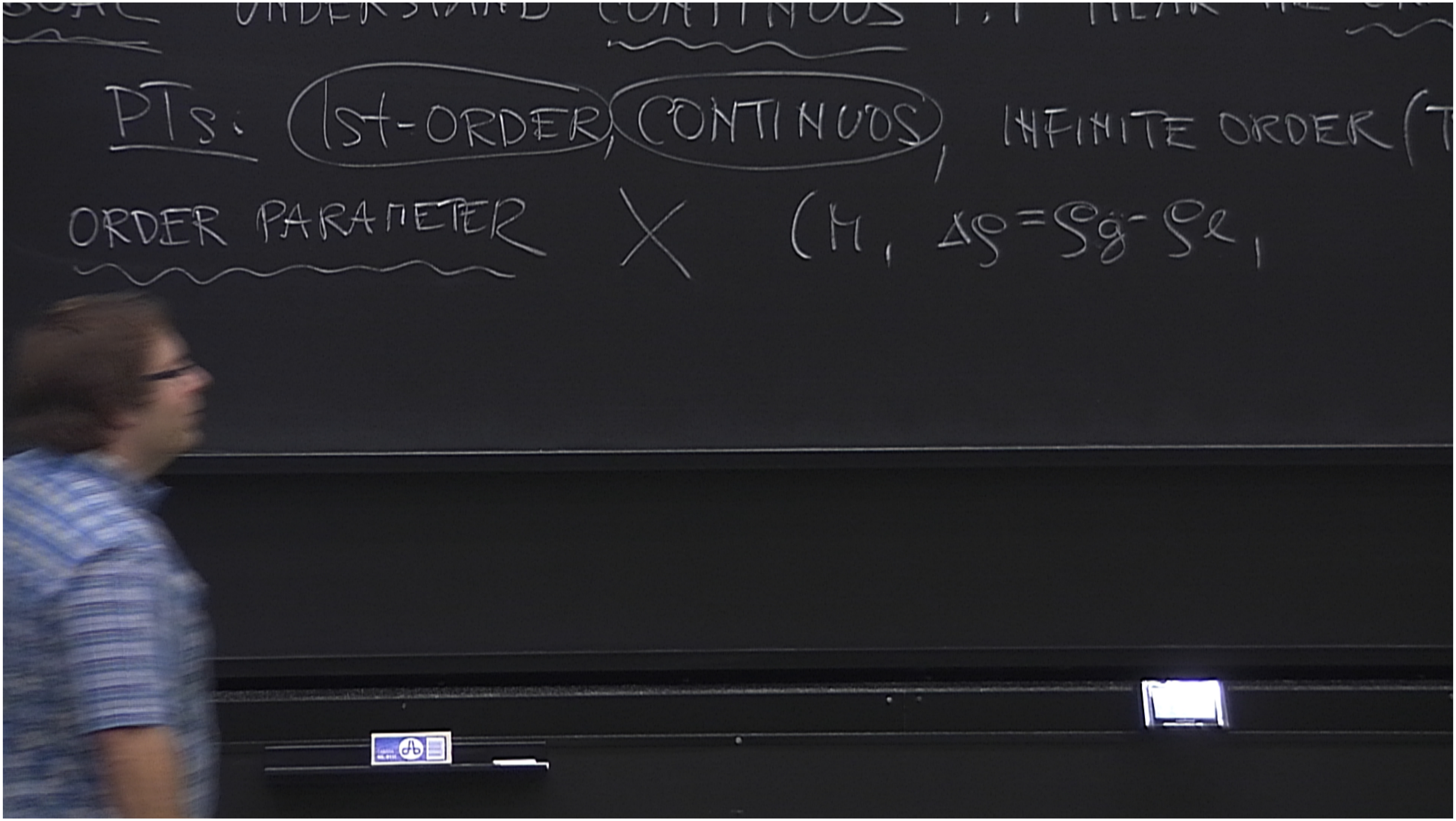

RENORMALIZATION

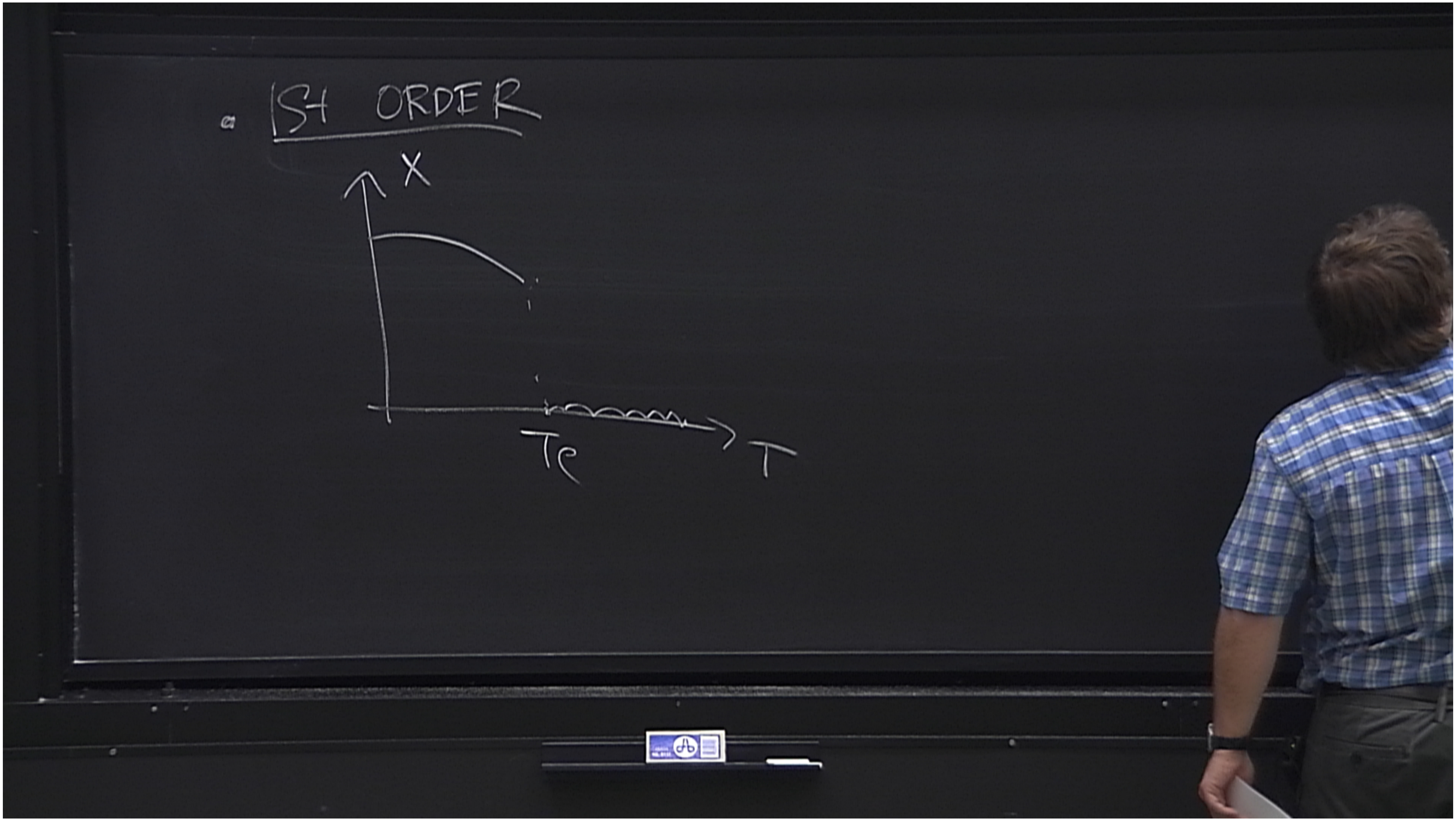
$\boxed{\text{RENORMALIZATION}}$

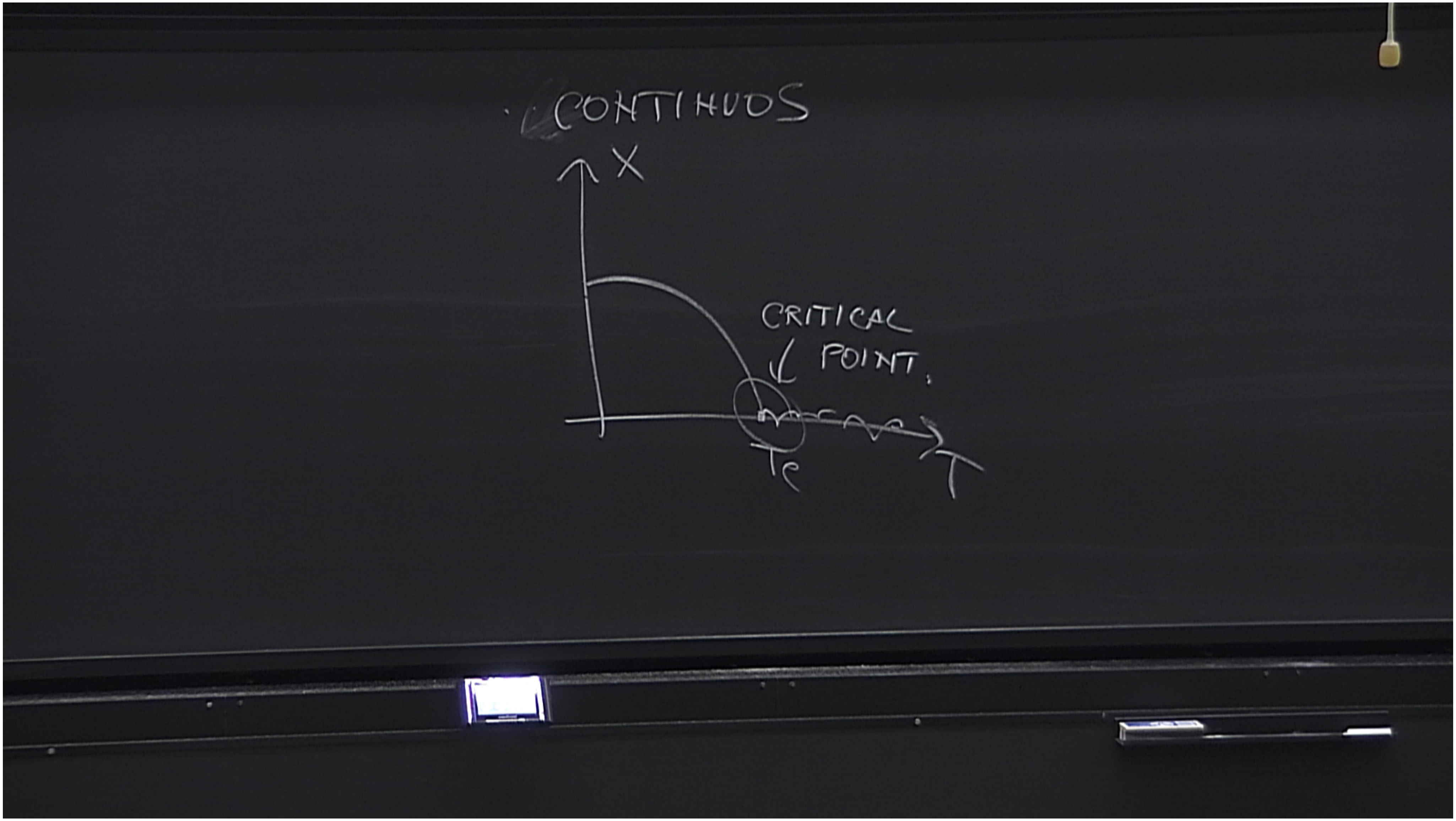
1) GOAL UNDERSTAND CONTINUOS P.T NEAR THE CRITI

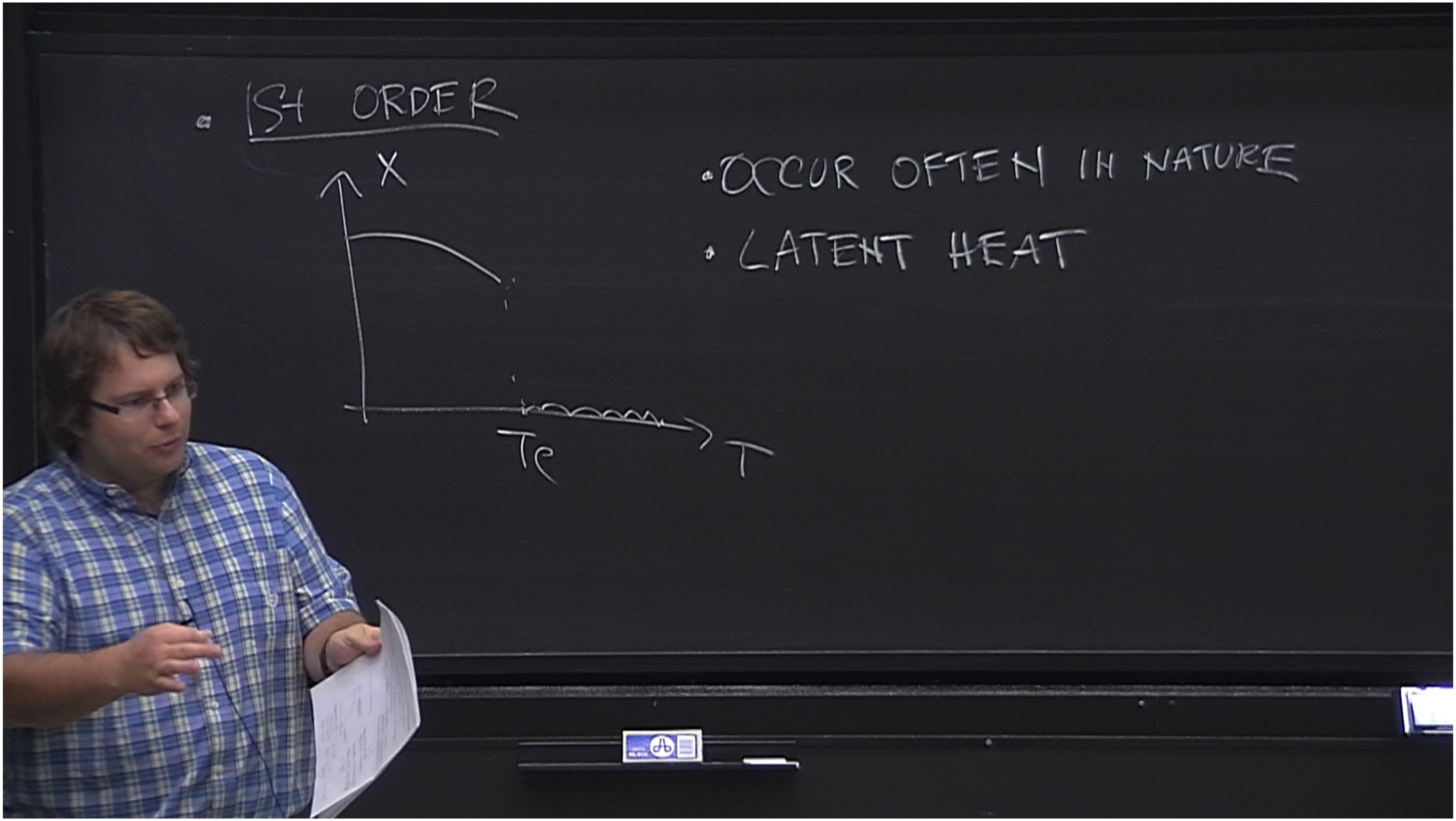
PTs: 1st-ORDER, CONTINUOS, INFINITE ORDER (


