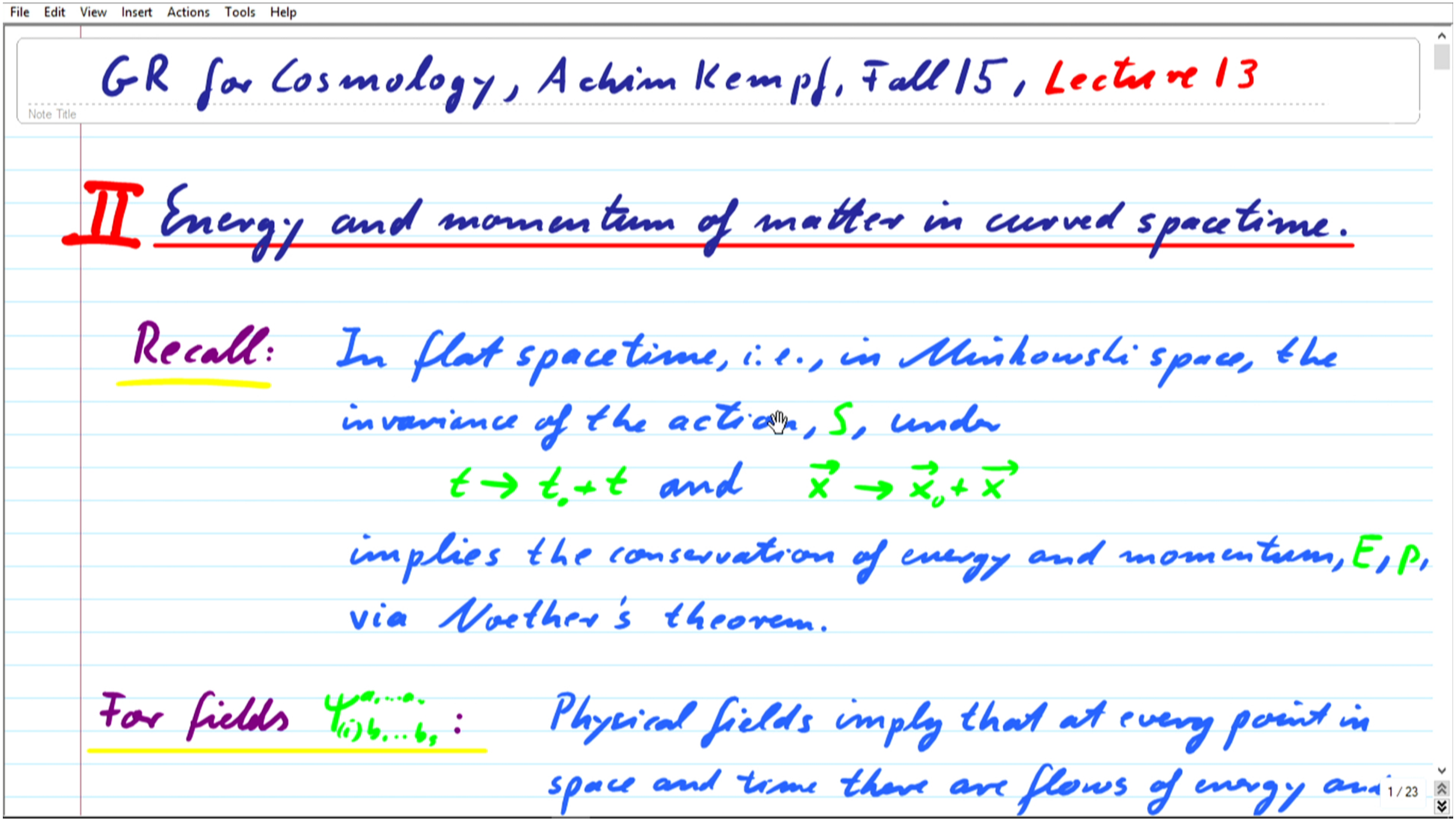
Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 13

Date: Oct 26, 2015 03:00 PM

URL: http://pirsa.org/15100068

Abstract:



GR for Cosmology, Achim Kempf, Fall 15, Lecture 13

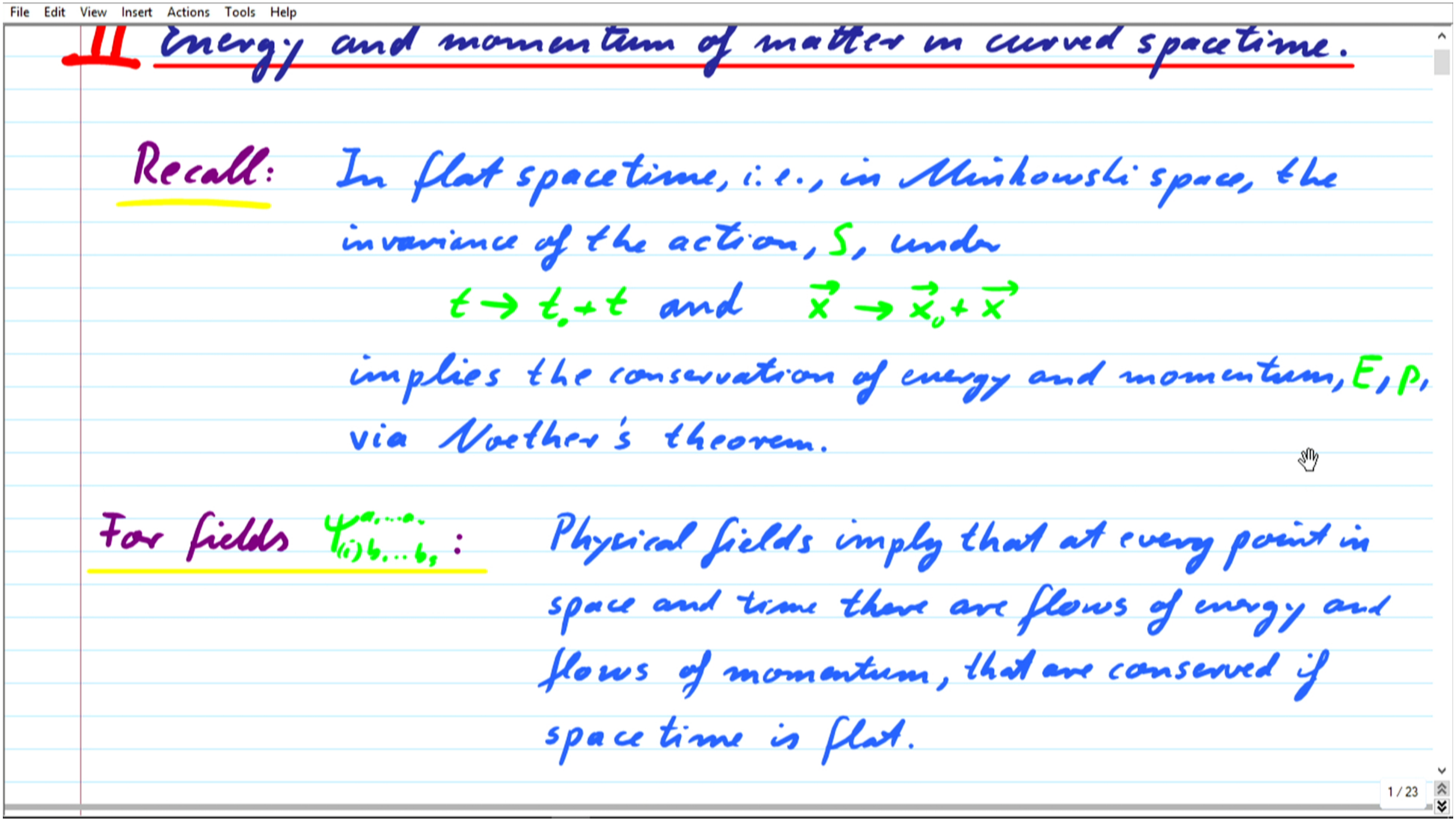
II Energy and momentum of matter in curved spacetime.

Recall: In flat spacetime, i.e., in Minkowski space, the invariance of the action, S, under

$$t \rightarrow t_0 + t \quad \text{and} \quad \vec{x} \rightarrow \vec{x}_0 + \vec{x}$$

implies the conservation of energy and momentum, E, p, via Noether's theorem.

For fields $\psi^{a_1 \dots a_r}_{(i)\, b_1 \dots b_s}$: Physical fields imply that at every point in space and time there are flows of energy an



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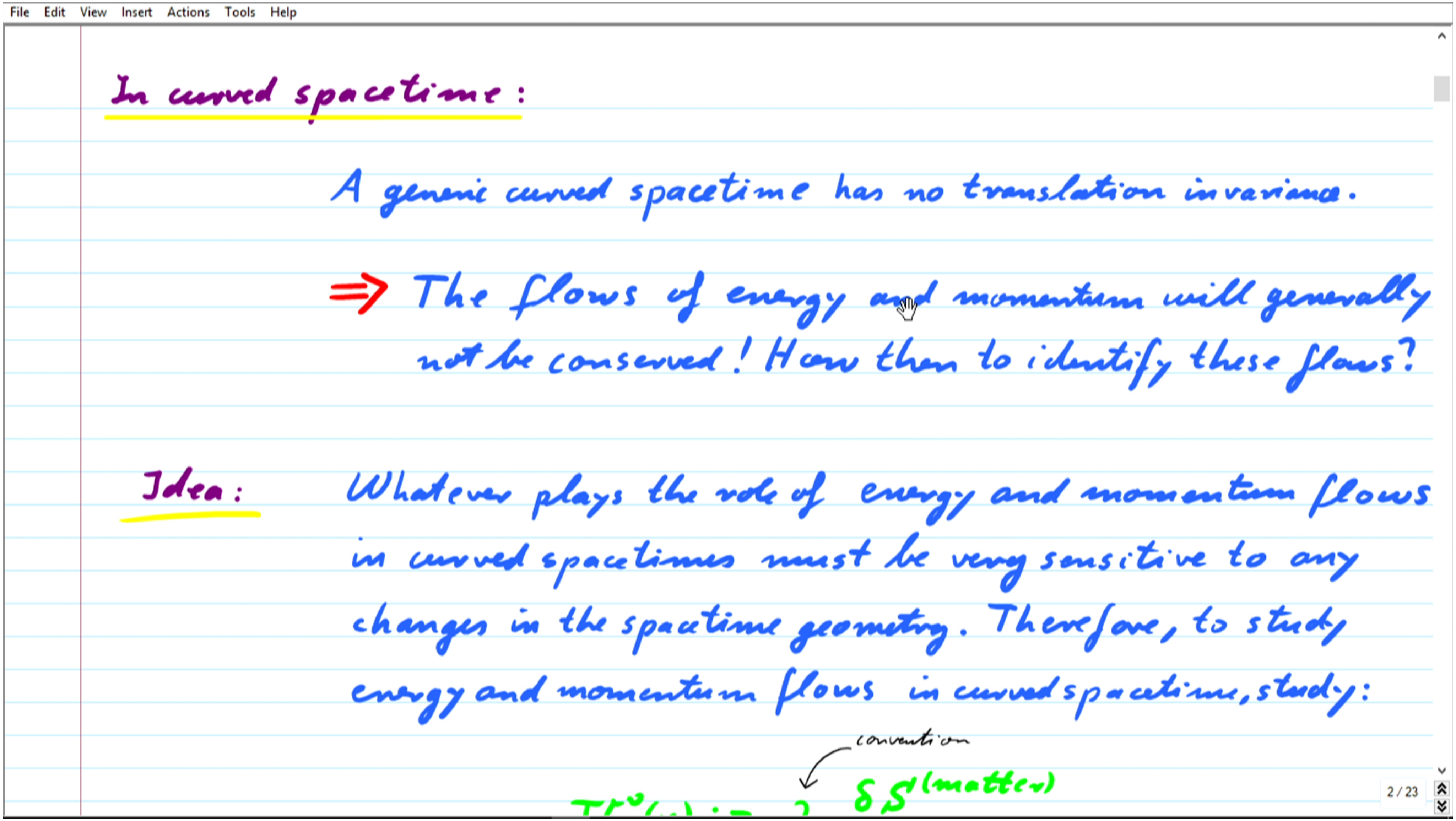


