Title: Conformal defects in gravity - backreacted Dirac delta sources

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Abstract: <p>In this talk I will describe numerical constructions of gravitational
duals of theories deformed by localized Dirac delta sources for scalar
operators. We perform two different constructions, one at zero and
the other at nonzero temperature. Surprisingly we find that imposing the preservation of scale
invariance at zero temperature requires the bulk scalar self-interaction potential to be
the one found in a certain Kaluza-Klein compactification of 11D supergravity.
The gravitational setup seems a-priori to be quite unusual and challenging
from the numerical relativity perspective.</p>

Conformal defects in gravity - backreacted Dirac delta sources

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Outline

- Motivation
- The AdS/CFT setup
- Linearized solution
- Backreacted solution at $T = 0$
- Backreacted solution for $T > 0$
- Entanglement entropy and tests
- The periodic lattice
- Conclusions

Motivation

- In recent years AdS/CFT correspondence has been extensively used to model condensed matter phenomena
- One of the important developments was to break translational invariance of the system — introduce a holographic version of a lattice...
- This was first done by Horowitz, Tong, Santos who introduced a *sinusoidal* source for the field theory:

$$S = S_{CFT} + A_0 \int d^3x \sin kx_1 \, \mathcal{O}(x)$$

- Our aim was to use a strictly localized source with a Dirac Delta shape to model defects/crystaline lattice
- In this talk I will concentrate on a single Dirac delta source:

$$S = S_{CFT} + A_0 \int d^3x \, \delta(x_1) \, \mathcal{O}(x)$$

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Problem:

Develop techniques to numerically solve Einstein's equations (with matter fields) with b.c. $\phi(x^\mu, z) \to z\,\delta(x) + z^2 \ldots$

What is known for discontinuous boundary conditions

- $\theta(x)$ with $m^2 = 0$: Janus solutions (analytical, 1D ODE)
- $\theta(x)$ with $m^2 < 0$ at $T \neq 0$: Janus BH – numerical PDE and analytical in 2+1 dimensions
- $\delta(x)$ with $m^2 = -2$ and additonal p-form fields – uses SUSY: analytical, scale invariant

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The AdS/CFT setup – linearized analysis

- What does it mean to have

$$\phi_{lin}(x,z) \sim z\,\delta(x) + \dots$$

- Construct first linearized modes with e^{ikx} source...

$$\phi_{lin,k}(x,z) \sim z\,e^{ikx} + \dots \quad \longrightarrow \quad z\,e^{-|k|z}\cdot e^{ikx}$$

- We get the solution by straightfoward integration

$$\phi_{lin}(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\phi_{lin,k}(x,z) = \frac{z^2}{\pi(x^2+z^2)} = z\cdot\frac{z}{\pi(x^2+z^2)}$$

- Note that this solution is **scale-invariant**
- Make a coordinate change $z = r\cos\alpha$, $x = r\sin\alpha$. Then

$$\phi_{lin}(r,\alpha) = \frac{1}{\pi}\cos^2\alpha$$

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Backreacted solution at $T = 0$

- We expect(would like?) to have a scale invariant backreacted solution
- We adopt an AdS_3 slicing (almost identical to the one used for Janus solutions)

$$ds^2 = \frac{1}{A(\alpha)^2}\left(\frac{d\alpha^2}{p^2} + \frac{-dt^2 + dy^2 + dr^2}{r^2}\right) \qquad \phi = \phi(\alpha)$$

- p is a constant which ensures that the AdS boundary is always at $\alpha = \pm\pi/2$ even for nontrivial $\phi(\alpha)$

$$A\left(\pm\frac{\pi}{2}\right) = 0 \quad \longrightarrow \quad \text{fixes } p$$

- We look for a perturbative solution

$$A(\alpha) = \sum_{n=0}^{\infty} A_n(\alpha)\,\epsilon^{2n} \qquad \phi(\alpha) = \sum_{n=0}^{\infty} f_n(\alpha)\,\epsilon^{2n+1} \qquad p = \sum_{n=0}^{\infty} p_n\,\epsilon^{2n},$$

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- The 0^{th} order solution is

$$A_0(\alpha) = \cos\alpha \quad p_0 = 1 \quad f_0(\alpha) = \cos\alpha$$

- Impose reflection symmetry around $x = 0$:

$$\partial_x A_n(0) = 0 \qquad \partial_x f_n(0) = 0$$

- Impose $f_n(0) = 0$ which sets the perturbation expansion parameter equal to the physical amplitude $\epsilon \equiv \phi(0)$
- Set the AdS boundary $A_n(\pi/2) = 0$
- For any given choice of scalar potential $V(\phi)$ we get a unique solution...