T NEAR THE CRITICAL POINT
INFINITE ORDER (TOPOLOGICA), LIQUID-GLASS, QUANTUM (T=0)

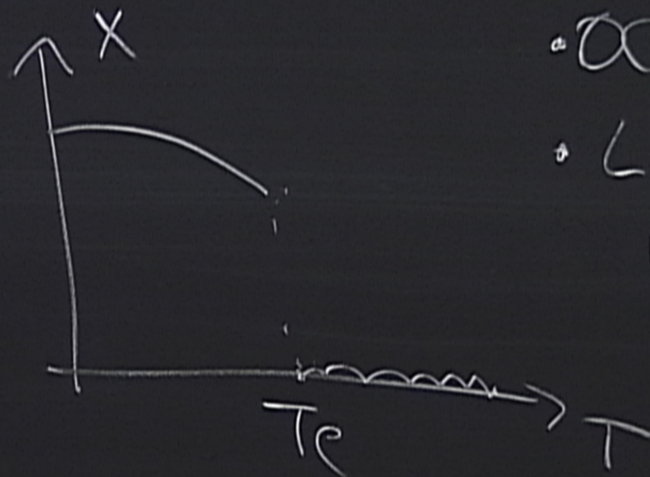

PTs: 1st-ORDER, CONTINUOS, INFINITE ORDER (T
ORDER PARAMETER X (M, Δϱ = ϱg - ϱℓ,


1st ORDER
X
Tc
T

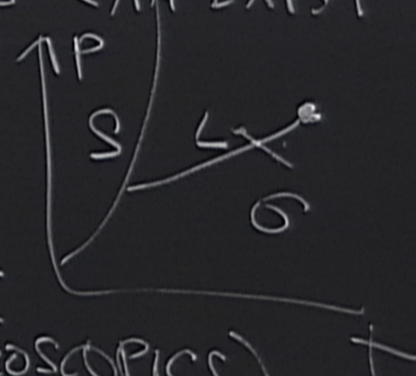

CONTINUOS
X
CRITICAL
POINT.
Tc
T


1st ORDER
X
Tc
T
OCCUR OFTEN IN NATURE
LATENT HEAT

- **1st ORDER**

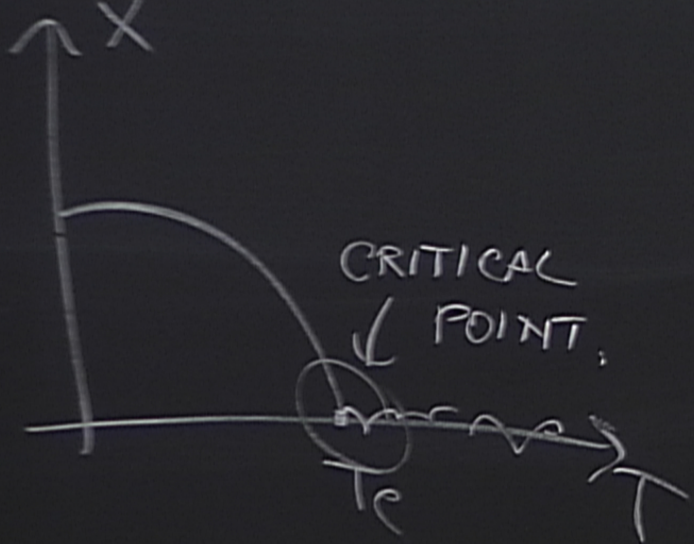


- OCCUR OFTEN IN NATURE
- LATENT HEAT ⇒ DISTINCT PHASES CAN EXIST IN EQ



- DIFFICULT TO DESCRIBE (DEPEND ON MICROSCOPICS)

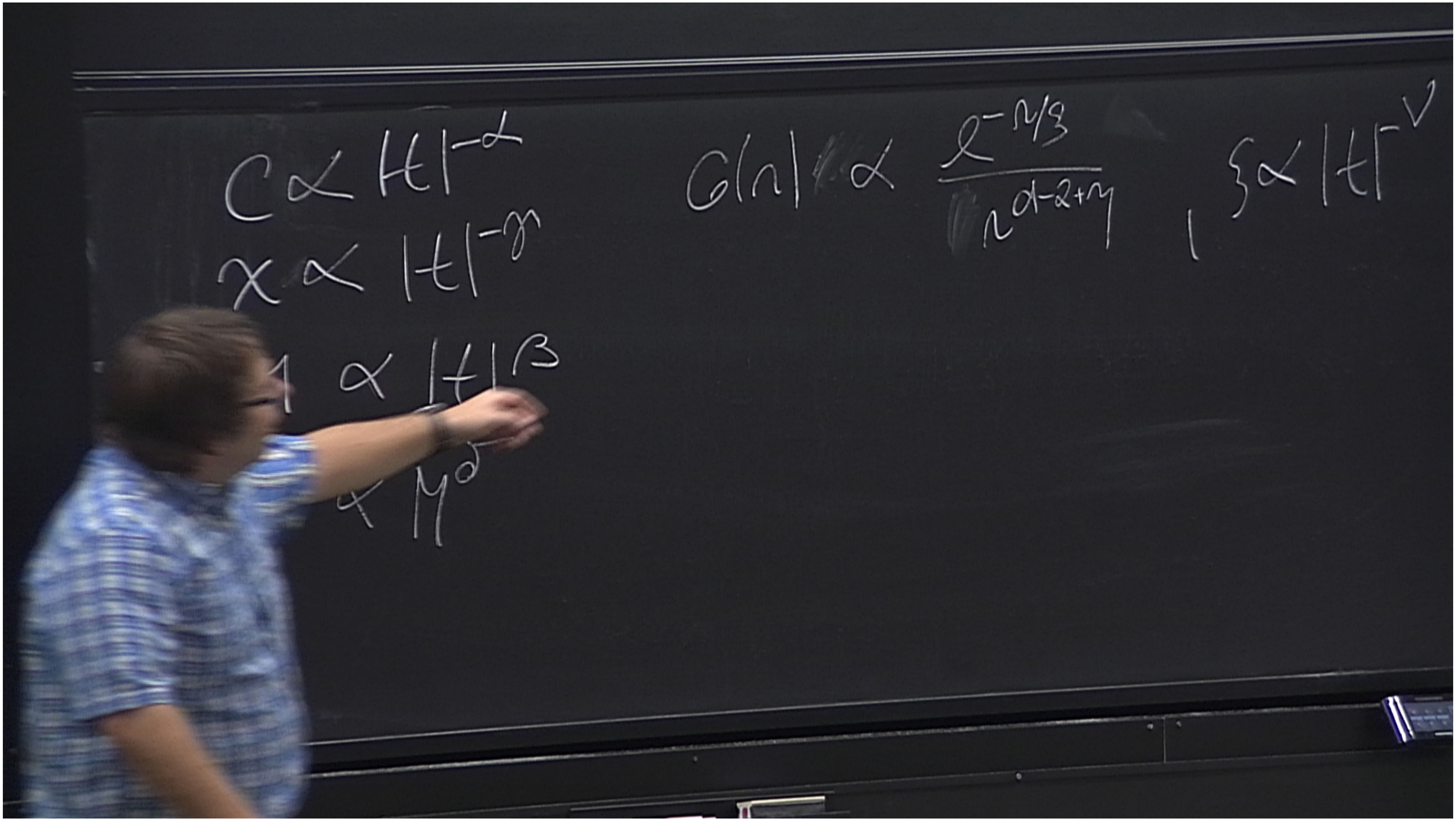
CONTINUOS



- LESS OFTEN (NEED SYMMETRY)
- DEMONSTRATE UNIVERSALITY

$$t = \frac{T - T_c}{T_c}$$

ORDERED PHASE $t < 0$


$C \propto |t|^{-\alpha}$
$\chi \propto |t|^{-\gamma}$
$M \propto |t|^{\beta}$
$\propto M^{\delta}$
$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}$
$\xi \propto |t|^{-\nu}$