In curved spacetime:

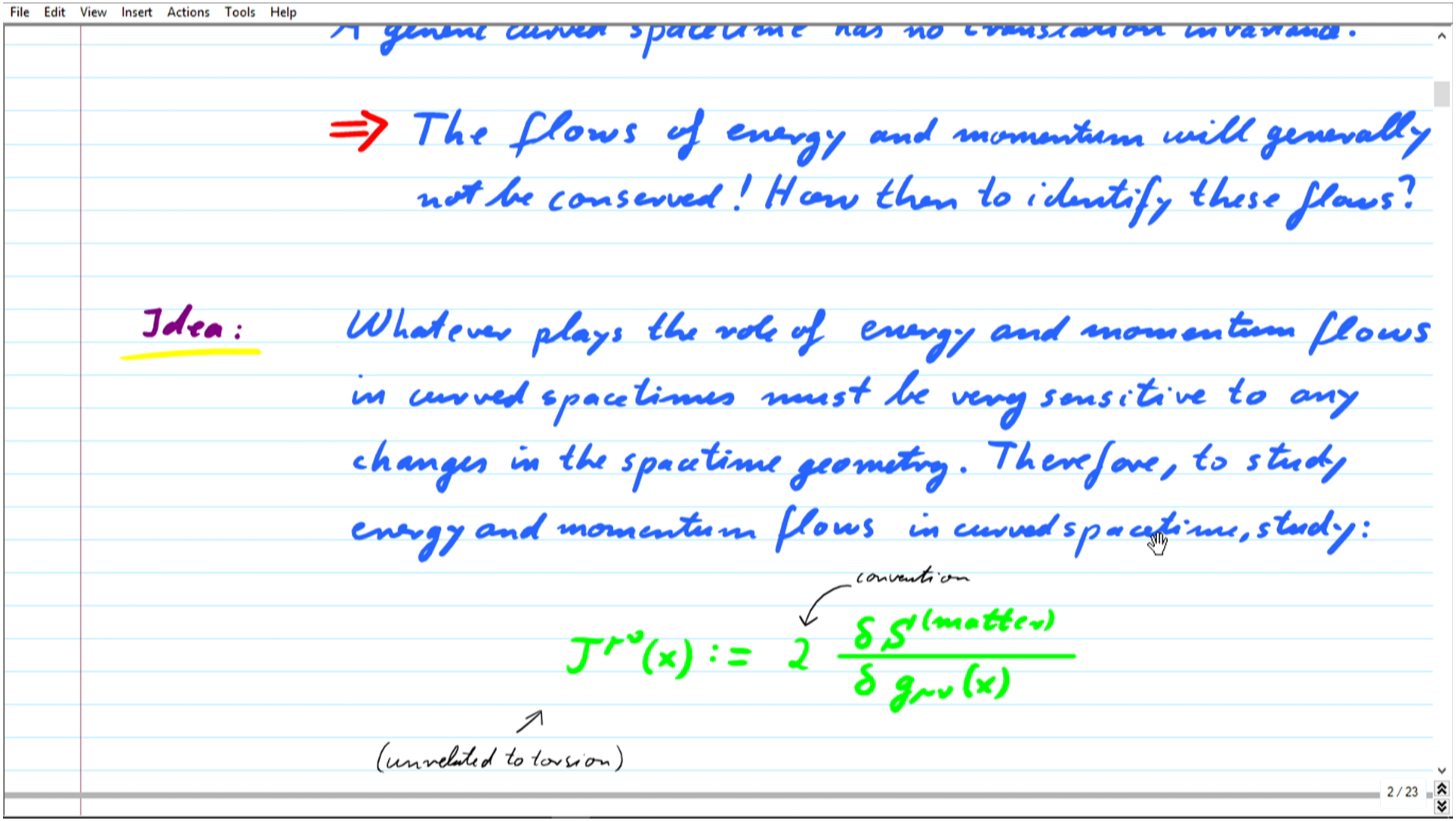
A generic curved spacetime has no translation invariance.

⇒ The flows of energy and momentum will generally not be conserved! How then to identify these flows?

Idea: Whatever plays the role of energy and momentum flows in curved spacetimes must be very sensitive to any changes in the spacetime geometry. Therefore, to study energy and momentum flows in curved spacetime, study:

convention

$$T^{\mu\nu}(x) := 2 \; \delta S^{(matter)}$$



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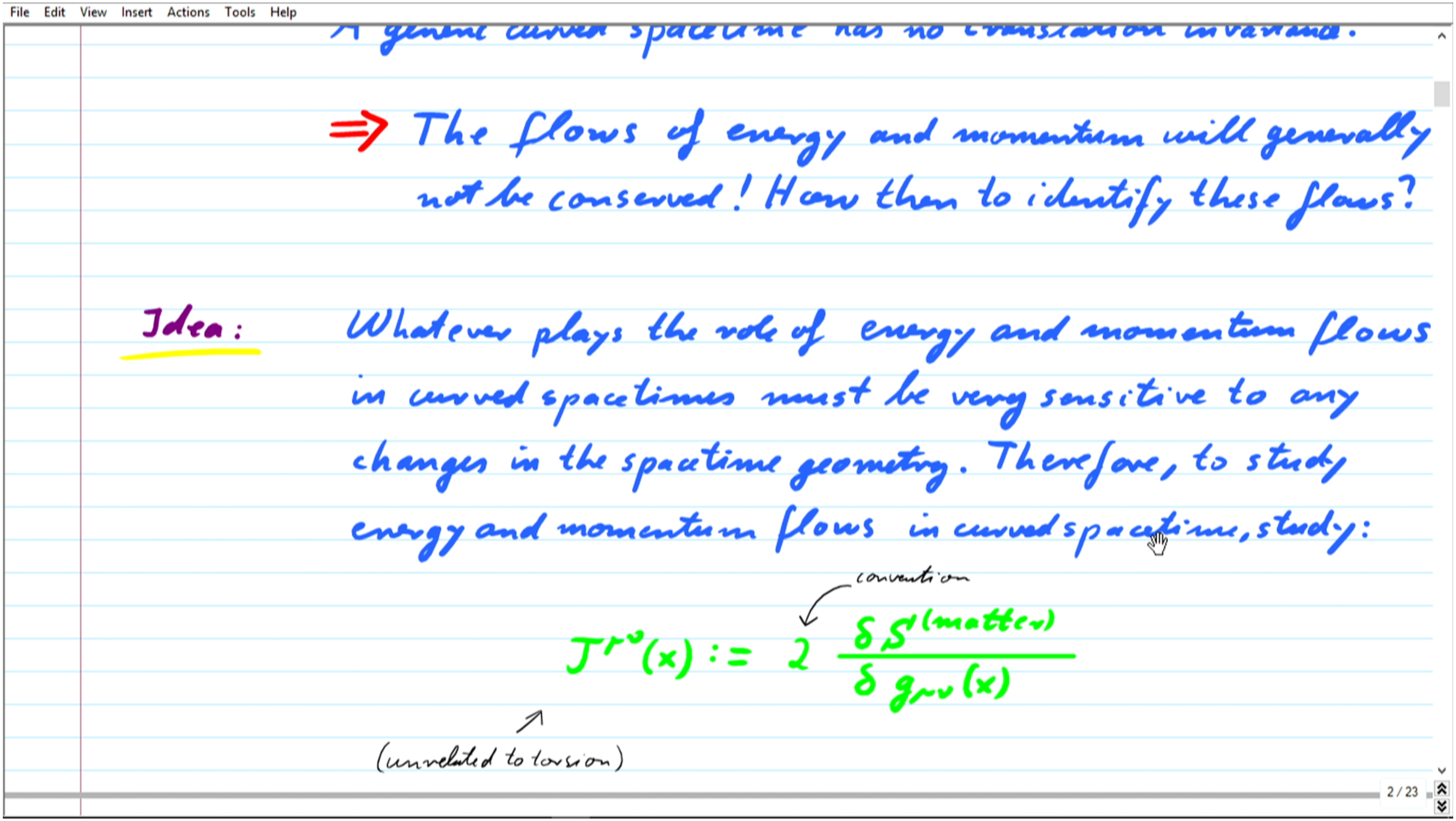
(unrelated to torsion)



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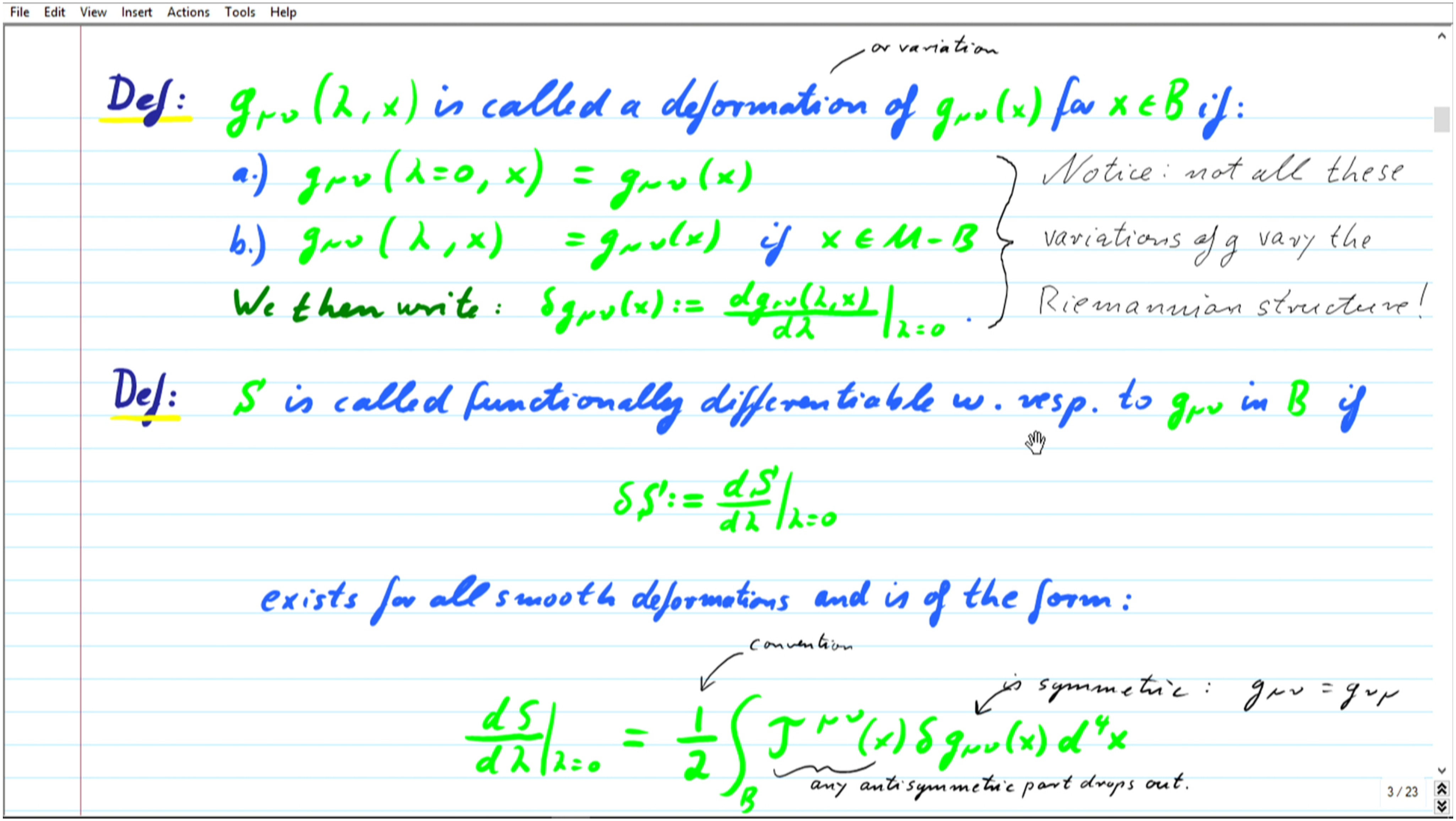

or variation

Def: $g_{\mu\nu}(\lambda,x)$ is called a deformation of $g_{\mu\nu}(x)$ for $x\in B$ if:

a.) $g_{\mu\nu}(\lambda=0,x) = g_{\mu\nu}(x)$

b.) $g_{\mu\nu}(\lambda,x) = g_{\mu\nu}(x)$ if $x \in \mathcal{M}-B$

We then write: $\delta g_{\mu\nu}(x) := \frac{d g_{\mu\nu}(\lambda,x)}{d\lambda}\Big|_{\lambda=0}$.