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Surprise:

Generically we find that

$$f_n'\left(\frac{\pi}{2}\right) \neq 0 \qquad \text{for } n > 0$$

- Translating back to Fefferman-Graham coordinates close to the boundary this means that the source for the scalar field is modified to

$$\varepsilon\,\delta(x) + (\varepsilon^3 + \ldots)\,\frac{1}{|x|}$$

at *generically* a localized source is **inconsistent** with
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$$\varepsilon\,\delta(x) + (\varepsilon^3 + \ldots)\,\frac{1}{|x|}$$
- This means that *generically* a localized source is **inconsistent** with scale invariance!!

Backreacted solution at $T = 0$

The above conclusions remain true beyond perturbation theory...

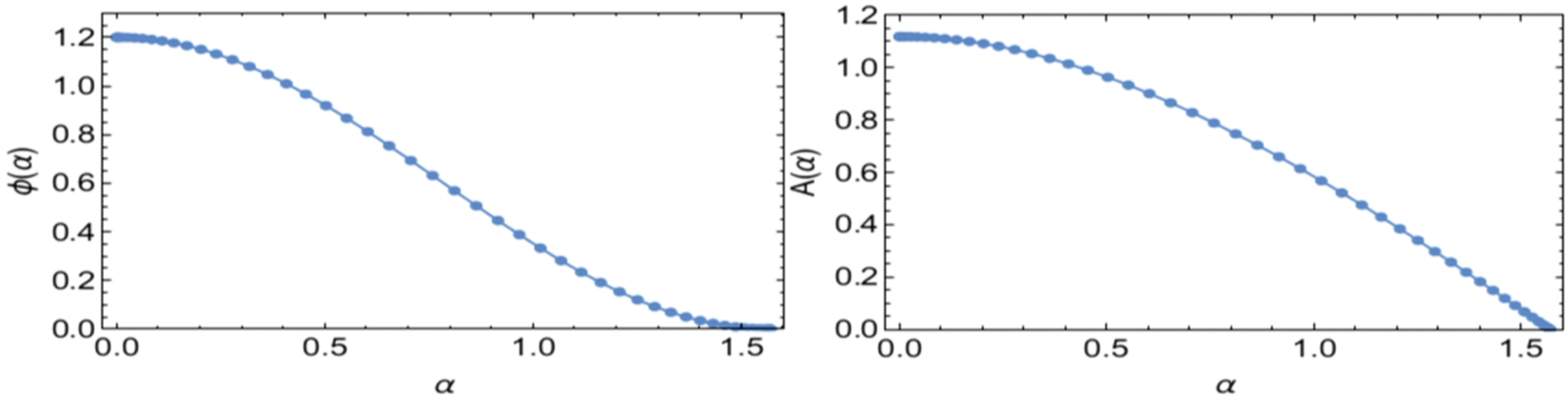

Figure : Metric and scalar field for $\phi(0) = 1.2$. Numerical solution (points) with $N = 47$ spectral grid. Lines correspond to fourth order perturbative solution.

How to interpret this result?

- Recall the distinction between marginal and exactly marginal operators/deformations..

$$S_\lambda = S_{CFT} + \lambda \int d^D x\, \mathcal{O}(x)$$

- $\mathcal{O}(x)$ is marginal if $\Delta = D$, then scale invariance is preserved at linear order in λ
- If $\mathcal{O}(x)$ is exactly marginal, then S_λ defines a CFT for any λ which remains dimensionless $\longleftarrow$ quite rare
- Otherwise a mass scale is dynamically generated...