$$C \propto |t|^{-\alpha}$$

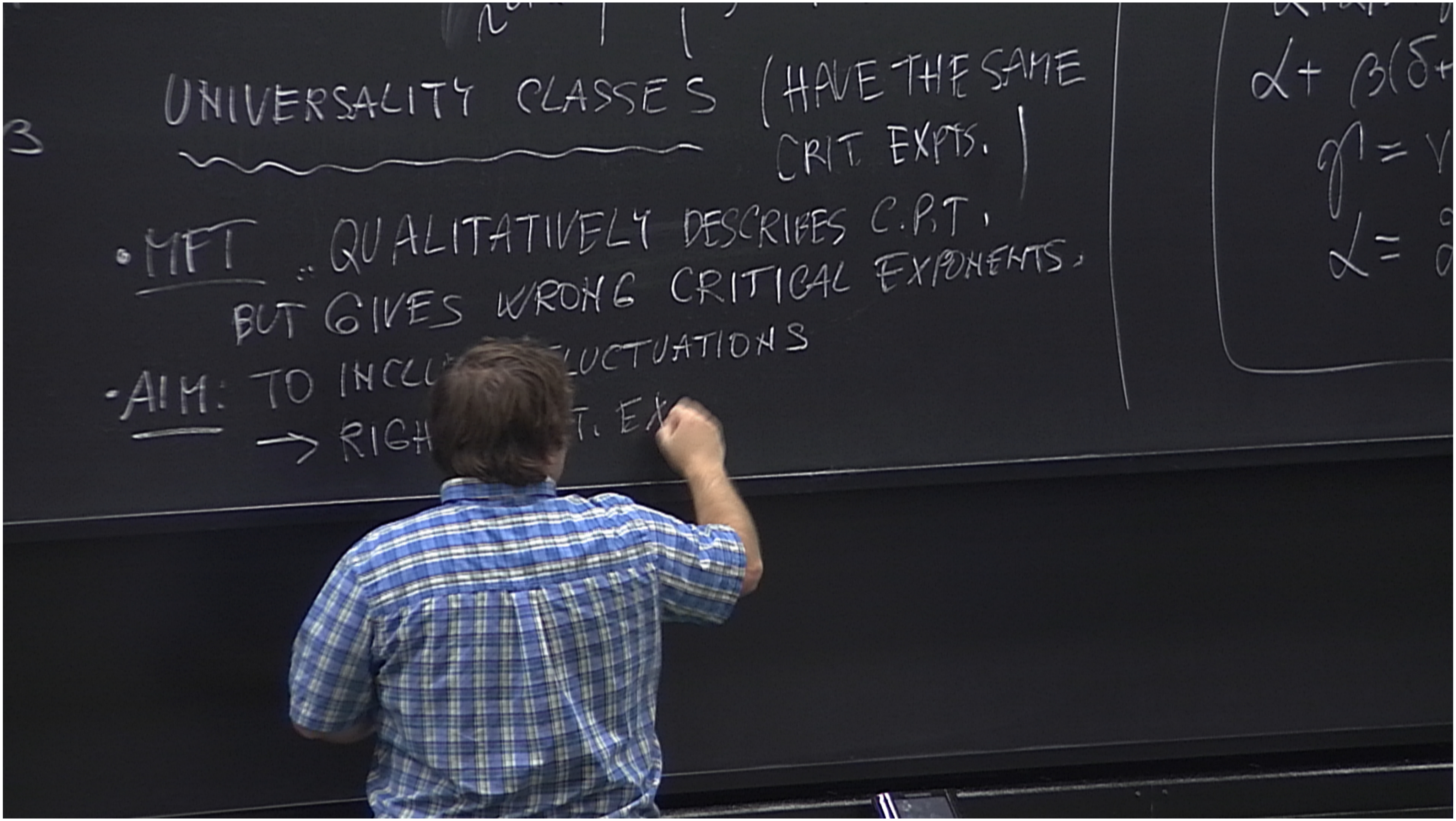
$$\chi \propto |t|^{-\gamma}$$

$$M \propto |t|^{\beta}$$

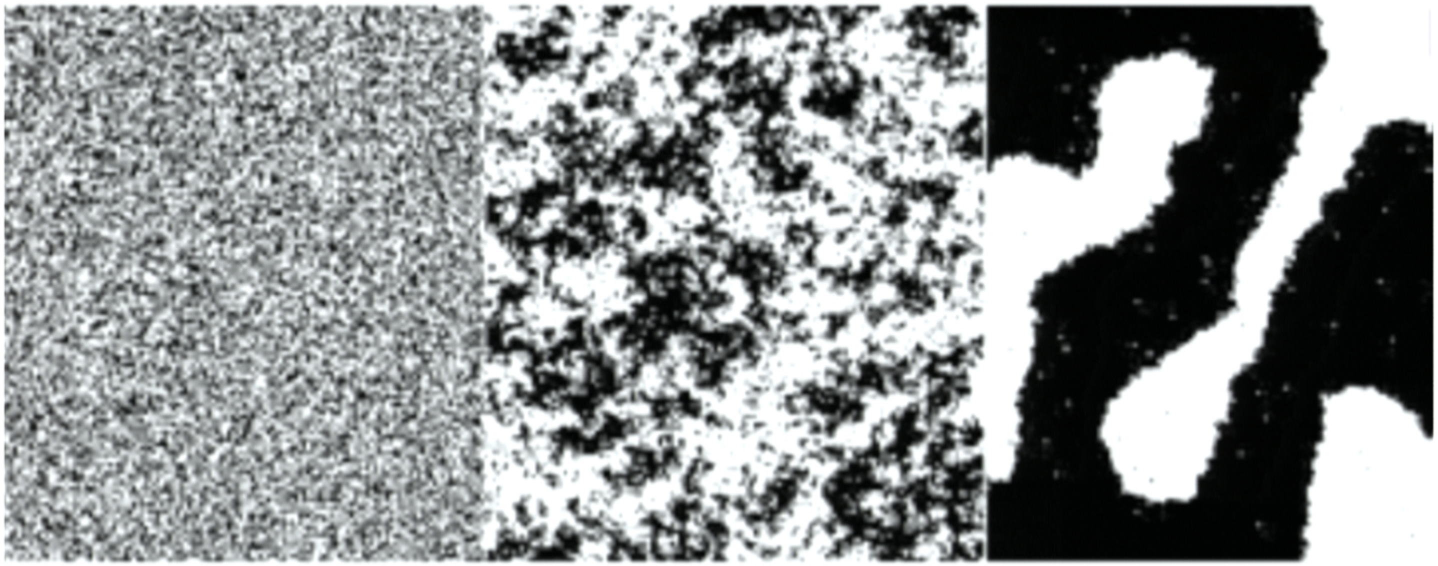
$$H \propto M^{\delta}$$

$$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}, \quad \xi \propto |t|^{-\nu}$$

<u>UNIVERSALITY CLASSES</u> (HAVE T


UNIVERSALITY CLASSES (HAVE THE SAME CRIT. EXPTS.)
• MFT ∴ QUALITATIVELY DESCRIBES C.P.T.,
BUT GIVES WRONG CRITICAL EXPONENTS.
• AIM: TO INCL
UCTUATIONS
→ RIGH
T. EX

ISING MODEL: ordering



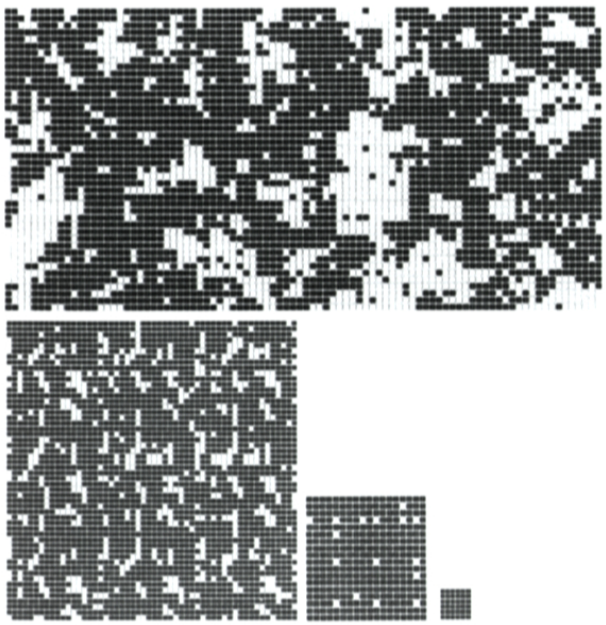
B=0, J=0 B=0, J=0.4406868 B=0, J=0.8

At Tc: scale invariant (fractal) structure

http://www.dhushara.com/DarkHeart/Ising/Ising.htm

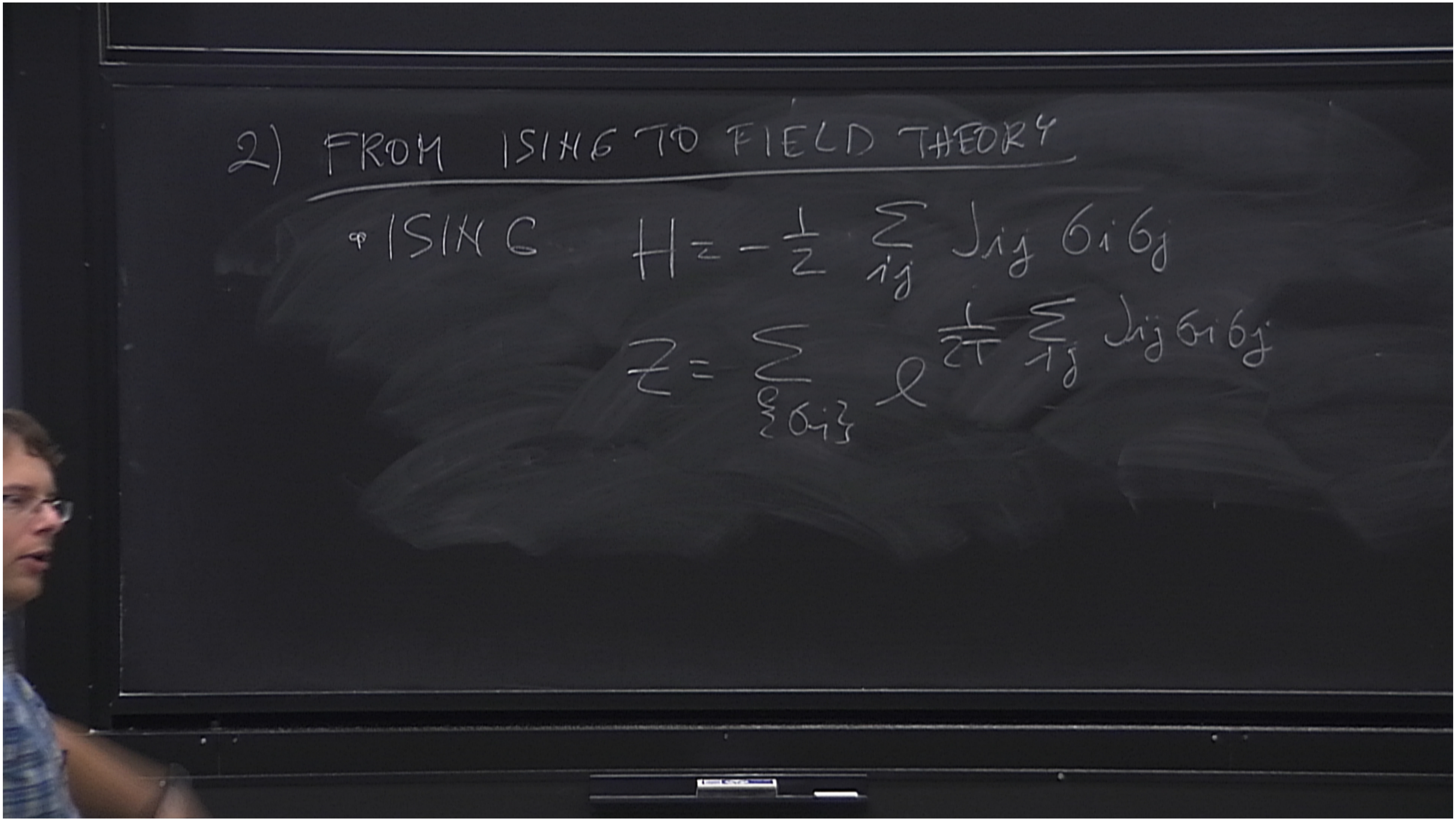
Block Spin RG

Below Tc (M=M0)



Above Tc (M=0)

https://plus.maths.org/content/going-flow-0

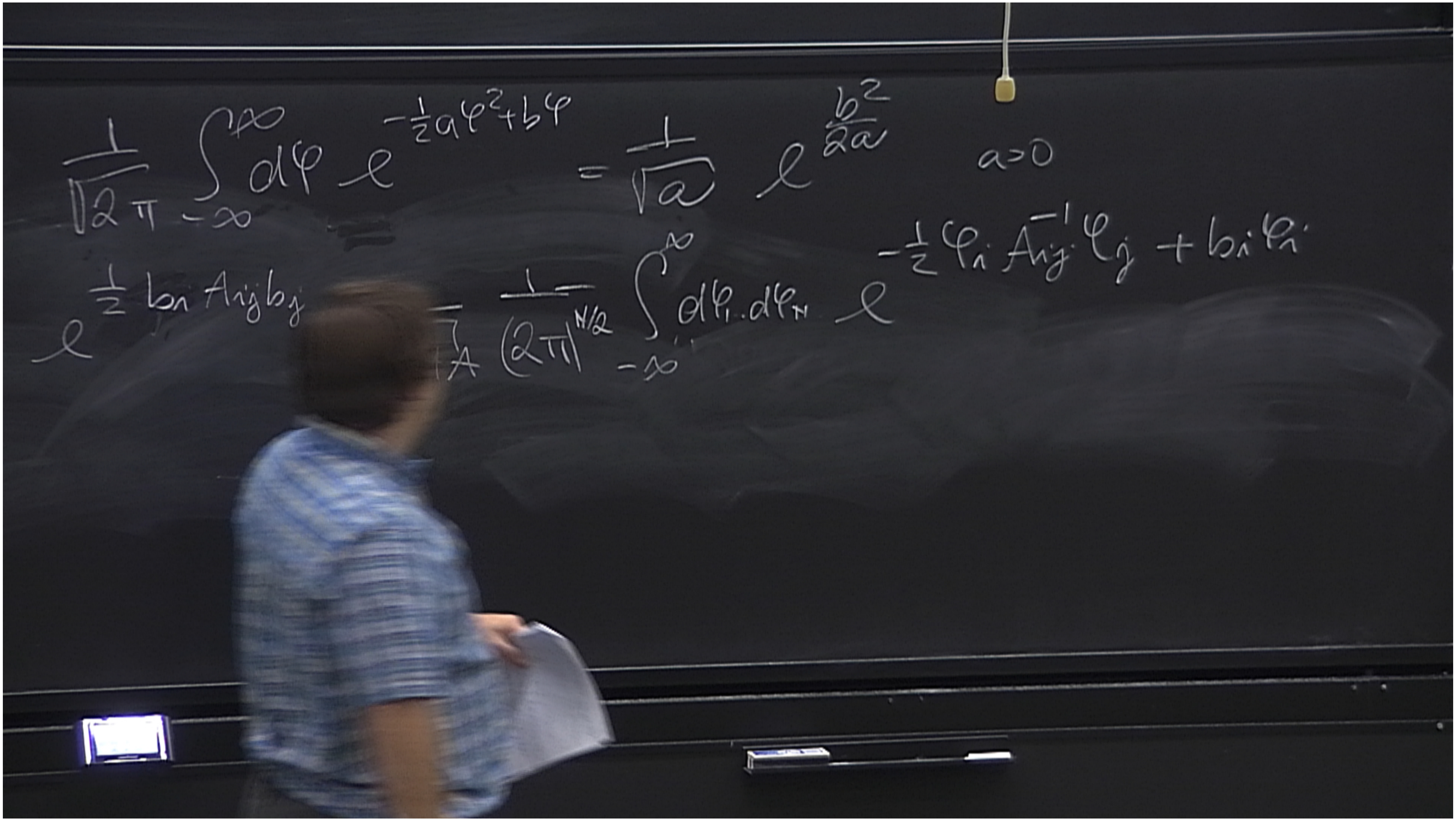

2) FROM ISING TO FIELD THEORY
• ISING $H = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j$
$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T}\sum_{ij} J_{ij}\sigma_i\sigma_j}$

2) FROM ISING TO FIELD THEORY

• ISING $$H = -\frac{1}{2}\sum_{ij} J_{ij}\,\sigma_i\sigma_j$$

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T}\sum_{ij} J_{ij}\,\sigma_i\sigma_j}$$

a) APPLY HUBBARD-STRATONOVICH TF.