Notice: not all these variations of g vary the Riemannian structure!

Def: S is called functionally differentiable w. resp. to $g_{\mu\nu}$ in B if

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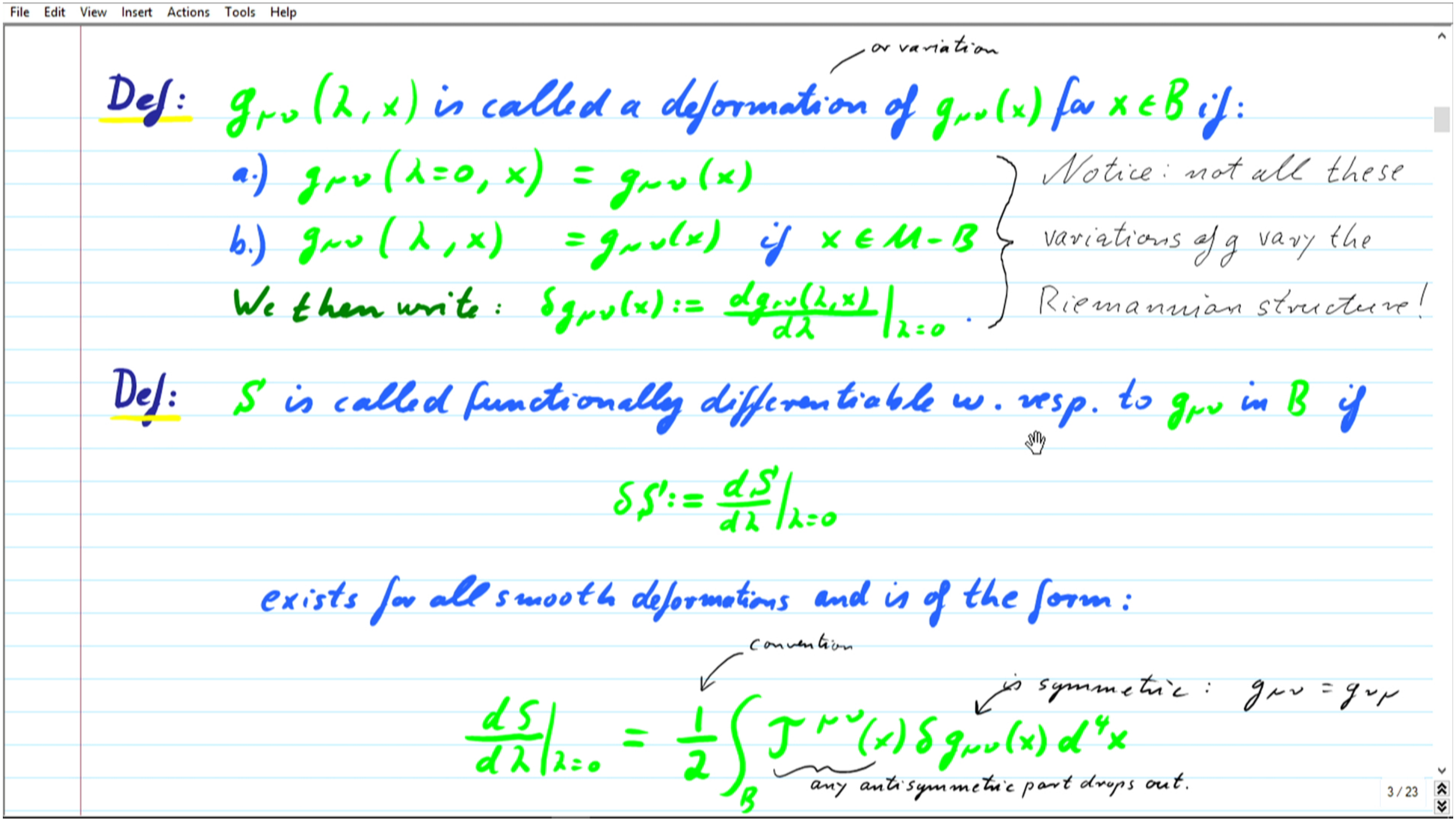

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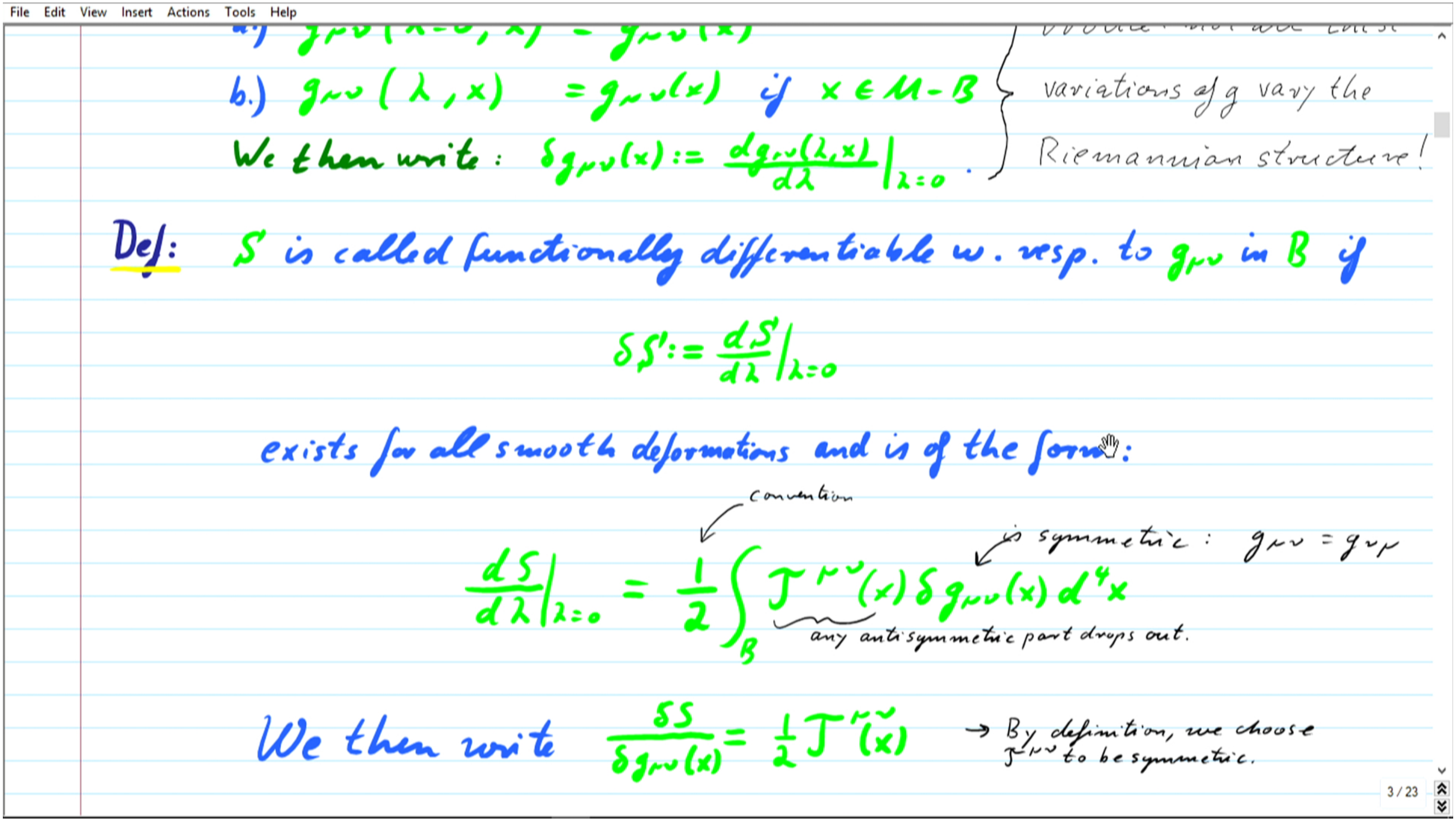

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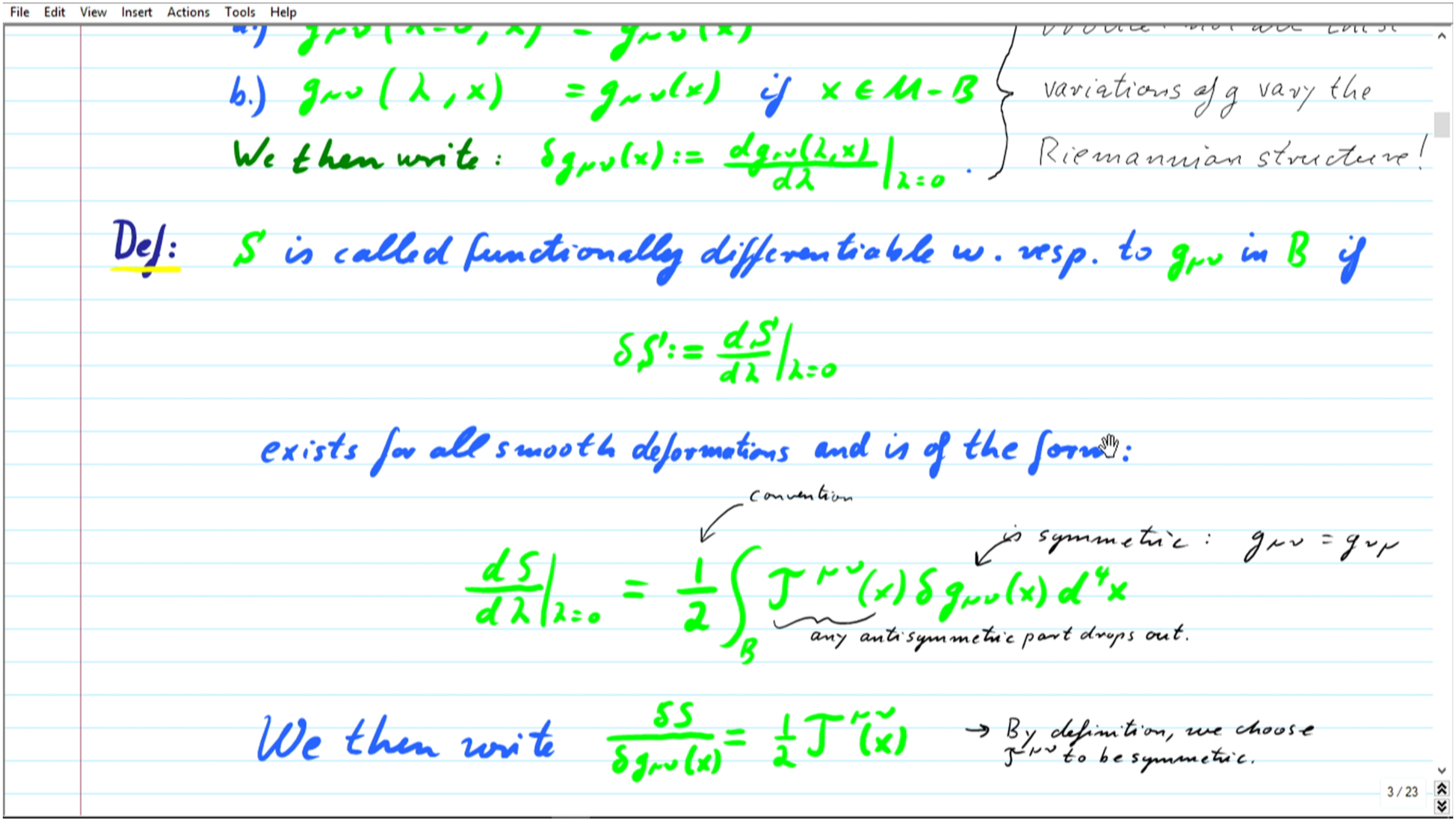