- Here we have an analogous situation but with an additional Dirac delta function:

$$S_\lambda = S_{CFT} + \lambda \int d^D x\, \delta(x_1)\, \mathcal{O}(x)$$

- We have Dirac delta exact marginality when the bulk theory is supersymmetric...

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Proceed to nonzero temperature...

- From now on we will deal exclusively with the supersymmetric potential

$$V(\phi) = -6\cosh\left(\frac{\phi}{\sqrt{3}}\right)$$

- Much more difficult: nonlinear PDE's
- We no longer have a chance of an analytical perturbative expansion...
- We should use some of the lessons from the $T = 0$ case:
  - we should somehow blow up the infinitesimal neighbourhood of $x = z = 0$
  - the coordinate system used there is not convenient as it stands due to the horizon which we want to keep at the edge of some rectangular grid

Backreacted solution for $T > 0$

- We will use the DeTurck method — DeTurck; Wiseman
- Define

$$G_{ab} \equiv R_{ab} - \frac{1}{2}\left(\nabla_a\phi\nabla_b\phi + V(\phi)g_{ab}\right)$$

- $G_{ab} = 0$ is not elliptic
- Difficult to make a good coordinate choice...
- Keep the metric coefficients completely general but solve instead

$$G_{ab} - \nabla_{(a}\xi_{b)} = 0 \qquad \nabla_a\nabla^a\phi - \frac{dV}{d\phi} = 0$$

where

$$\xi^a = g^{cd}[\Gamma^a_{cd} - \bar{\Gamma}^a_{cd}]$$

and $\bar{\Gamma}(\bar{g})$ are Christoffel symbols of some fixed given reference metric

- Check *a-posteriori* that $\xi^a = 0$
- One can interpret $\xi^a = 0$ as a dynamically chosen implicit coordinate choice

Backreacted solution for $T > 0$

Apparent problem:

1. The DeTurck method requires a completely general ansatz for the spacetime metric
2. The final choice of coordinates $\xi^a = 0$ is very much implicit (given in terms of Christoffel symbols Γ^a_{bc})
   — these are differential conditions for the metric coefficient functions...

Question: How to adapt coordinates to the Delta function at the boundary?

Answer: Choose appropriate coordinates in the reference metric

Backreacted solution for $T > 0$

- The reference metric will be the planar black hole (with $T = \frac{3}{4\pi}$)

$$ds^2 = \frac{1}{z^2}\left[-\frac{dt^2}{1-z^3} + dx^2 + dy^2 + \frac{dz^2}{1-z^3}\right]$$

- Close to the AdS boundary the *linearized* scalar field will look like

$$\phi(x,z) \sim \phi_0 \frac{z^2}{x^2+z^2}$$

- We will replace x by

$$\alpha = \arctan \frac{x}{z}$$

  and keep z.
- The behaviour of the scalar field $\phi(x,z)$ for small z will now be perfectly smooth...

Backreacted solution for $T > 0$

- The form of the metric is

$$ds^2 = \frac{1}{z^2}\Bigg[-(1-z)G(z)H_1(\alpha,z)dt^2 + \frac{H_2(\alpha,z)dz^2}{(1-z)G(z)} + S_1(\alpha,z)(dx + F(\alpha,z)dz)^2 + S_2(\alpha,z)dy^2\Bigg],$$

  with $x \equiv z\tan\alpha$ and $G(z) = 1 + z + z^2$

- The scalar field is $\phi(\alpha, z)$
- The numerical grid is a rectangle $\alpha \in [0, \pi/2]$, $z \in [0, 1]$
- The boundary conditions are as follows:
  1. $\alpha = 0$: reflection symmetry
  2. $\alpha = \pi/2$: infinity: $g_{\mu\nu} \to g^{BH}_{\mu\nu}$ and $\phi \to 0$
  3. $z = 1$: $H_1(\alpha, 1) = 1$ (fixed temperature); regularity at the horizon
  4. $z = 0$: **Subtlety:** the AdS boundary is only located at $\alpha = \pi/2$; $\alpha \in (0, \pi/2)$ is an infinitesimal neighbourhood of $x = z = 0$