$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\varphi \, e^{-\frac{1}{2}a\varphi^2+b\varphi} = \frac{1}{\sqrt{a}} e^{\frac{b^2}{2a}}$
$a>0$
$e^{\frac{1}{2} b_i A_{ij} b_j} = \frac{1}{\sqrt{A}\,(2\pi)^{N/2}} \int_{-\infty}^{\infty} d\varphi_1 . d\varphi_N \, e^{-\frac{1}{2}\varphi_i A^{-1}_{ij}\varphi_j + b_i\varphi_i}$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\varphi \, e^{-\frac{1}{2}a\varphi^2+b\varphi} = \frac{1}{\sqrt{a}} \, e^{\frac{b^2}{2a}}$$

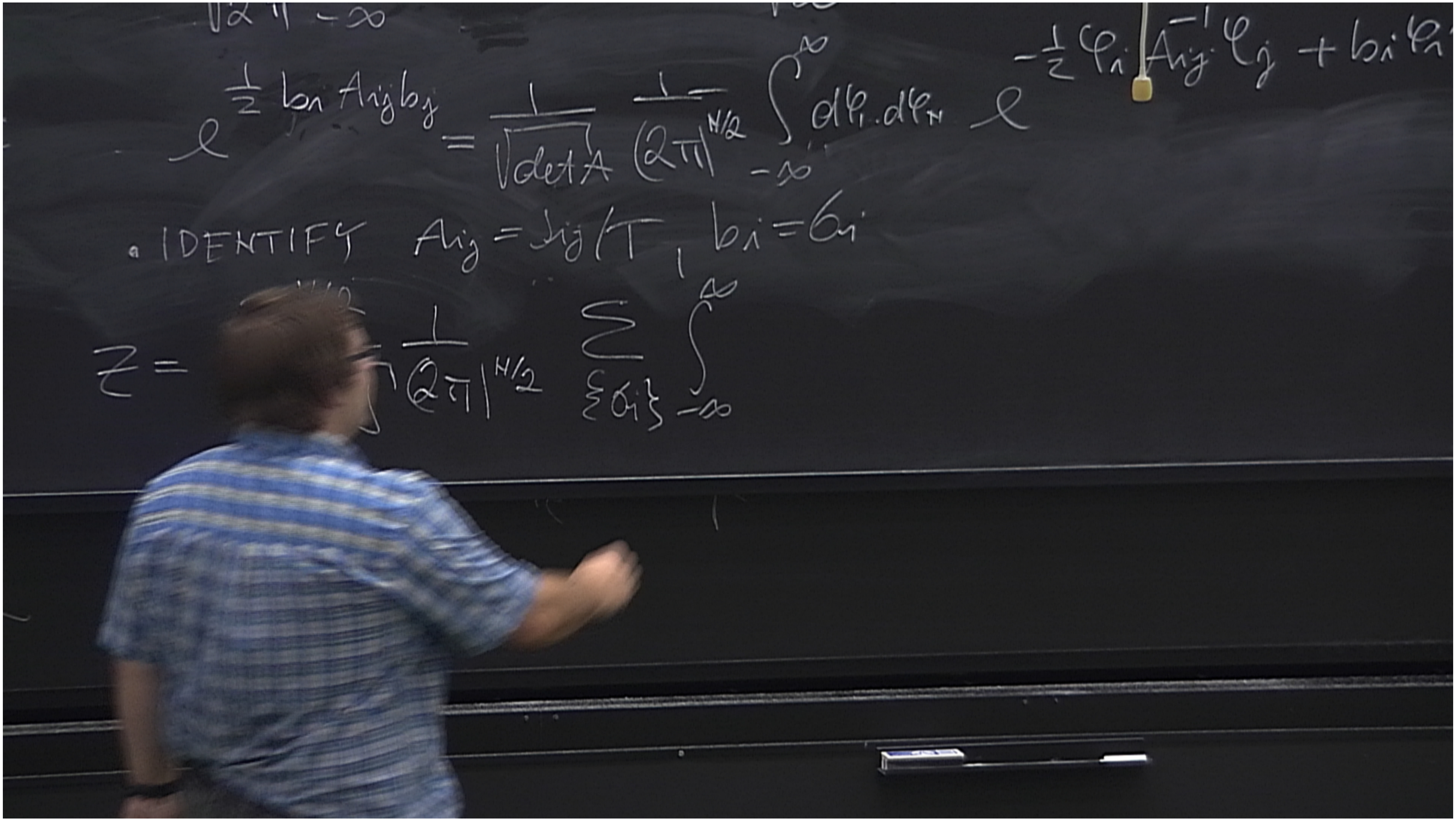
$a>0$

$$e^{\frac{1}{2} b_i A_{ij} b_j} = \frac{1}{\sqrt{\det A}} \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} d\varphi_1 \ldots d\varphi_N \, e^{-\frac{1}{2}\varphi_i A^{-1}_{ij}\varphi_j + b_i\varphi_i}$$

$\sqrt{2\pi} \quad -\infty$

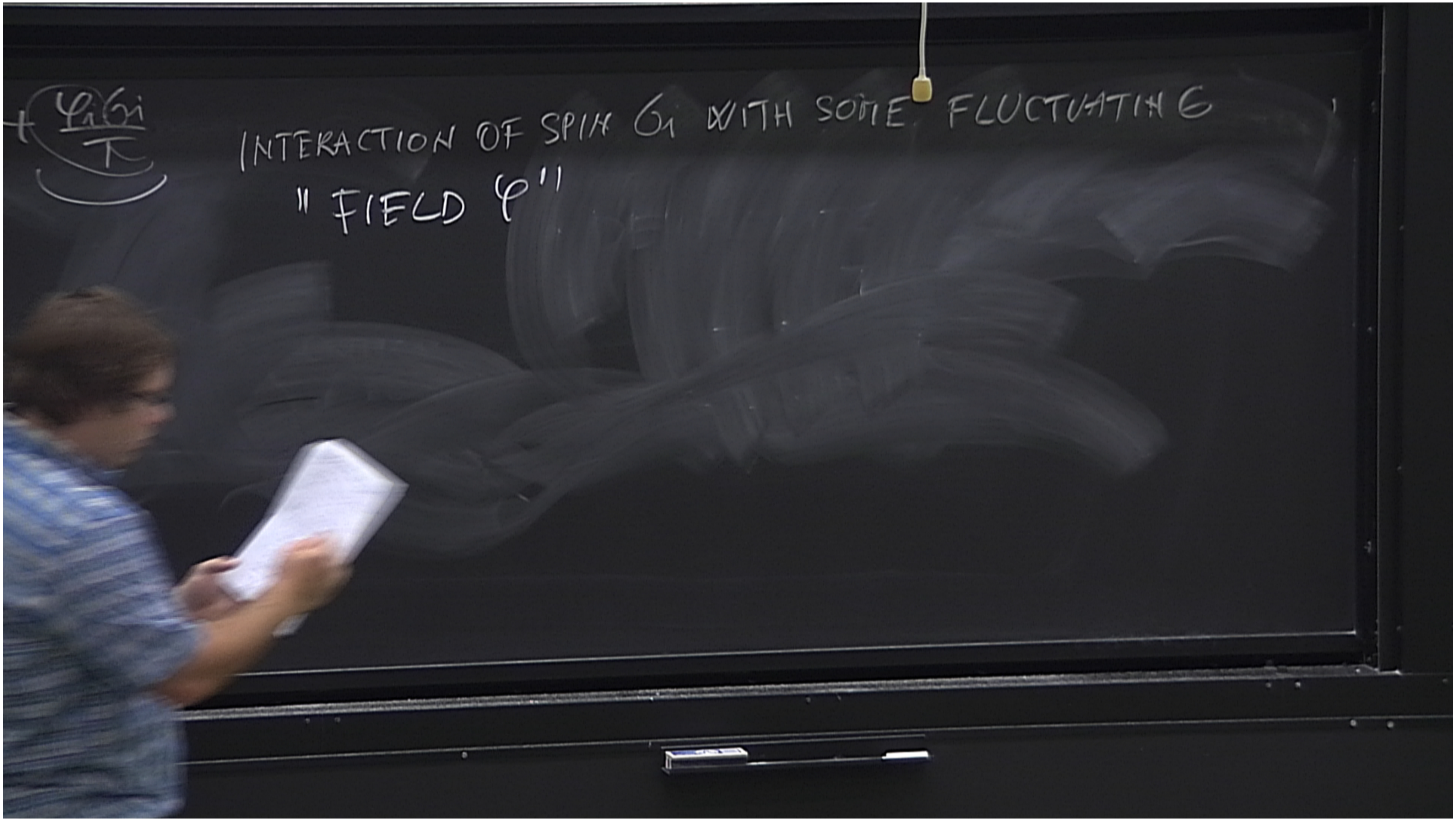
$$e^{\frac{1}{2} b_i A_{ij} b_j} = \frac{1}{\sqrt{\det A}} \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} d\varphi_1 . d\varphi_N \; e^{-\frac{1}{2}\varphi_i A^{-1}_{ij} \varphi_j + b_i \varphi_i}$$

- IDENTIFY $A_{ij} = J_{ij}/T$, $b_i = \sigma_i$


$e^{\frac{1}{2} b_i A_{ij} b_j} = \frac{1}{\sqrt{\det A}} \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} d\varphi_1 . d\varphi_N \, e^{-\frac{1}{2} \varphi_i A^{-1}_{ij} \varphi_j + b_i \varphi_i}$
IDENTIFY $A_{ij} = J_{ij}/T$, $b_i = \sigma_i$
$Z = \frac{1}{(2\pi)^{N/2}} \sum_{\{\sigma_i\}} \int_{-\infty}^{\infty}$

$\varphi_i \to \varphi_i / T$

$$= \frac{1}{\sqrt{\det J}} \frac{1}{(2\pi T)^{N/2}} \sum_{\{S_i\}} \int_{-\infty}^{\infty} D\varphi \; e^{-\frac{1}{2T}\varphi_i J_{ij}^{-1} \varphi_j + \frac{\varphi_i S_i}{T}}$$


INTERACTION OF SPIN σi WITH SOME FLUCTUATING
"FIELD φ"

$+\frac{\varphi_i \sigma_i}{T}$

INTERACTION OF SPIN σ_i WITH SOME FLUCTUATING "FIELD φ" (MFT $\varphi \approx \langle \varphi \rangle \approx M$)

$$\sum_{\sigma_i=\pm 1} e^{\frac{1}{T}\varphi_i \sigma_i} = e^{\log[2\cosh}$$

$$Z = \int D\varphi \, e^{-S[\varphi]}$$

$$S[\varphi] = \frac{1}{2T}\varphi_i J^{-1}_{ij}\varphi_j - \sum_i \log\left[2\cosh \varphi_i/T\right]$$

NOTE: MFT $\Longleftrightarrow$ SADDLE POINT

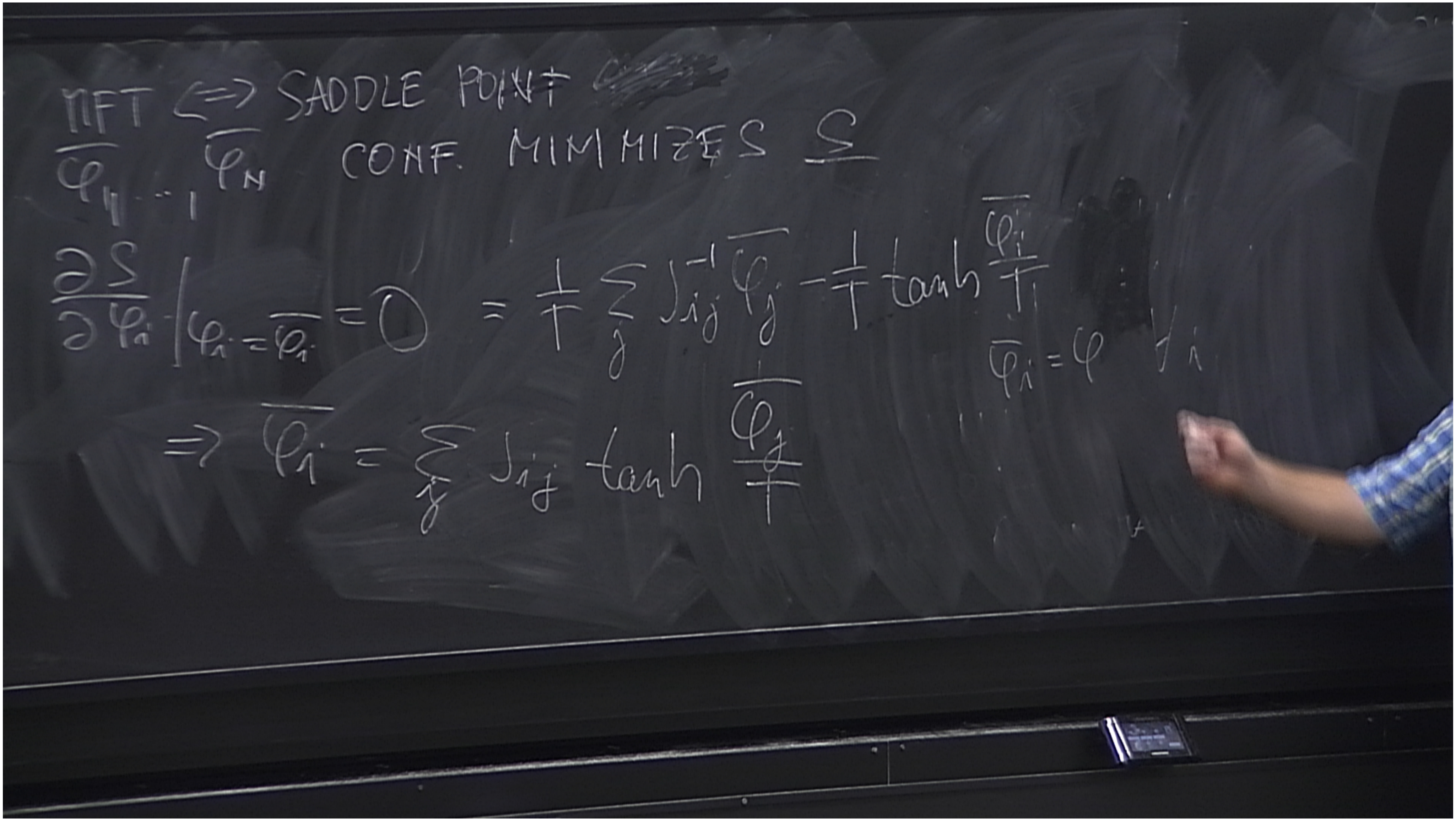
LET $\overline{\varphi_1}, \dots, \overline{\varphi_N}$ CONF. MINIMIZES $\underline{S}$