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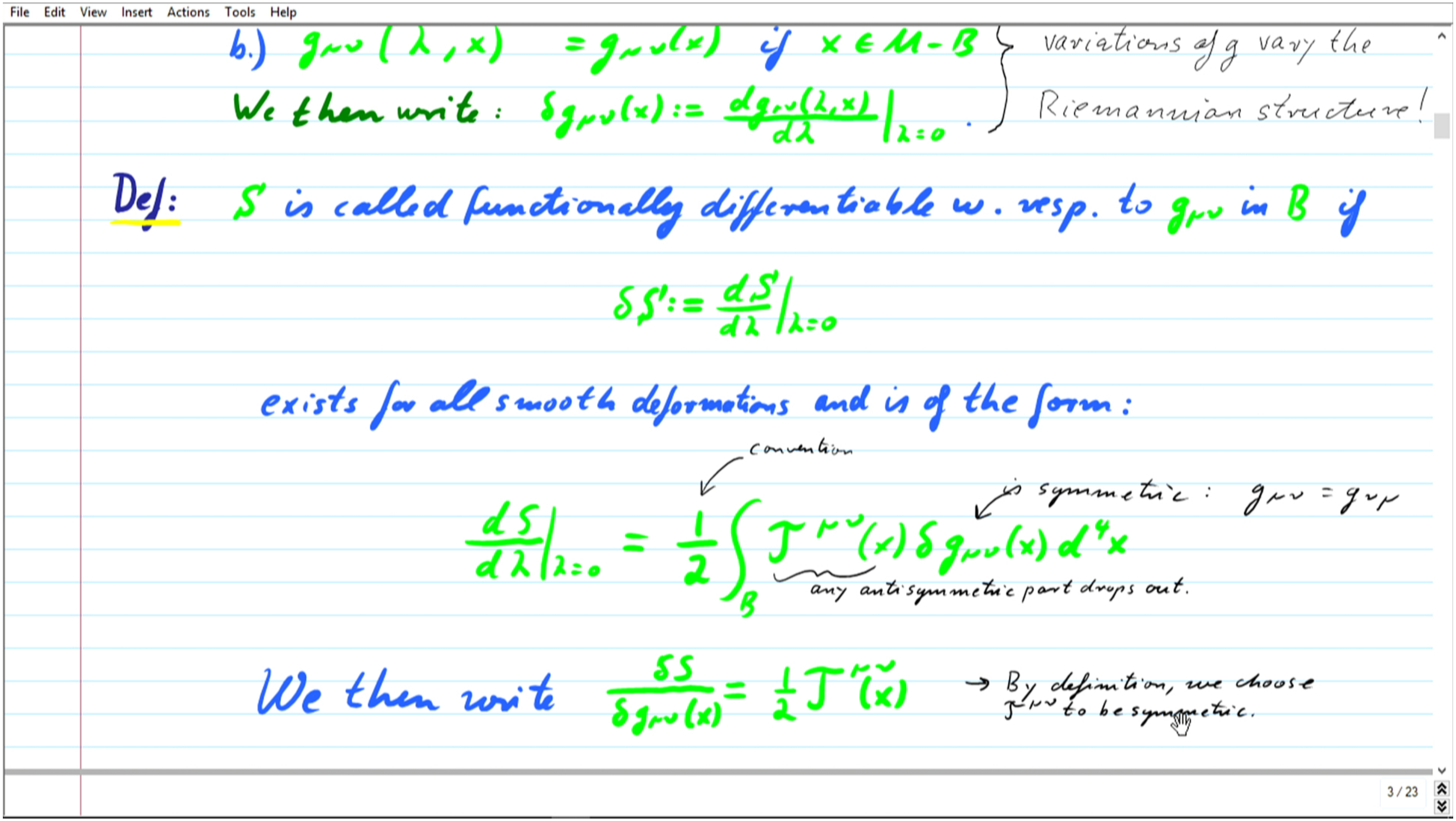

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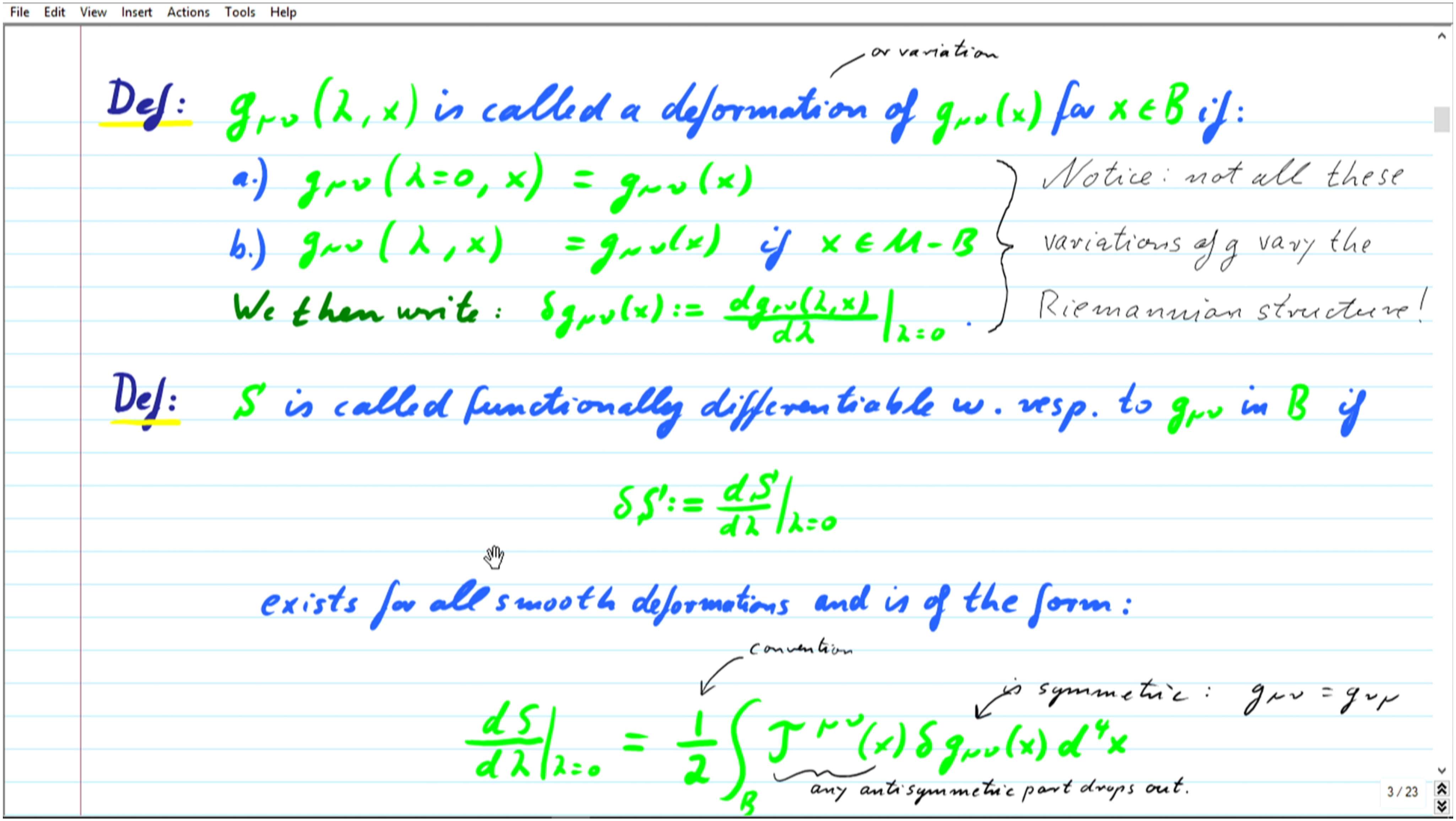

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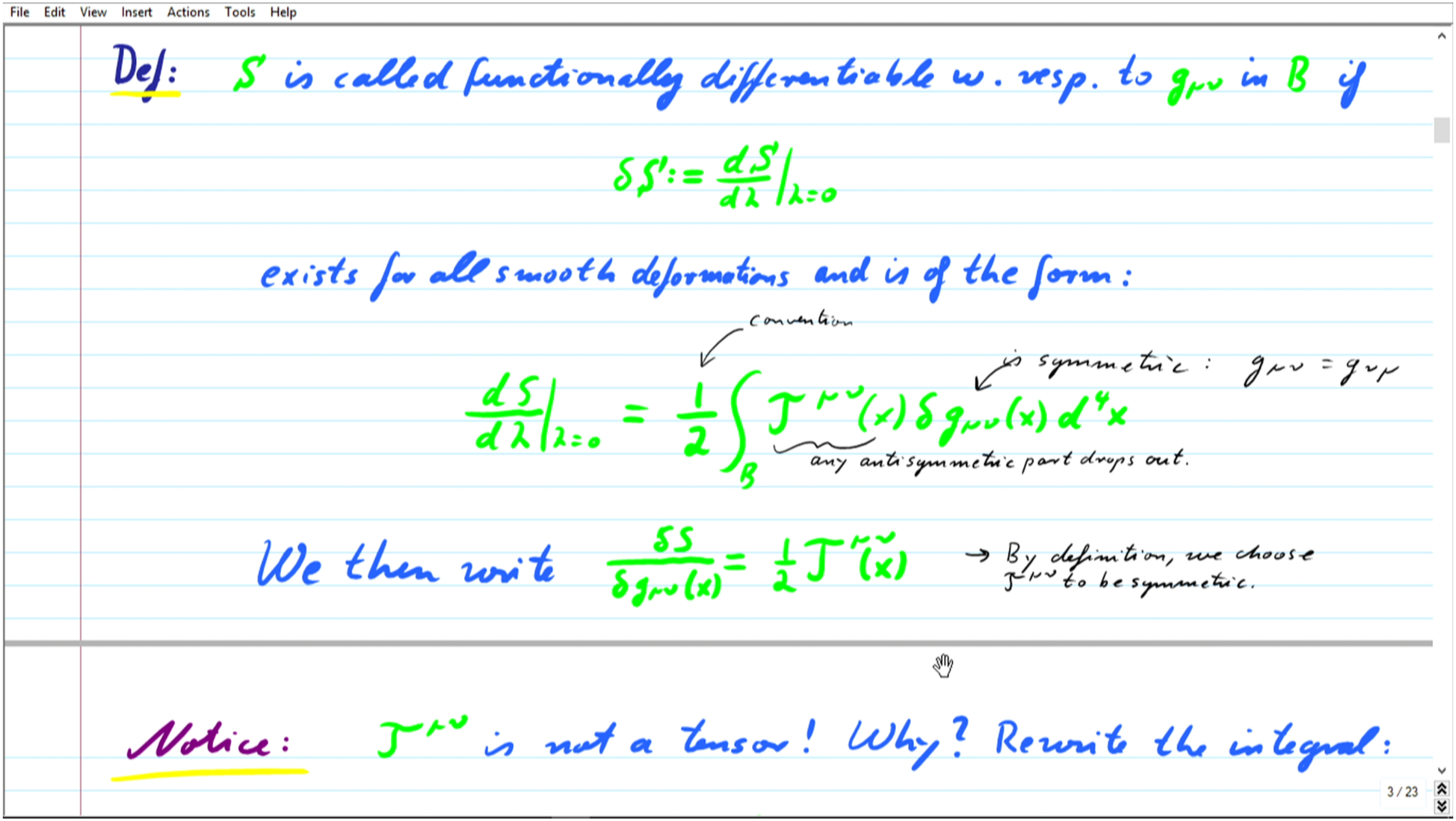