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Backreacted solution for $T > 0$

- The DeTurck equations at $z = 0$ reduce to ODE in α
- Solve them with $\phi \to \phi(0)$ at $\alpha = 0$ and empty AdS at $\alpha = \pi/2$
- Use the numerical solution as boundary conditions at $z = 0$
- Solve the PDE's on the full recatngular grid
- The ξ^a indeed is zero

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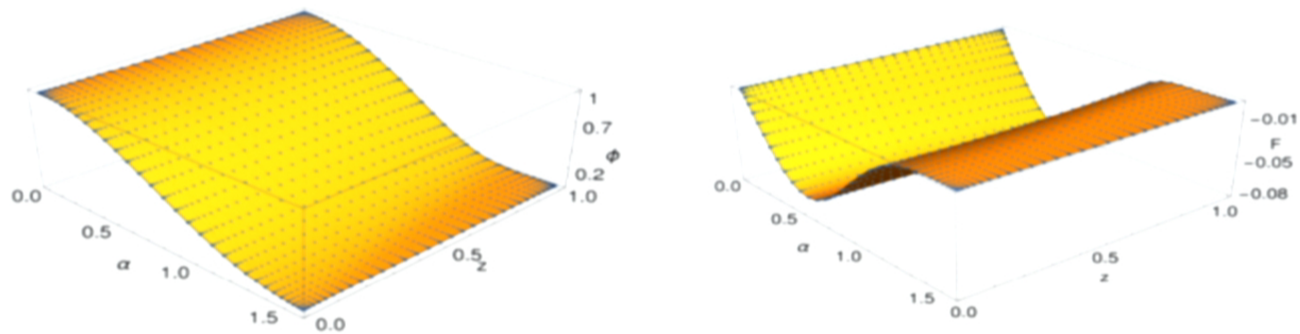

Figure : Metric and scalar field for $\phi(0) = 1.0$. Numerical solution (points) with $N_\alpha = N_z = 35$ spectral grid.

Entanglement entropy and tests

In the $T = 0$ geometry the evaluation of the cutoff requires quite complicated constructions

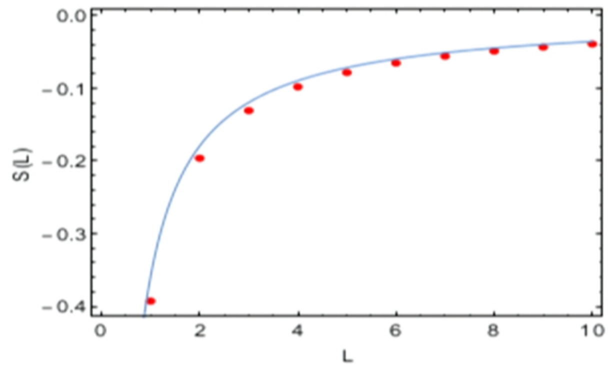

Figure : Regular part of the entanglement entropy for the defect configuration (dots) and for case of empty AdS_4 (line). Here the defect amplitude corresponds to $\phi(0) = 2$.

$$S_{defect} - S_{AdS} \approx \frac{0.0107}{L} \quad \text{for } \phi(0) = 1$$

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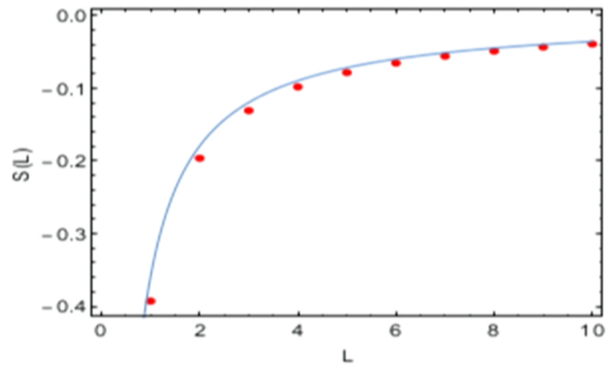