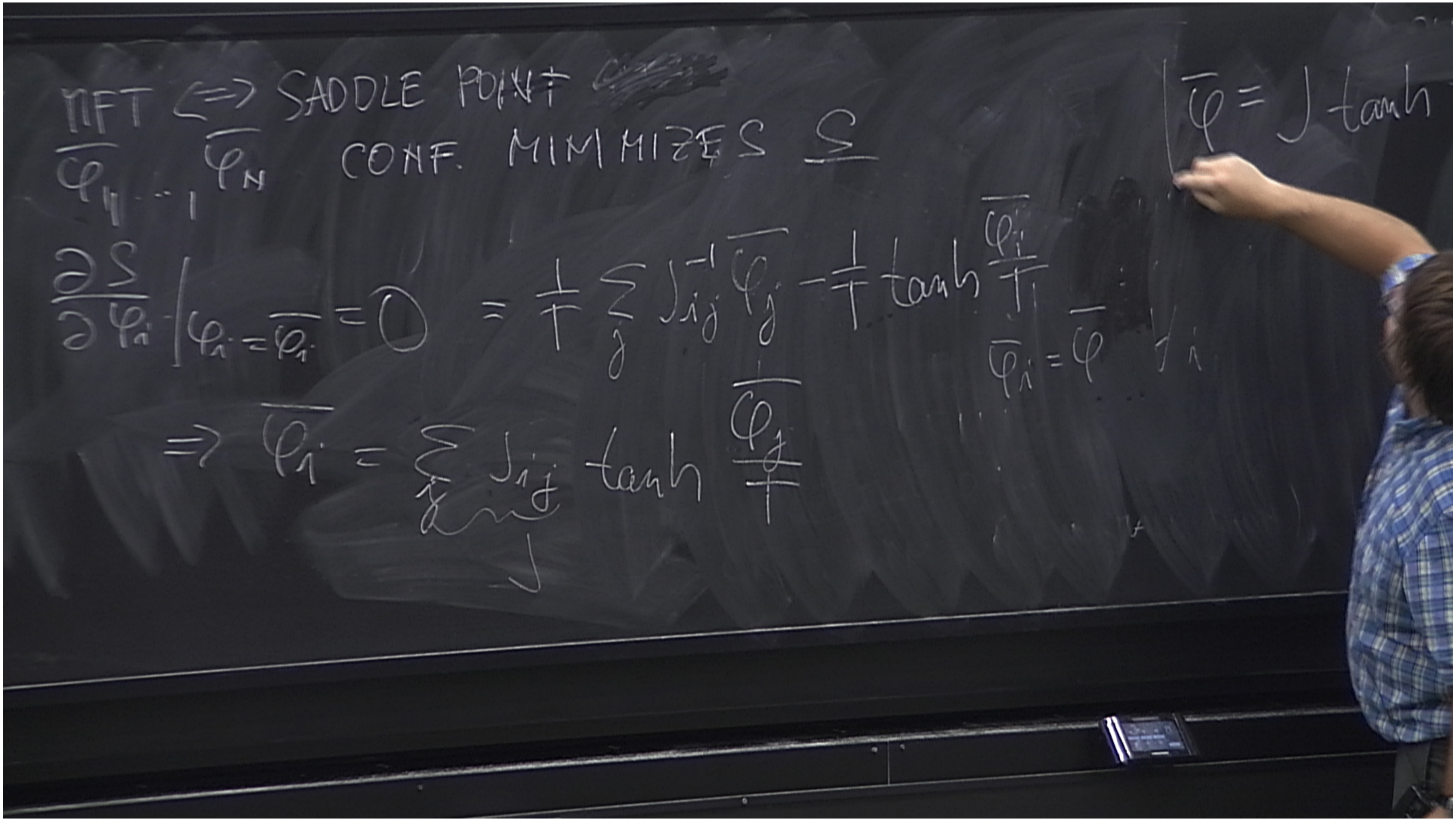
$$\left.\frac{\partial S}{\partial \varphi_1}\right|_{\varphi_1 = \overline{\varphi_1}} = 0$$

NOTE: MFT $\Longleftrightarrow$ SADDLE POINT

LET $\bar{\varphi}_1, \dots, \bar{\varphi}_N$ CONF. MINIMIZES $\underline{S}$

$$\frac{\partial S}{\partial \varphi_i}\Big|_{\varphi_i = \bar{\varphi}_i} = 0 = \frac{1}{T}\sum_j J^{-1}_{ij}\,\bar{\varphi}_j - \frac{1}{T}\tanh\frac{\bar{\varphi}_i}{T}$$


MFT ⟺ SADDLE POINT
$\overline{\varphi_1}, \dots, \overline{\varphi_N}$ CONF. MINIMIZES $\underline{S}$
$\frac{\partial S}{\partial \varphi_i}\Big|_{\varphi_i = \overline{\varphi_i}} = 0 = \frac{1}{T} \sum_j J^{-1}_{ij} \overline{\varphi_j} - \frac{1}{T} \tanh \frac{\overline{\varphi_i}}{T}$
$\overline{\varphi_i} = \varphi \quad \forall i$
$\Rightarrow \overline{\varphi_i} = \sum_j J_{ij} \tanh \frac{\overline{\varphi_j}}{T}$


MFT ⟺ SADDLE POINT
$\bar\varphi_1, \dots, \bar\varphi_N$ CONF. MINIMIZES $\underline{S}$
$\frac{\partial S}{\partial \varphi_i}\Big|_{\varphi_i=\bar\varphi_i} = 0 = \frac{1}{T}\sum_j J^{-1}_{ij}\bar\varphi_j - \frac{1}{T}\tanh\frac{\bar\varphi_i}{T}$
$\Rightarrow \bar\varphi_i = \sum_j J_{ij}\tanh\frac{\bar\varphi_j}{T}$
$\bar\varphi_i = \bar\varphi \quad \forall i$
$\bar\varphi = J\tanh$

b) "CONTINUUM LIMIT" φ_i . DEFINED AT SITE $i \to \varphi(x)$

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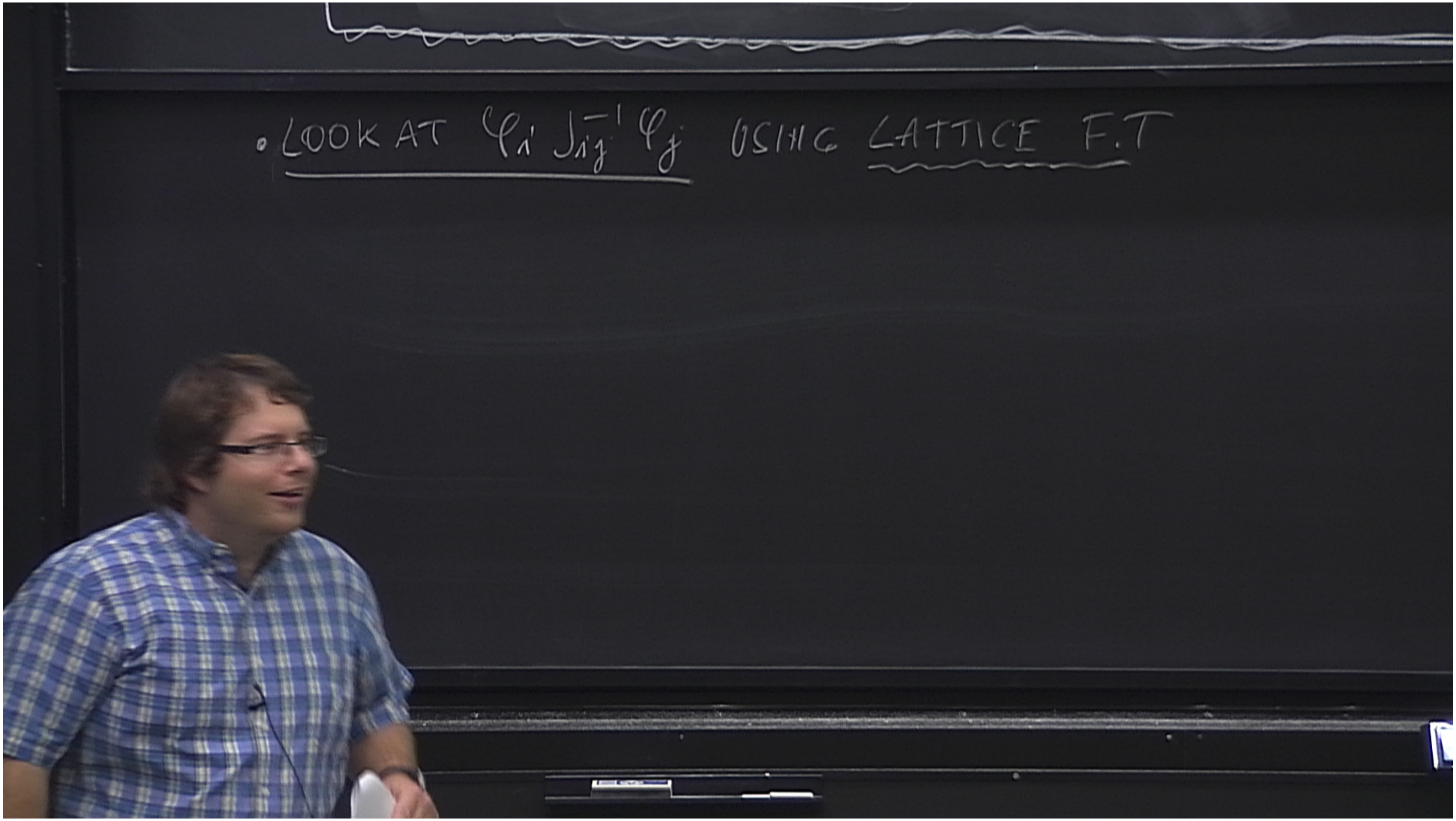
"ZOOM OUT" $\leftrightarrow$ LOOSE INFO ABOUT UNDERLYING LATTICE.

$$\sum_{\sigma_i=\pm 1} e^{\frac{1}{T}\varphi_i \sigma_i} = e^{\log[2}$$

$$\boxed{\begin{aligned} Z &= \int D\varphi \, e^{-S[\varphi]} \\ S[\varphi] &= \frac{1}{2T}\varphi_i J^{-1}_{ij}\varphi_j - \sum_i \log\left[2\cosh \varphi_i/T\right] \end{aligned}}$$


• LOOK AT $\varphi_i J_{ij}^{-1} \varphi_j$ USING LATTICE F.T

• LOOK AT $\varphi_i J_{ij}^{-1} \varphi_j$ USING LATTICE F.T

$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q})\, e^{i\vec{q}\cdot\vec{r}_i} \qquad \frac{1}{N}$$

$$J_{ij} = \frac{1}{N} \sum_{\vec{q}} J(\vec{q})\, e^{i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)}$$

USING LATTICE F.T

$$(\vec{q})\, e^{i\vec{q}\cdot\vec{r}_i}$$

$$\frac{1}{N}\sum_i e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}_i} = \delta_{\vec{k},\vec{k}'}$$

$$\vec{q})\, e^{i\vec{q}\cdot(\vec{r}_i-\vec{r}_j)}$$

$$\frac{1}{N}\sum_{\vec{q}} e^{+i\vec{q}(\vec{r}_i-\vec{r}_j)} = \delta_{ij}$$

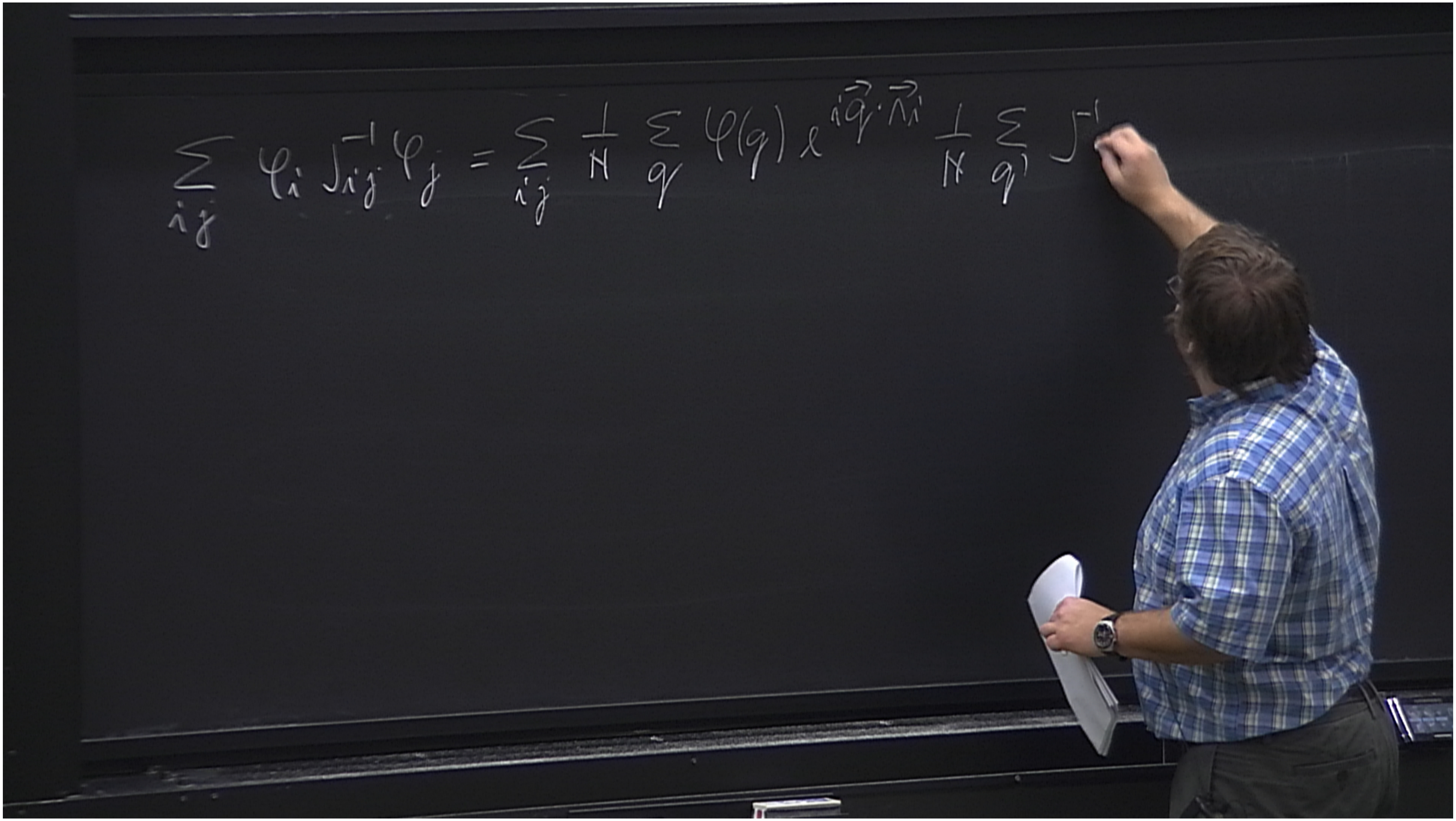
USING LATTICE F.T

$$(\vec{q})\, e^{i\vec{q}\cdot\vec{r}_i}$$

$$(\vec{q})\, e^{i\vec{q}\cdot(\vec{r}_i-\vec{r}_j)}$$

$$\frac{1}{N}\sum_i e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}_i} = \delta_{\vec{k},\vec{k}'}$$