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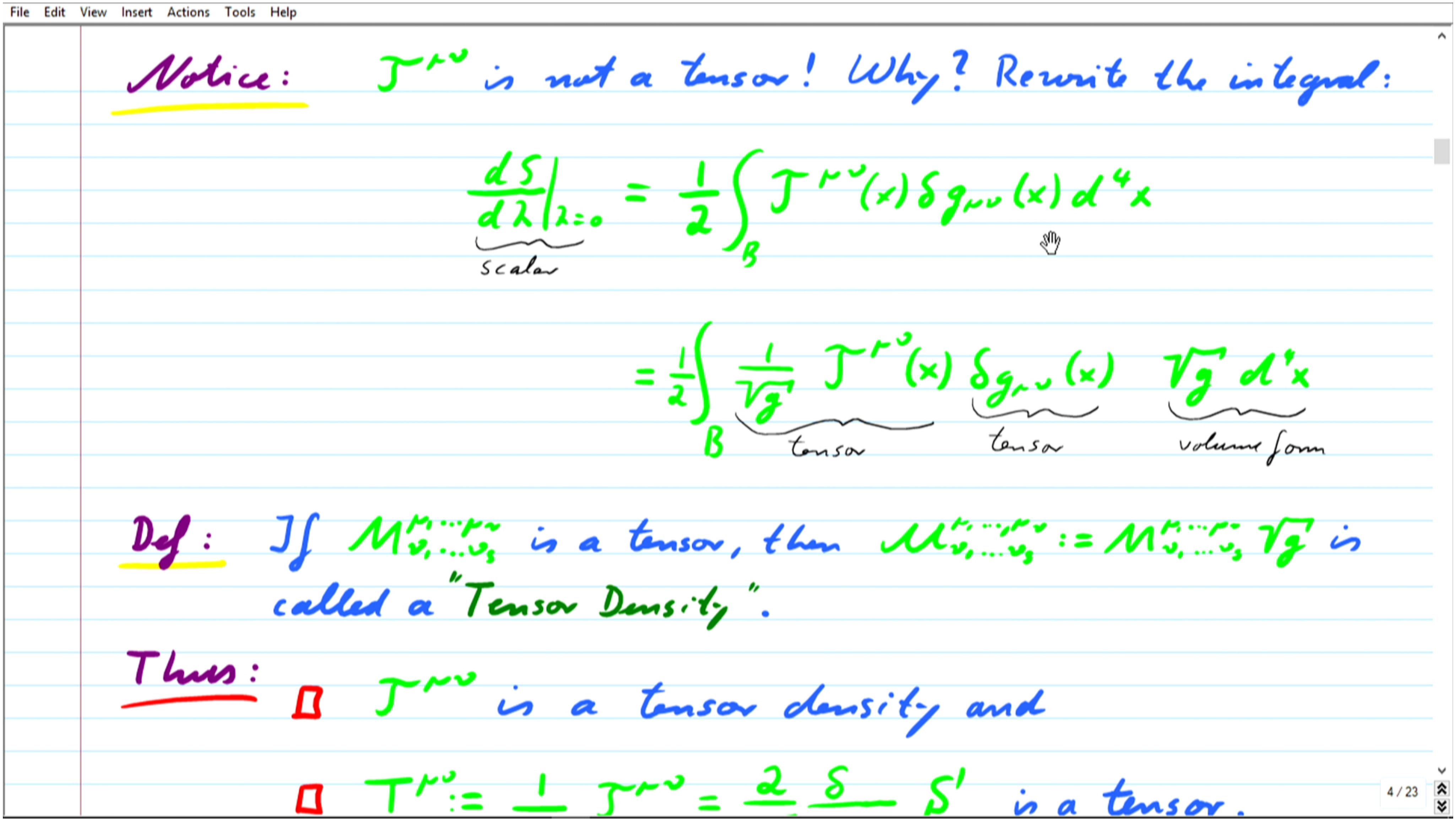
Notice: $T^{\mu\nu}$ is not a tensor! Why? Rewrite the integral:



Notice: $J^{\mu\nu}$ is not a tensor! Why? Rewrite the integral:

$$\underbrace{\frac{dS}{d\lambda}\Big|_{\lambda=0}}_{\text{scalar}} = \frac{1}{2}\int_B J^{\mu\nu}(x)\,\delta g_{\mu\nu}(x)\,d^4x$$

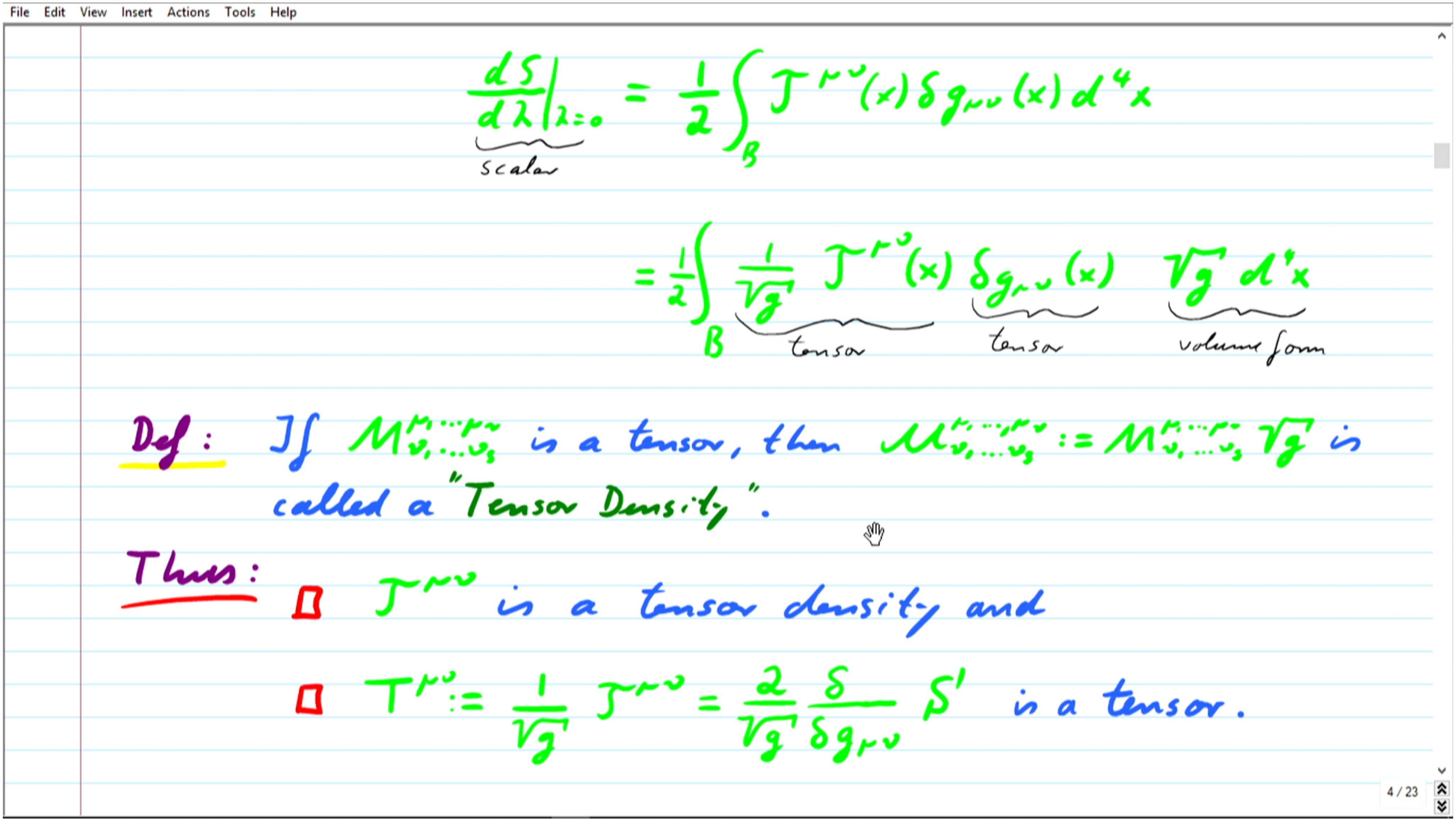
$$= \frac{1}{2}\int_B \underbrace{\frac{1}{\sqrt{g}}\,J^{\mu\nu}(x)}_{\text{tensor}}\;\underbrace{\delta g_{\mu\nu}(x)}_{\text{tensor}}\;\underbrace{\sqrt{g}\,d^4x}_{\text{volume form}}$$

Def: If $M^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}$ is a tensor, then $\mathcal{M}^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s} := M^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}\sqrt{g}$ is called a "Tensor Density".

Thus:

☐ $J^{\mu\nu}$ is a tensor density and

☐ $T^{\mu\nu} := \frac{1}{\;}\,J^{\mu\nu} = \frac{2}{\;}\,\frac{\delta}{\;}\,S'$ is a tensor.



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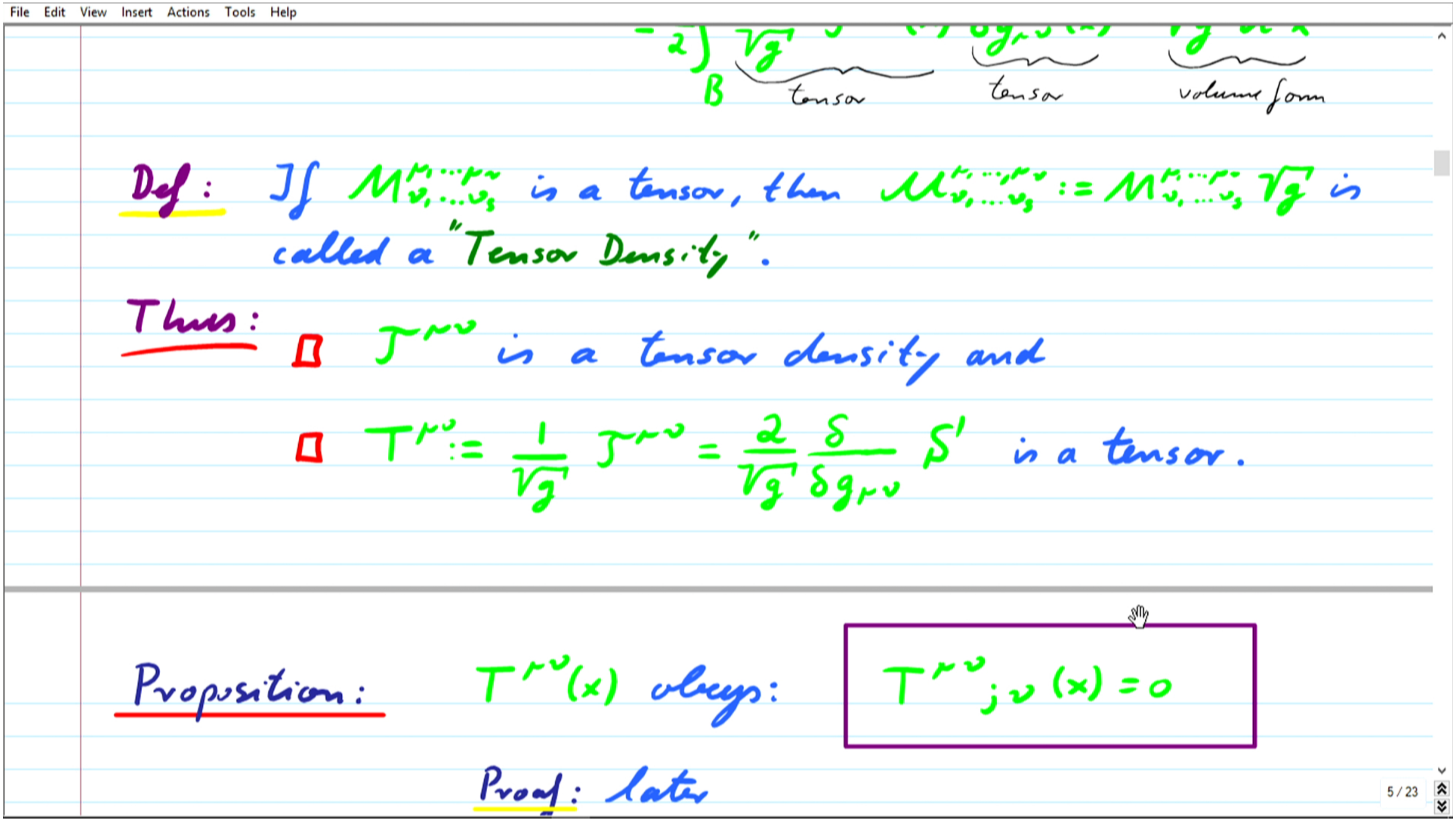