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Entanglement entropy and tests

- The $T > 0$ case requires quite different geometrical considerations
- On dimensional grounds we have

$$S_{def} \equiv S_{\text{defect}} - S_{\text{AdS}} = \frac{B(LT)}{L}$$

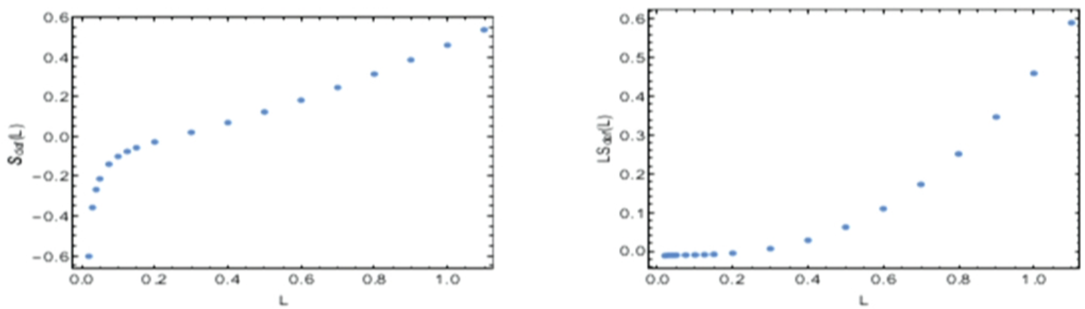


Figure : Entanglement entropy for a Dirac delta defect with $\phi(0) = 1.0$ at finite temperature with $N = 50$ spectral grid.

- When $L \to 0$, $B(TL)$ should reduce to the $T = 0$ answer. Indeed we get

$$B(0) = -0.0107$$

The periodic lattice

Work in progress

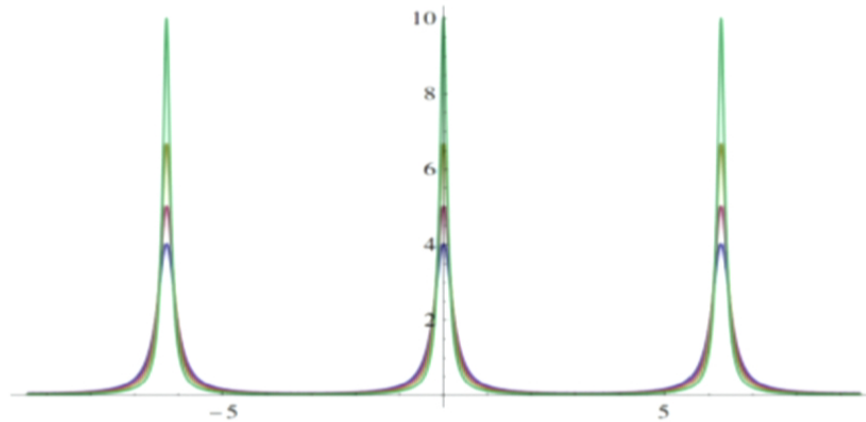
- We started looking at the periodic case i.e.

$$\delta(x) \longrightarrow \sum_{n=-\infty}^{\infty} \delta(x - 2n\pi)$$

- For the moment we used a regularized periodic source

$$\frac{1}{2\pi} \frac{\sinh(z + \varepsilon)}{\cosh(z + \varepsilon) - \cos x}$$

- Huge gradients — we need to adapt an appropriate numerical grid
- Exact Dirac delta sources should be easy to incorporate