$$\frac{1}{N}\sum_{\vec{q}} e^{+i\vec{q}(\vec{r}_i-\vec{r}_j)} = \delta_{ij}$$


$\sum_{ij} \varphi_i J^{-1}_{ij} \varphi_j = \sum_{ij} \frac{1}{N} \sum_{q} \varphi(q)\, e^{i\vec{q}\cdot\vec{r}_i} \frac{1}{N} \sum_{q'} J^{-1}$

$$\sum_{ij} \varphi_i J^{-1}_{ij} \varphi_j = \sum_{ij} \frac{1}{N} \sum_{q} \varphi(q)\, e^{i\vec{q}\cdot\vec{r}_i} \frac{1}{N} \sum_{q'} J^{-1}(q')\, e^{iq'(\vec{r}_i-\vec{r}_j)} \frac{1}{N} \sum_{q''} \varphi(\ldots$$

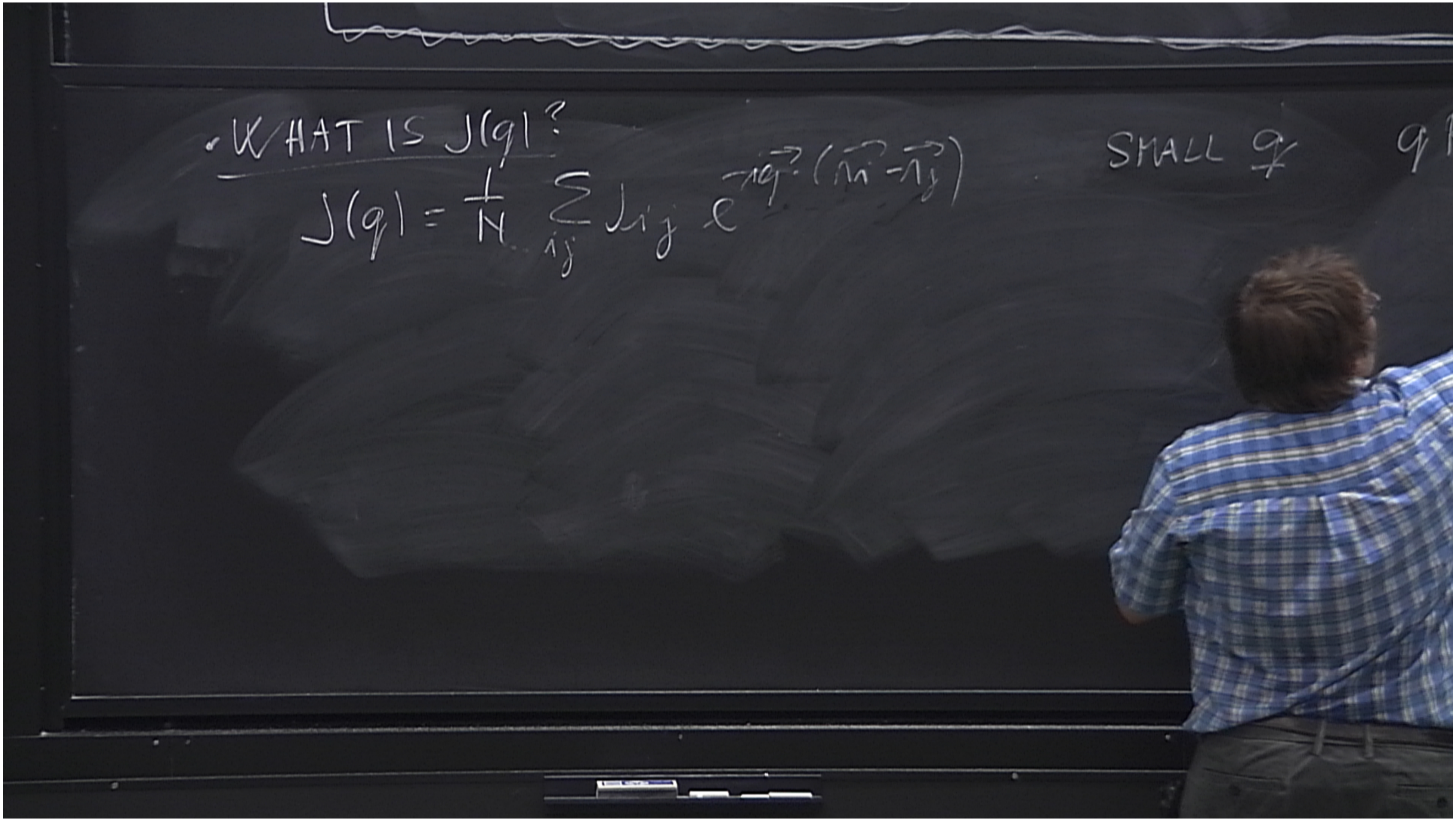
$$= \frac{1}{N} \sum_{q,q',q''} \varphi(q)\, J^{-1}(q')\, \varphi(q'')\, \delta_{q,-q'}\, \delta_{q',q''}$$

$$= \frac{1}{N} \sum_{q} \varphi(-q)\, J^{-1}(q)\, \varphi(q)$$

$$\sum_{ij} \varphi_i J^{-1}_{ij} \varphi_j = \sum_{ij} \frac{1}{N} \sum_{q} \varphi(q)\, e^{i\vec{q}\cdot\vec{r}_i} \frac{1}{N} \sum_{q'} J^{-1}(q')\, e^{iq'(\vec{r}_i - \vec{r}_j)} \frac{1}{N} \sum_{q''} \varphi(\ldots$$

$$= \frac{1}{N} \sum_{q,q',q''} \varphi(q)\, J^{-1}(q')\, \varphi(q'')\, \delta_{q,-q'}\, \delta_{q',q''}$$

$$= \underline{\frac{1}{N} \sum_{q} \varphi(-q)\, J^{-1}(q)\, \varphi(q)}$$


• WHAT IS J(q)?
$J(q) = \frac{1}{N} \sum_{ij} J_{ij} e^{-i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)}$
SMALL q

• WHAT IS $J(q)$?

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij}\, e^{-i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)}$$

SMALL q (ZOOM OUT) $\quad qR_a \ll 1$

$$\simeq \frac{1}{N} \sum_{ij} J_{ij} \left(1 - i\vec{q}\cdot(\vec{r}_i - \vec{r}_j) - \frac{1}{2}\left[\vec{q}\cdot(\vec{r}_i - \vec{r}_j)\right]^2 + \ldots \right.$$

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$$\simeq \frac{1}{N} \sum_{ij} J_{ij} \left(1 - i\vec{q}\cdot(\vec{r}_i - \vec{r}_j) - \frac{1}{2}\left[\vec{q}\cdot(\vec{r}_i - \vec{r}_j)\right]^2 + \dots \right)$$

NJ

• WHAT IS $J(q)$?

$$J(q) = \frac{1}{N}\sum_{ij} J_{ij}\, e^{-i\vec{q}\cdot(\vec{r}_i-\vec{r}_j)}$$

SMALL q (ZOOM OUT)

$qR \ll 1$, $R \sim a$

$$\simeq \frac{1}{N}\sum_{ij} \underbrace{J_{ij}}_{NJ}\left(1 - \cancel{i\vec{q}\cdot(\vec{r}_i-\vec{r}_j)} - \frac{1}{2}\left[\vec{q}\cdot(\vec{r}_i-\vec{r}_j)\right]^2 + \dots\right)$$

$$\simeq J - \frac{1}{2N}\sum_{ij}\left[\vec{q}\cdot(\vec{r}_i-\vec{r}_j)\right]^2$$

• WHAT IS $J(q)$?

SMALL q (ZOOM OUT) $\quad qR_a \ll 1$

$$J(q) = \frac{1}{N} \sum_{ij} J_{ij}\, e^{-i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)}$$

$$\simeq \frac{1}{N} \sum_{ij} J_{ij} \Big(\underbrace{1}_{NJ} - \cancel{i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} - \frac{1}{2}\left[\vec{q}\cdot(\vec{r}_i - \vec{r}_j)\right]^2 + \dots$$

$$\simeq J - \frac{1}{2N} \sum_{ij} \underbrace{\left[\vec{q}\cdot(\vec{r}_i - \vec{r}_j)\right]^2}_{q^2R^2} J_{ij}$$

χ ... SPIN STIFFNES

$$\chi \simeq \frac{R^2 J}{d}$$

$\ll 1$

$\mathcal{K}$ ... SPIN STIFFNES

$$\mathcal{K} \simeq \frac{R^2 J}{d}$$

$+ \ldots$

$R|q|^2$

WHAT IS $J(q)$?

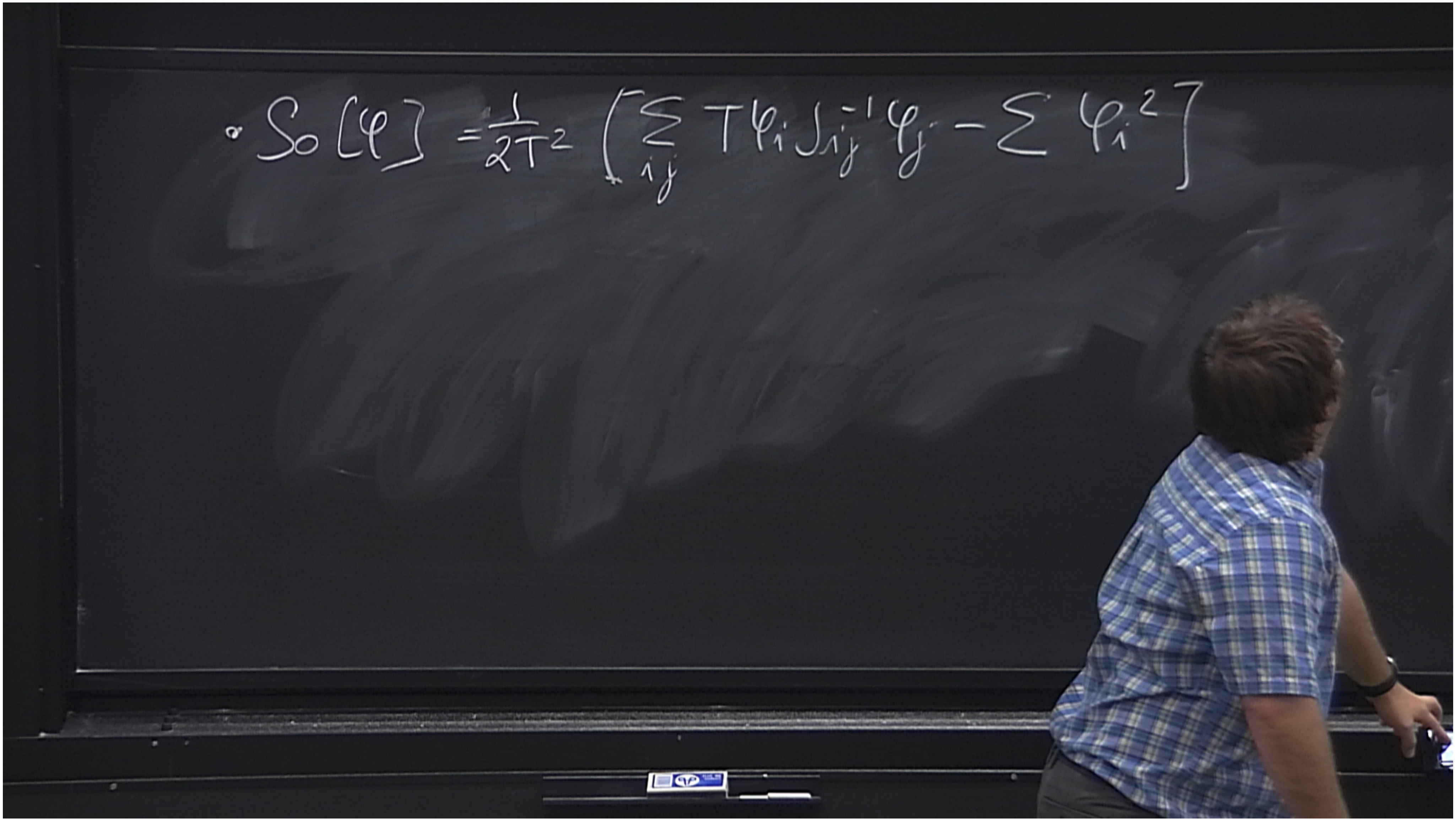
SMALL q (ZOOM OUT)