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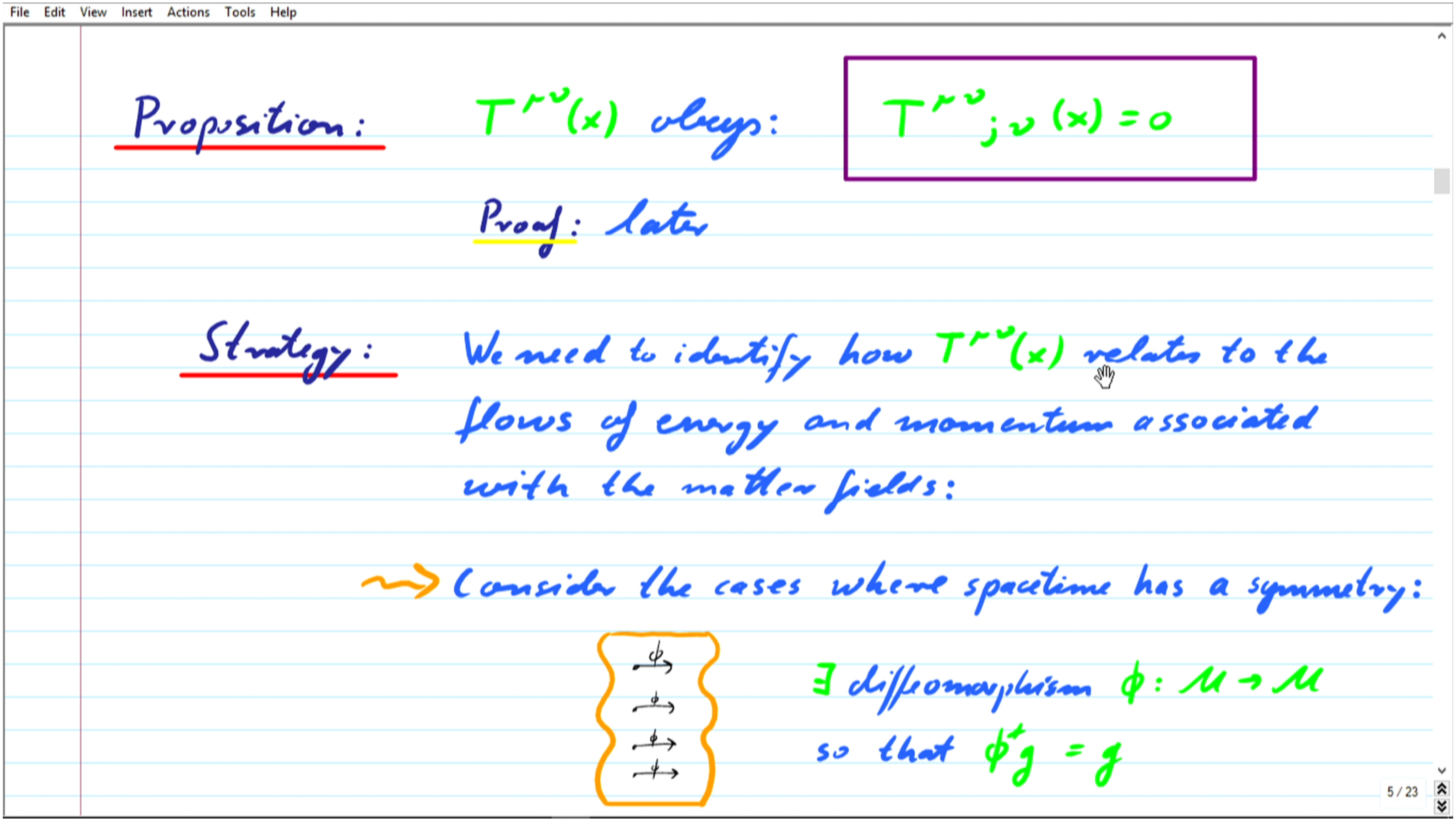
Proposition: $T^{\mu\nu}(x)$ obeys: $\boxed{T^{\mu\nu}{}_{;\nu}(x) = 0}$

Proof: later



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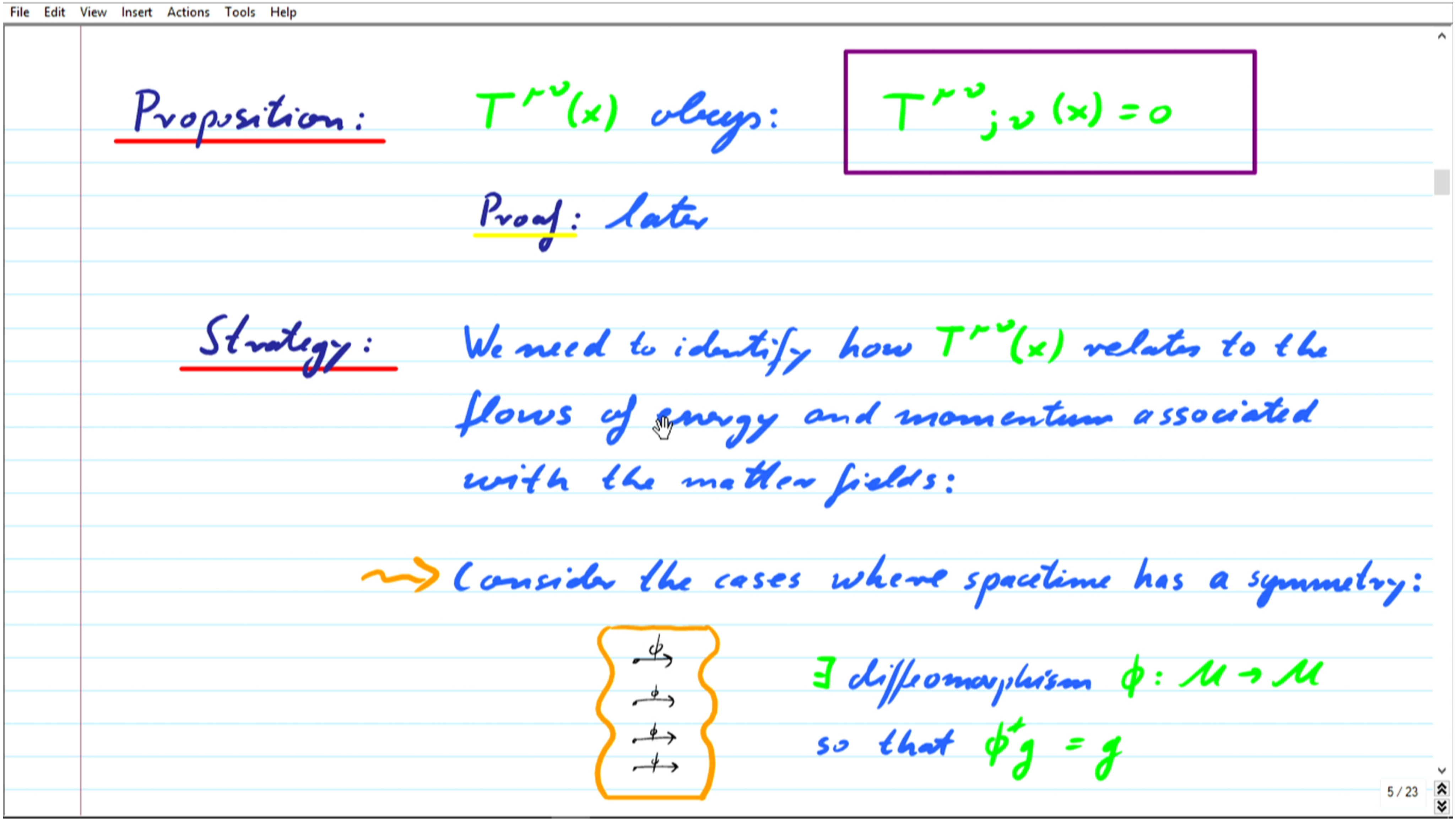
Strategy: We need to identify how $T^{\mu\nu}(x)$ relates to the flows of energy and momentum associated with the matter fields:

⇝ Consider the cases where spacetime has a symmetry:

$\exists$ diffeomorphism $\phi: \mathcal{M} \to \mathcal{M}$ so that $\phi^* g = g$



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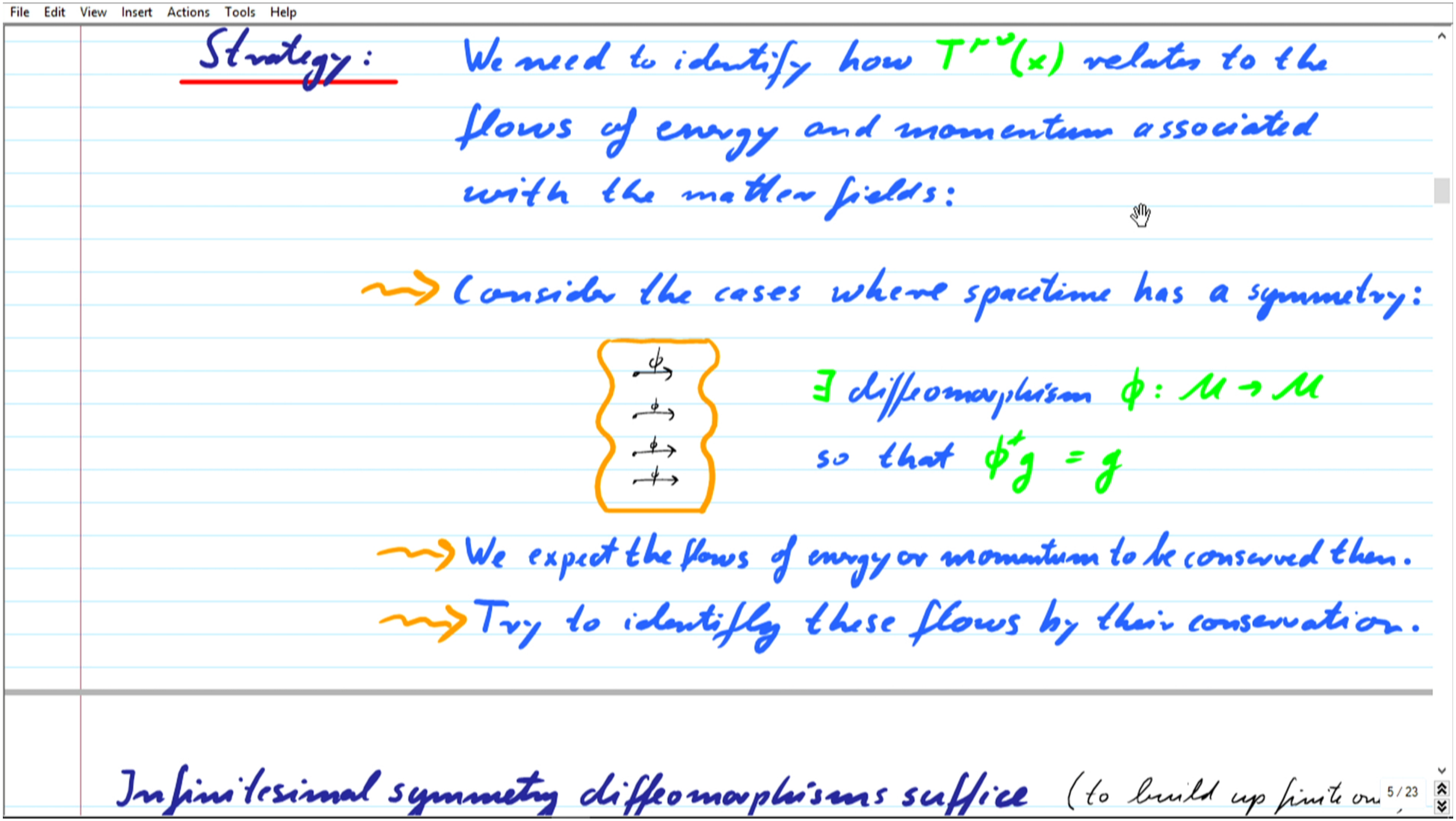