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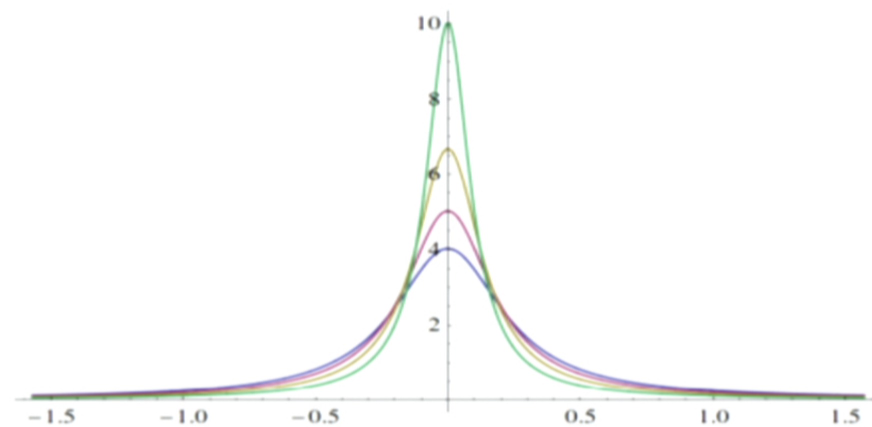
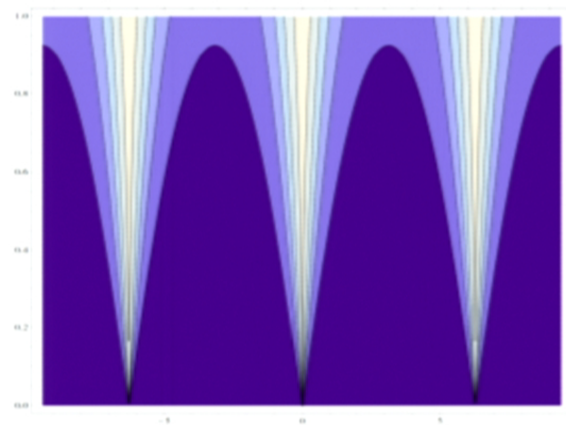


The periodic lattice

- The linearized solution (at $T = 0$) can be constructed analytically

$$\phi_{lin}(x, z) = \frac{1}{2\pi} \frac{z \sinh z}{\cosh z - \cos x}$$

- We used an adapted lattice (periodic generalization of the α coordinate)
- We constructed the background with nonzero ... chemical



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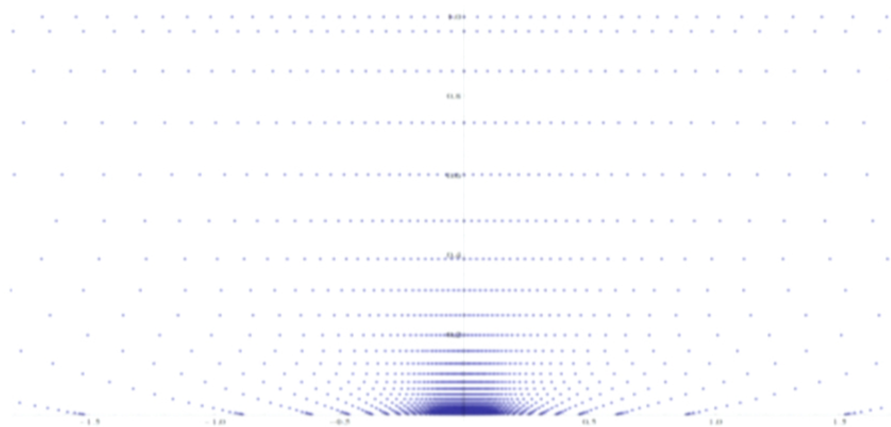
- ...



Open question:

What happens in the pure massive case without self-interaction?

- It is not clear whether the backreacted Dirac delta solution exists...
- We could not find so far a consistent scaling close to the defect...

Conclusions

- We developed techniques to construct numerically gravitational solutions with a Dirac delta asymptotics at the boundary ($\equiv$ this corresponds to a Dirac delta source in the dual field theory)
- We found that a scale invariant defect requires a very specific form of scalar self-interaction:

$$V(\phi) = -6\cosh\left(\frac{\phi}{\sqrt{3}}\right)$$

- This is exactly the scalar potential appearing in a KK reduction of supergravity
- Generalized notions of defect exact marginality?
- We used DeTurck method with an adapted choice of reference metric to construct the solution at nonzero temperature
- We computed entanglement entropy and did some cross-checks
- Interesting to study a lattice and compute optical conductivity...
- The nonconformal case remains quite mysterious...