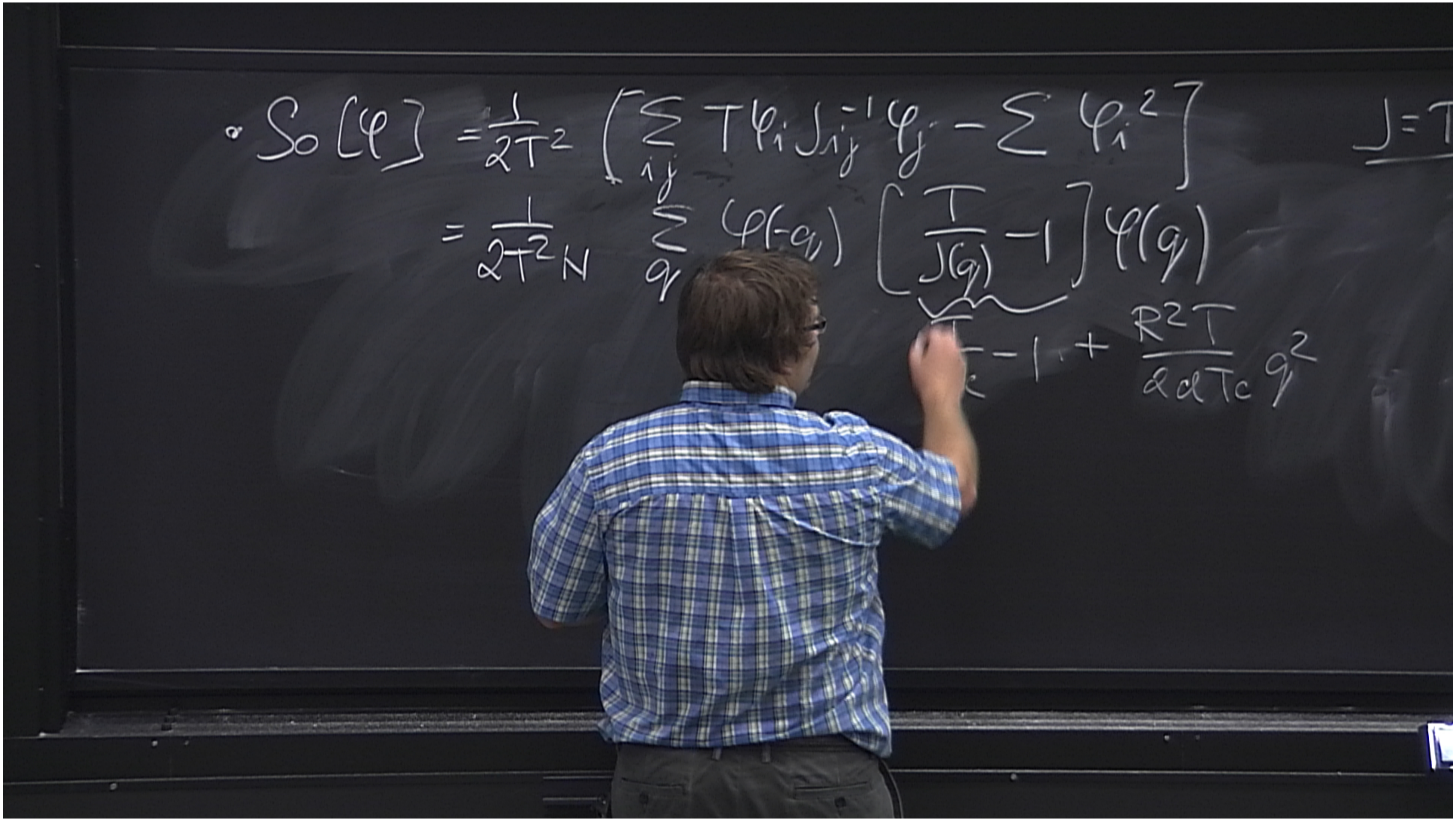
$q R_a \ll$

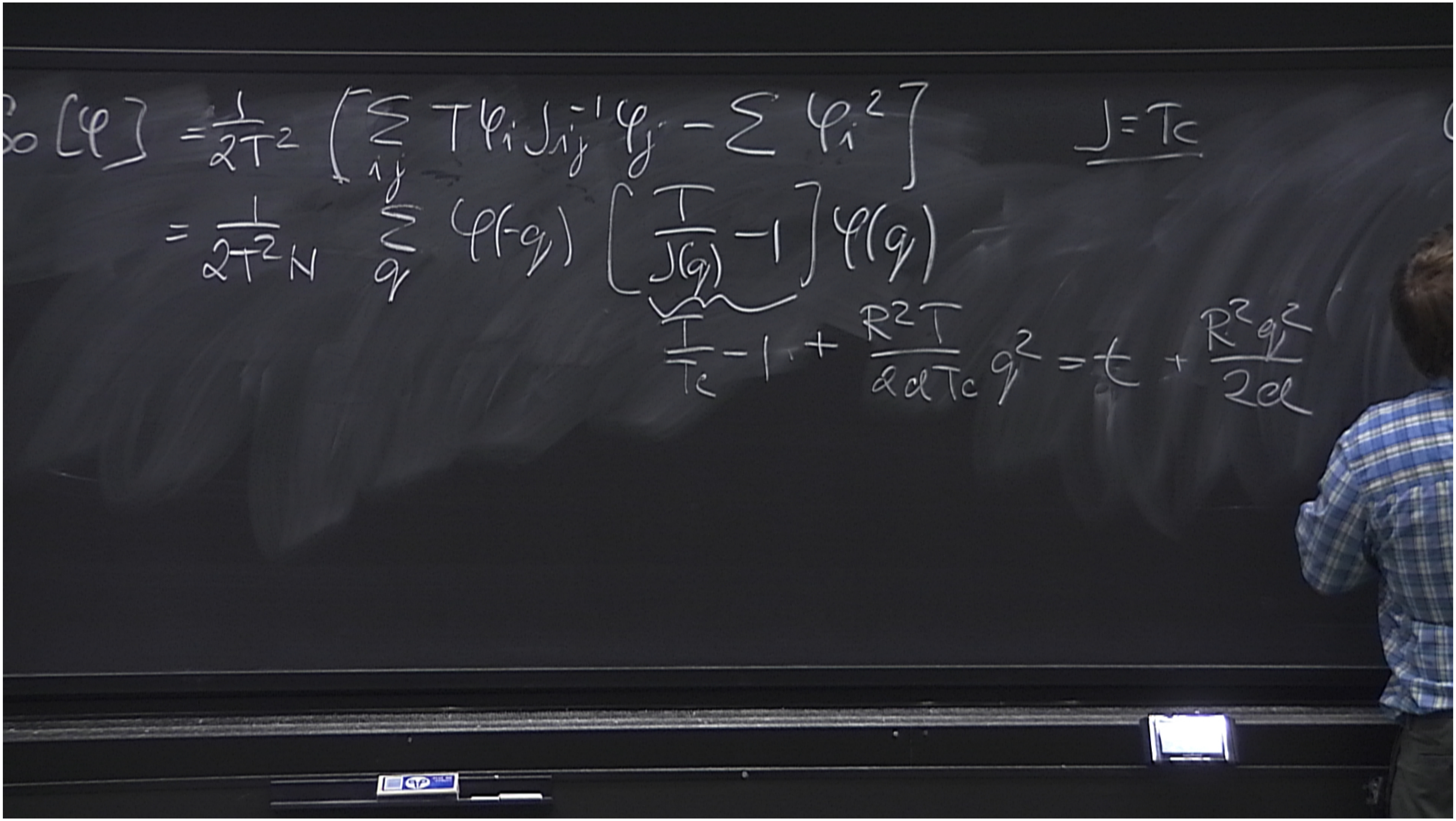
$$J(q) = \frac{1}{N}\sum_{ij} J_{ij}\, e^{-i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)}$$

$$\simeq \frac{1}{N}\sum_{ij} J_{ij}\left(\underbrace{1}_{NJ} - \cancel{i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} - \frac{1}{2}\left[\vec{q}\cdot(\vec{r}_i - \vec{r}_j)\right]^2 + \dots\right)$$

$$\simeq J - \frac{1}{2N}\sum_{ij}\underbrace{\left[\vec{q}\cdot(\vec{r}_i - \vec{r}_j)\right]^2 J_{ij}}_{\frac{q^2 R^2 J N}{d}} \simeq J - \frac{1}{2}\mathcal{H}(q$$


$S_0[\varphi] = \frac{1}{2T^2}\left[\sum_{ij} T\varphi_i J_{ij}^{-1}\varphi_j - \sum \varphi_i^2\right]$


$S_0[\varphi] = \frac{1}{2T^2}\left[\sum_{ij} T\varphi_i J^{-1}_{ij}\varphi_j - \sum \varphi_i^2\right]$
$= \frac{1}{2T^2 N}\sum_q \varphi(-q)\left[\frac{T}{J(q)} - 1\right]\varphi(q)$
$- 1 + \frac{R^2 T}{2\alpha T_c} q^2$
$J = T$


$S_0[\varphi] = \frac{1}{2T^2}\left[\sum_{ij} T\varphi_i J_{ij}^{-1}\varphi_j - \sum \varphi_i^2\right]$
$J = T_c$
$= \frac{1}{2T^2 N}\sum_q \varphi(-q)\left[\frac{T}{J(q)} - 1\right]\varphi(q)$
$\frac{T}{T_c} - 1 + \frac{R^2 T}{2d T_c} q^2 = t + \frac{R^2 q^2}{2d}$

$J = T_c$

$$\varphi \to \frac{T_c\sqrt{2\alpha}}{R\,a^{d/2}}\,\varphi, \qquad r \equiv \frac{2\alpha t}{R^2}$$

$$t + \frac{R^2 q^2}{2\alpha}$$

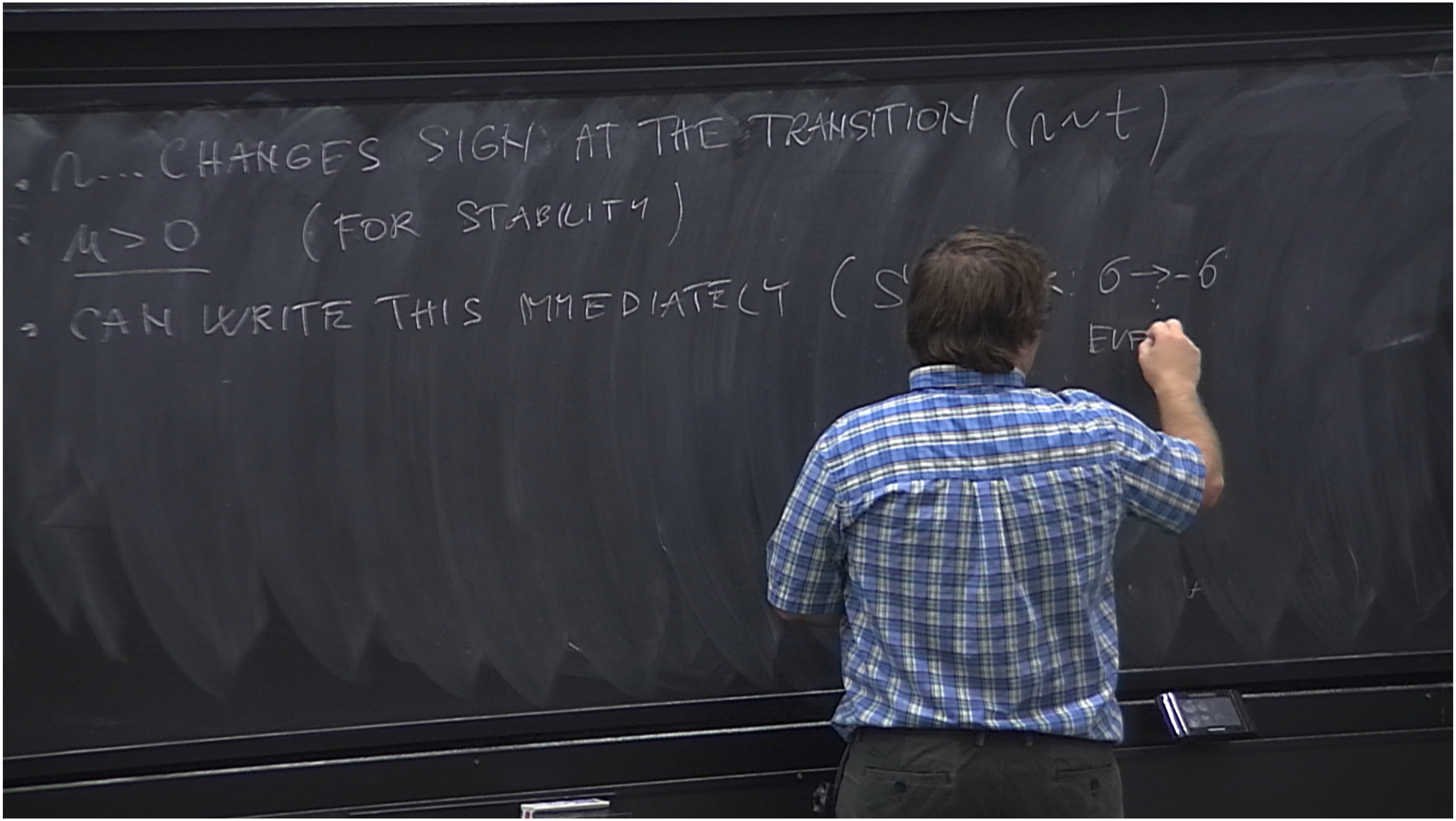
$$\left[\sum_{ij} T\varphi_i J_{ij}^{-1}\varphi_j - \sum \varphi_i^2\right] \qquad \underline{J = T_c} \qquad \varphi \to \frac{T_c\sqrt{2d}}{R a^{d/2}}\varphi$$