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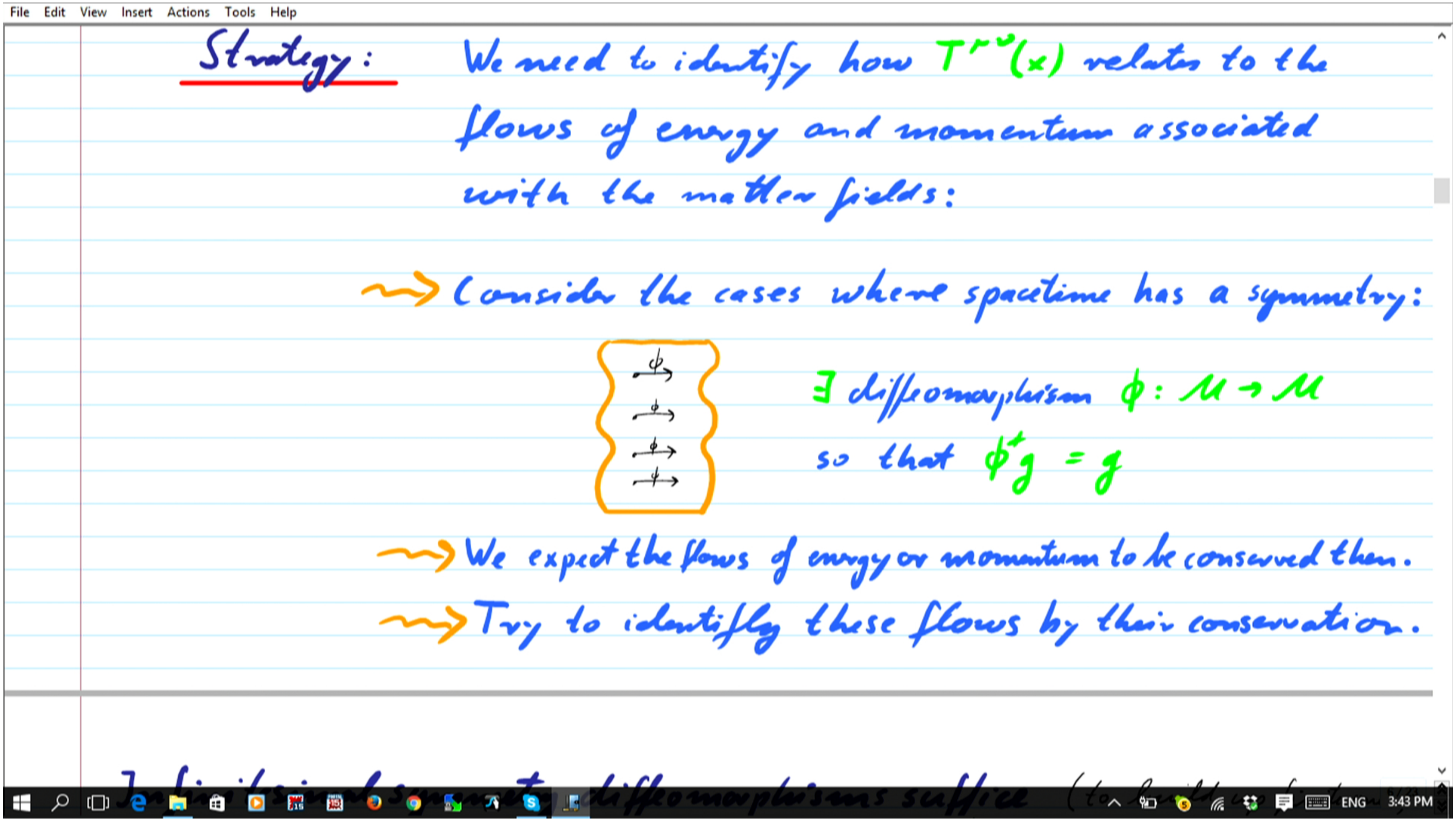
⇝ We expect the flows of energy or momentum to be conserved then.

⇝ Try to identify these flows by their conservation.

Infinitesimal symmetry diffeomorphisms suffice (to build up finite ones



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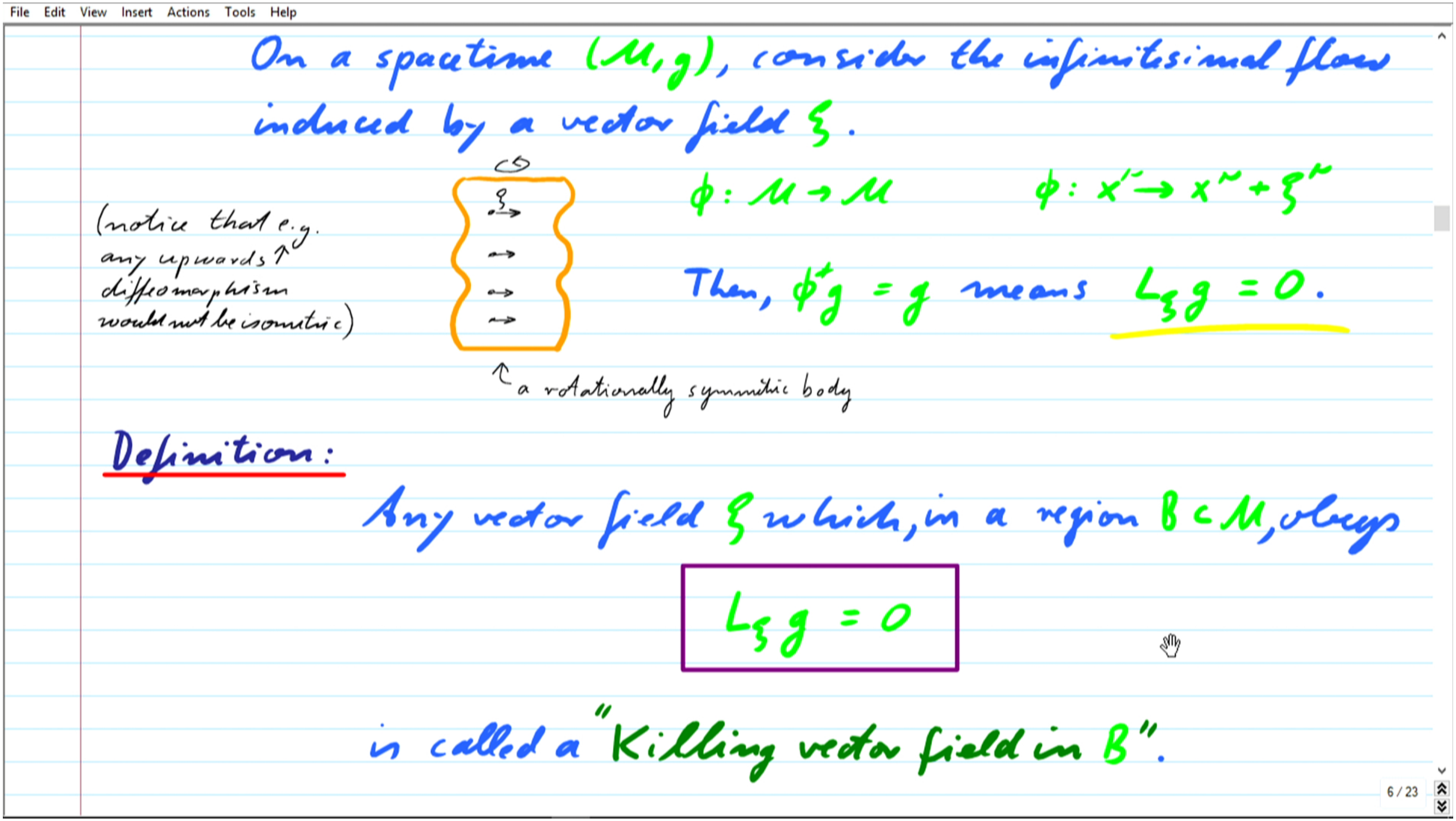


On a spacetime $(\mathcal{M}, g)$, consider the infinitesimal flow induced by a vector field ξ.

(notice that e.g. any upwards ↑ diffeomorphism would not be isometric)

$\phi : \mathcal{M} \to \mathcal{M}$ $\phi : x^\mu \to x^\mu + \xi^\mu$

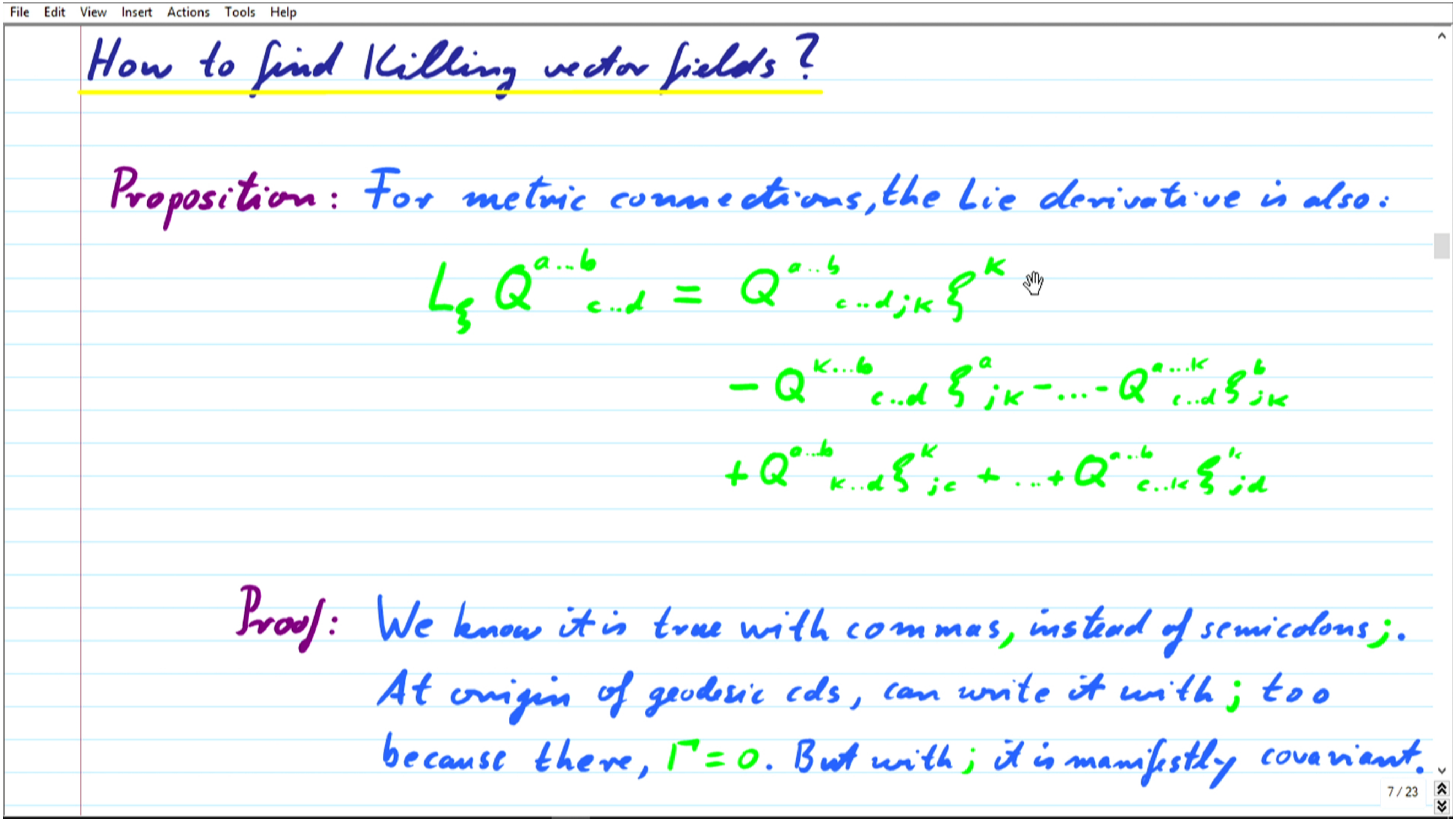
Then, $\phi^* g = g$ means $\mathcal{L}_\xi g = 0$.

Definition:

Any vector field ξ which, in a region $B \subset \mathcal{M}$, obeys

$$\boxed{\mathcal{L}_\xi g = 0}$$

is called a "Killing vector field in B".



How to find Killing vector fields?