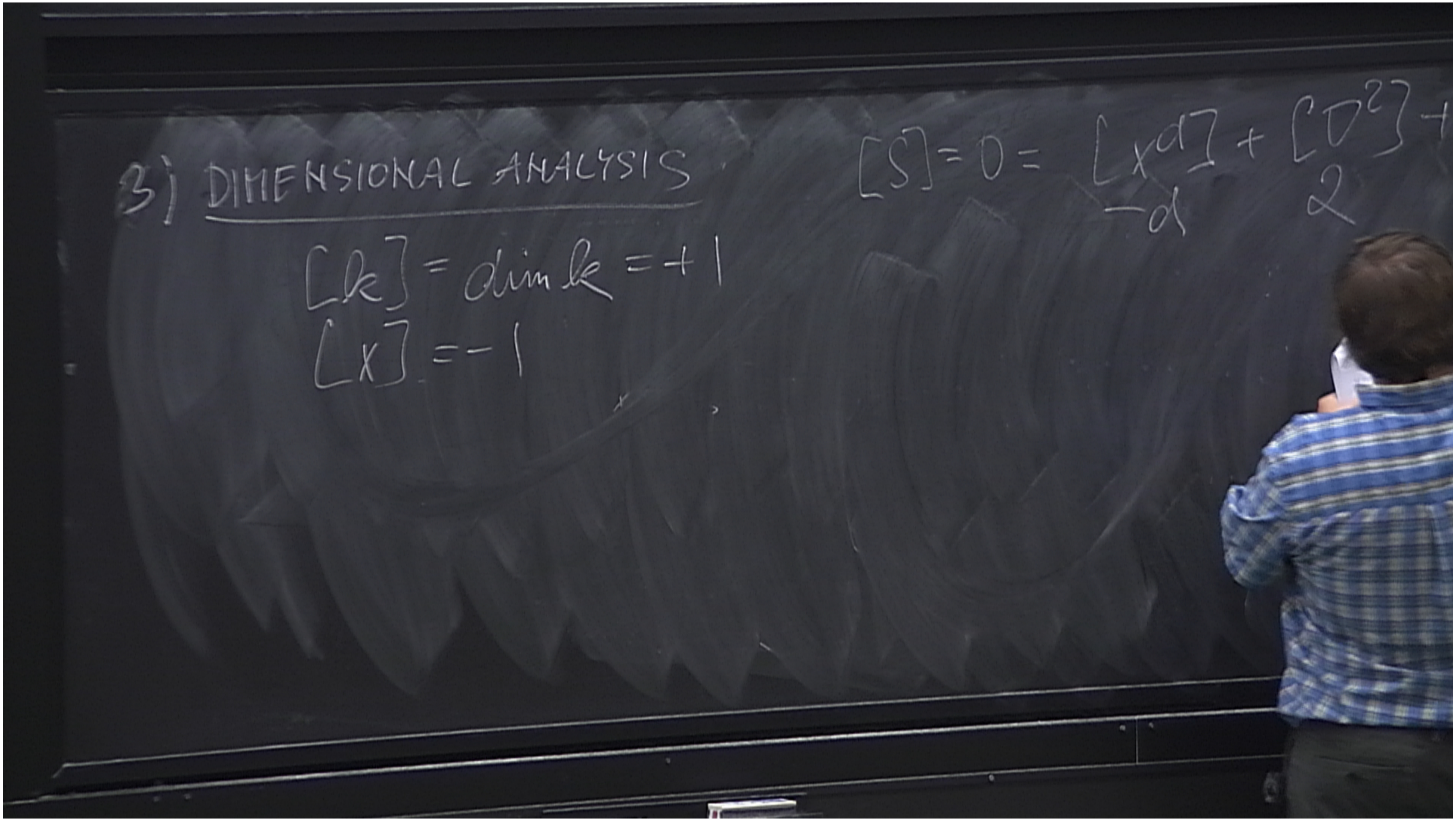
$$\sum_q \varphi(-q)\Bigg[\underbrace{\frac{T}{J(q)}}_{\frac{T}{T_c}-1+\frac{R^2T}{2dT_c}q^2} - 1\Bigg]\varphi(q)$$

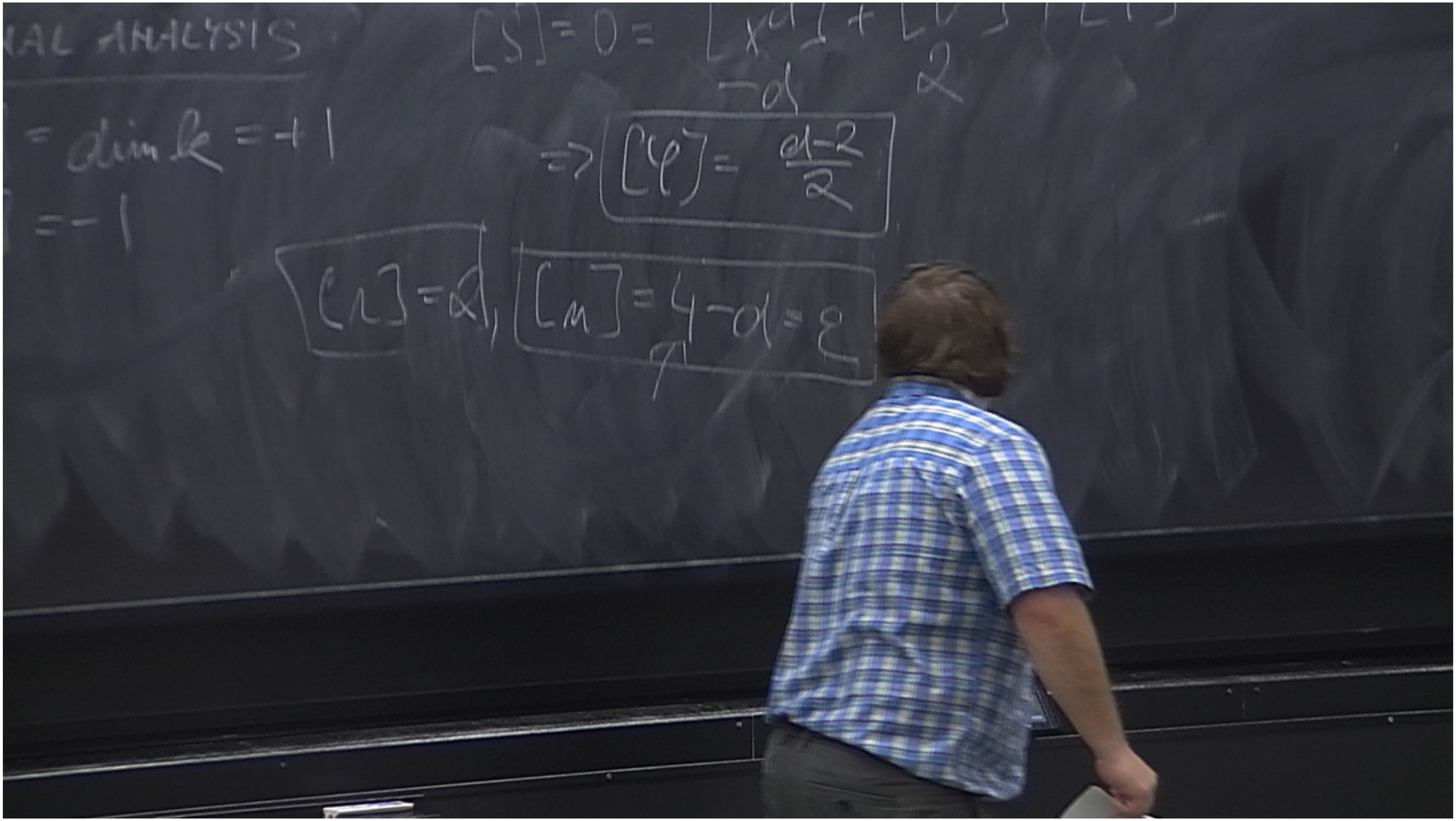
$$\frac{T}{T_c} - 1 + \frac{R^2 T}{2dT_c}q^2 = t + \frac{R^2q^2}{2d}$$

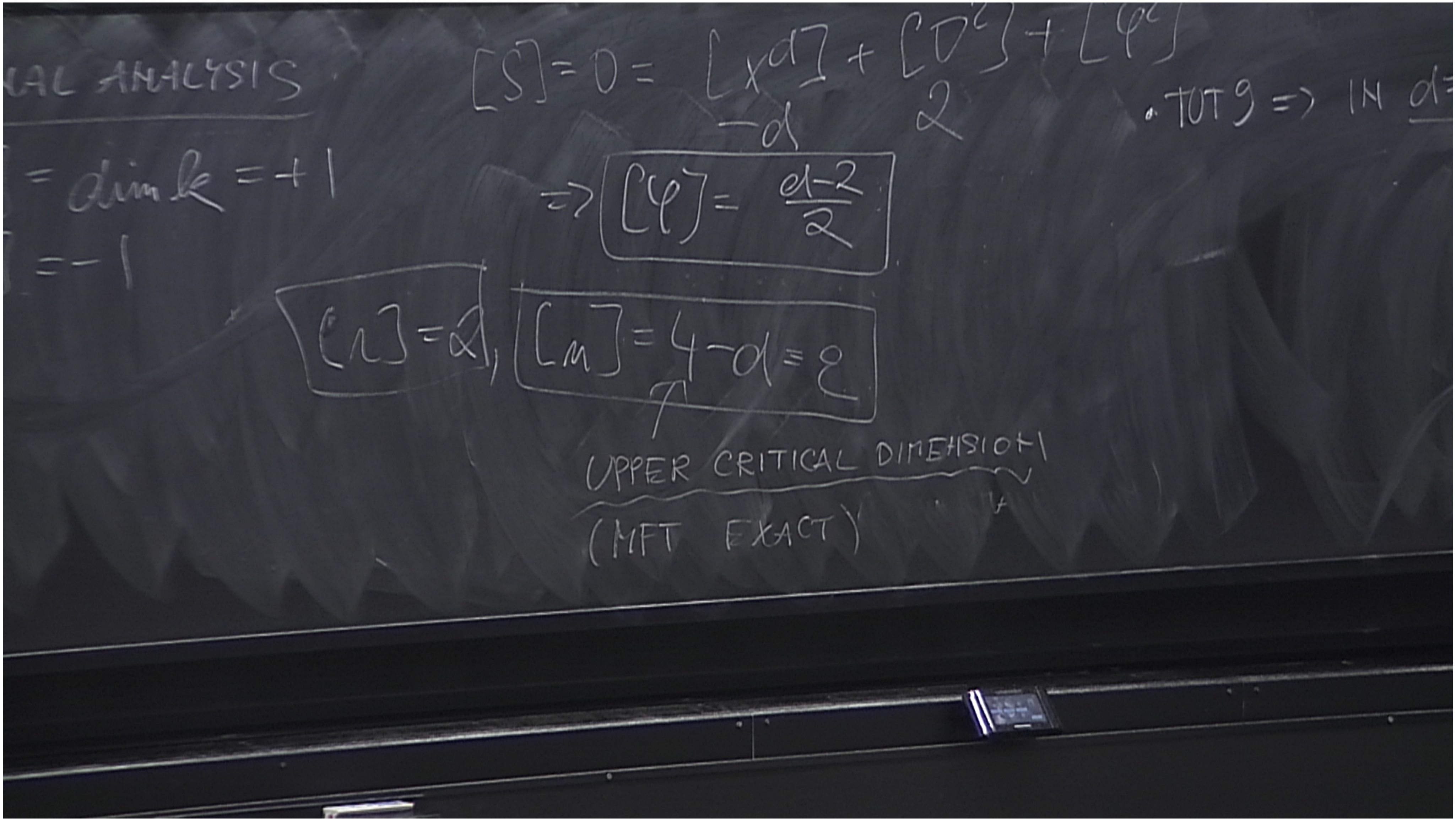
$$\sum_q \varphi(-q)\left(r+q^2\right)\varphi(q) = \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\varphi(-q)\left(r+q^2\right)\varphi(q)$$

- r... CHANGES SIGN AT THE TRANSITION ($r \sim t$)
- $\underline{u > 0}$


• r ... CHANGES SIGN AT THE TRANSITION (r ~ t)
• u > 0 (FOR STABILITY)
• CAN WRITE THIS IMMEDIATELY (
σ → -σ


3) DIMENSIONAL ANALYSIS
[k] = dim k = +1
[x] = -1
[S] = 0 = [x^d] + [∂^2] +
-d
2


NAL ANALYSIS
[S] = 0 =
= dim k = +1
= -1
[φ] = (d-2)/2
[λ] = d, [m] = 4-d = ε


AL ANALYSIS
[S] = 0 = [x^d] + [∂^2] + [φ^2]
-d 2
= dim k = +1
= -1
• TOT 9 => IN d=
=> [φ] = (d-2)/2
[λ] = 2, [u] = 4-d = ε
UPPER CRITICAL DIMENSION
(MFT EXACT)