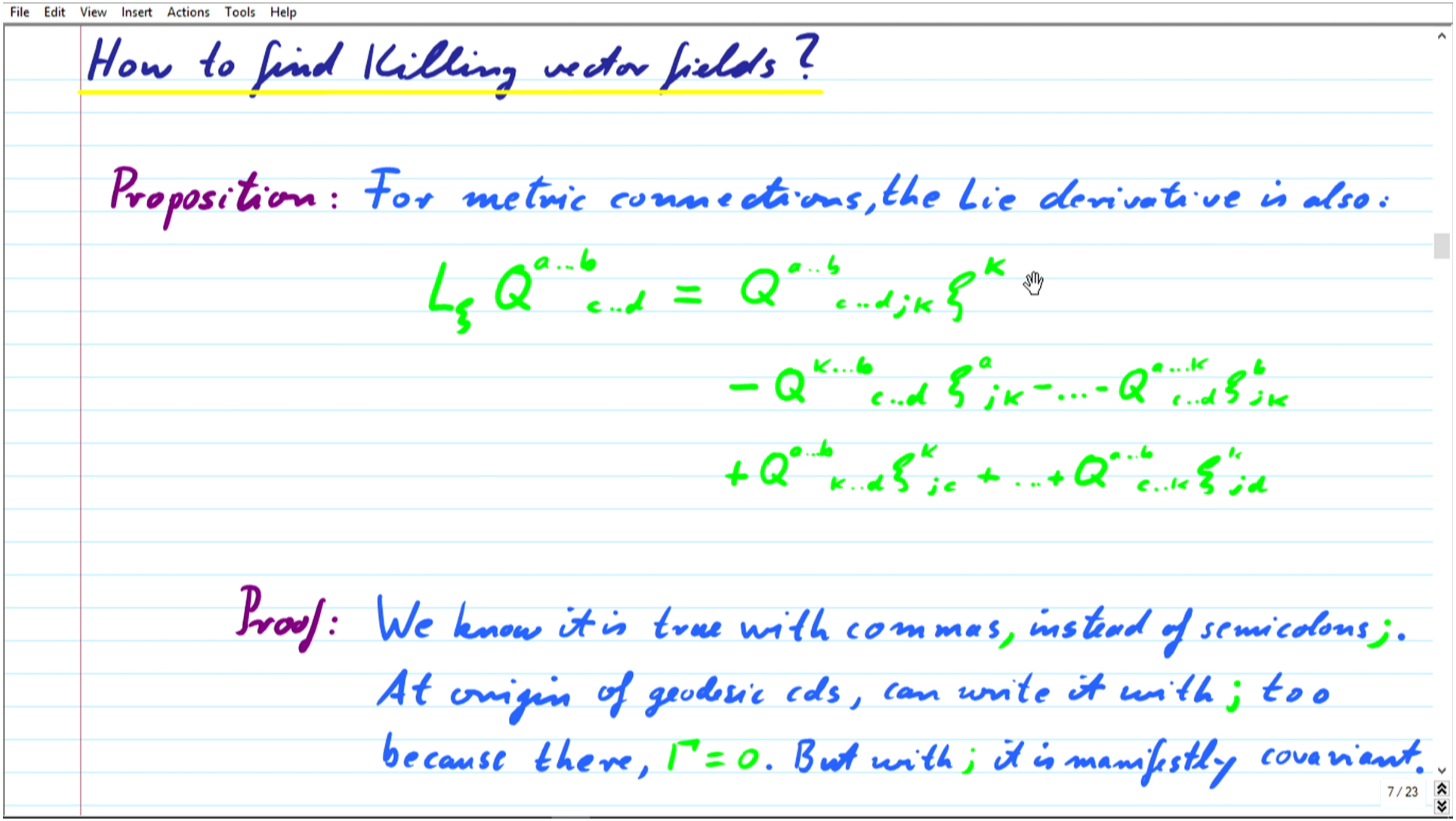
Proposition: For metric connections, the Lie derivative is also:

$$L_\xi Q^{a..b}{}_{c..d} = Q^{a..b}{}_{c..d;k}\,\xi^k$$

$$- Q^{k..b}{}_{c..d}\,\xi^a{}_{;k} - \dots - Q^{a..k}{}_{c..d}\,\xi^b{}_{;k}$$

$$+ Q^{a..b}{}_{k..d}\,\xi^k{}_{;c} + \dots + Q^{a..b}{}_{c..k}\,\xi^k{}_{;d}$$

Proof: We know it is true with commas, instead of semicolons ;. At origin of geodesic cds, can write it with ; too because there, $\Gamma = 0$. But with ; it is manifestly covariant.



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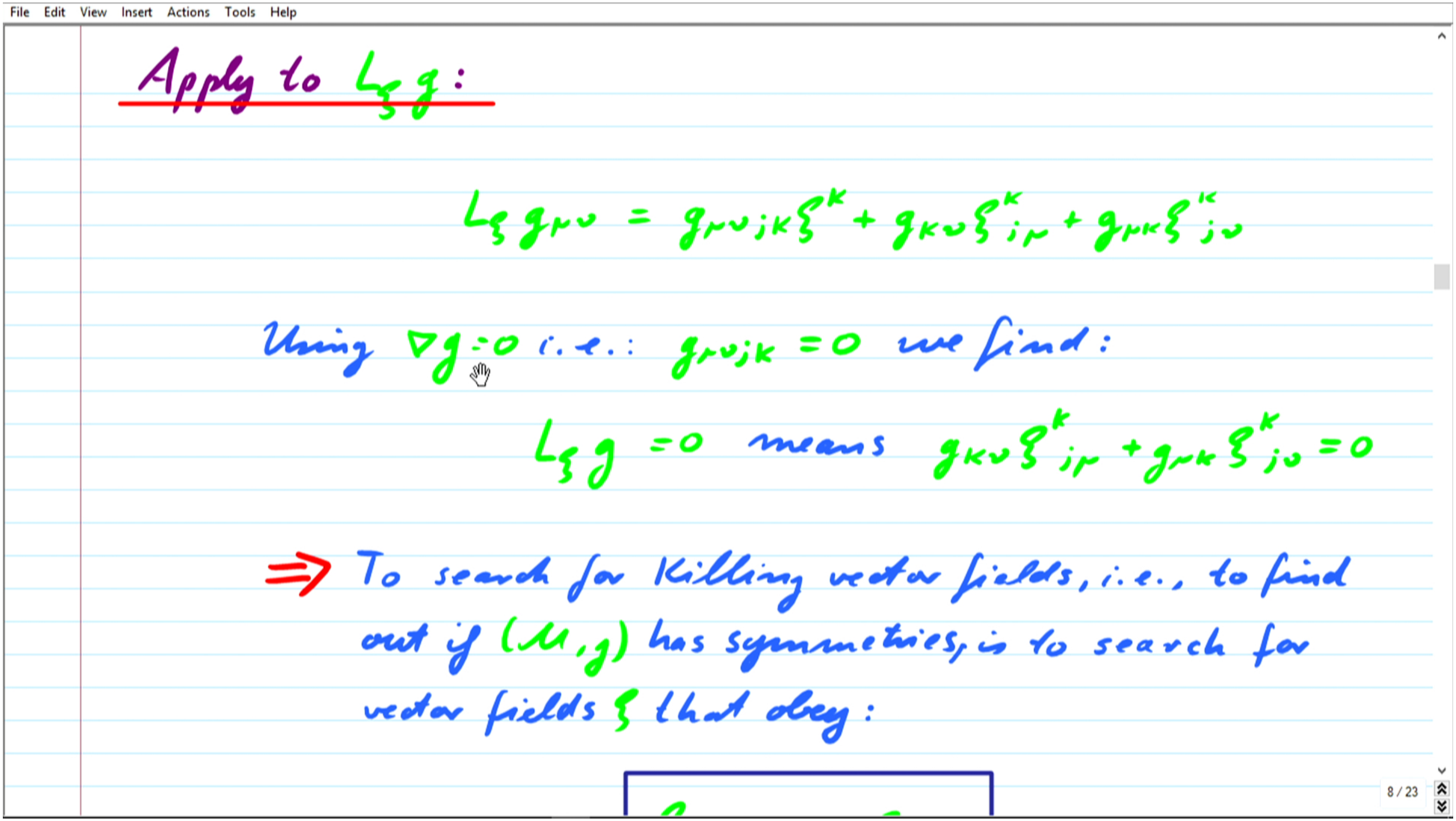


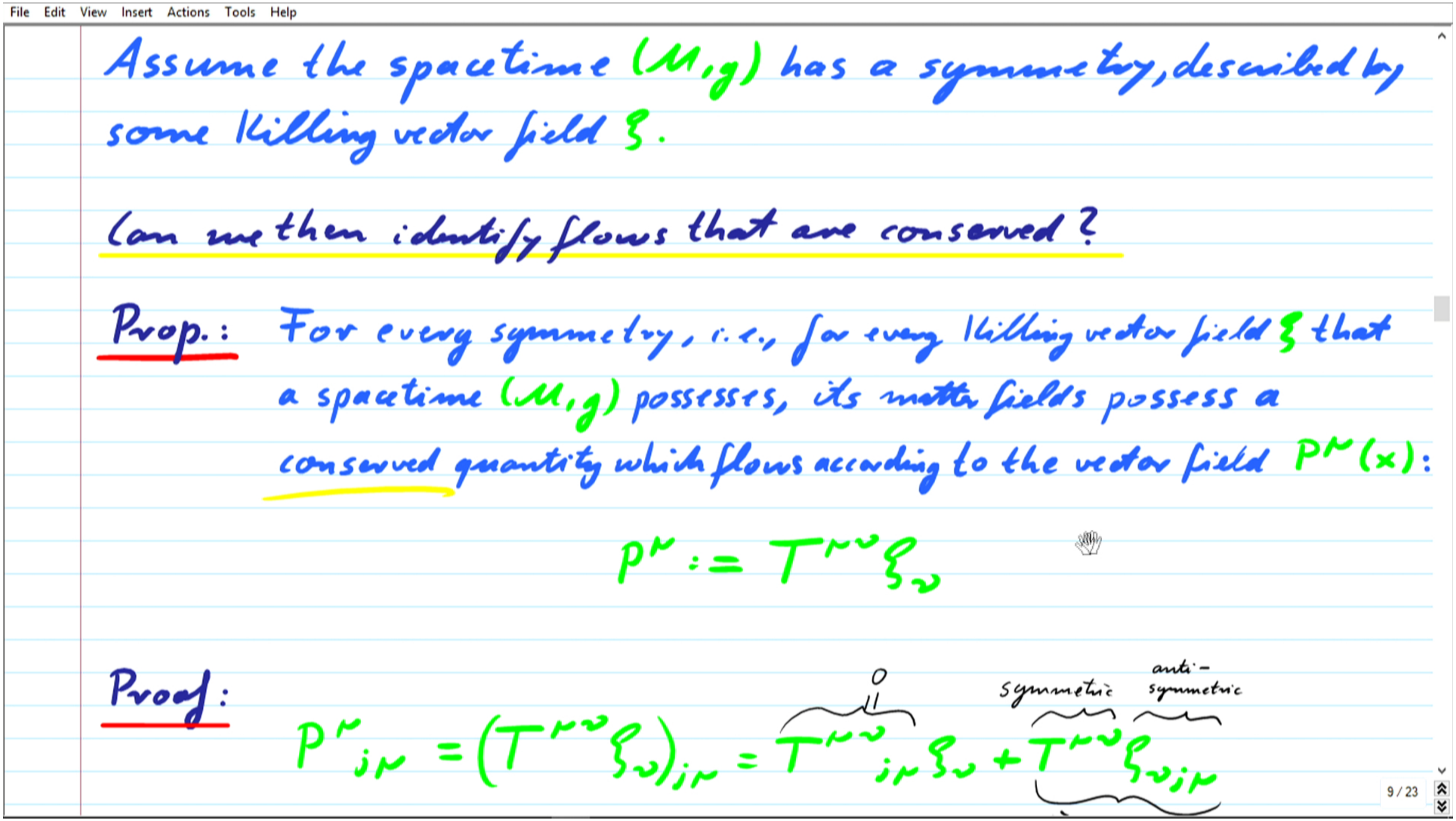
Apply to $L_\xi g$:

$$L_\xi g_{\mu\nu} = g_{\mu\nu;\kappa}\xi^\kappa + g_{\kappa\nu}\xi^\kappa{}_{;\mu} + g_{\mu\kappa}\xi^\kappa{}_{;\nu}$$

Using $\nabla g = 0$ i.e.: $g_{\mu\nu;\kappa} = 0$ we find:

$$L_\xi g = 0 \text{ means } g_{\kappa\nu}\xi^\kappa{}_{;\mu} + g_{\mu\kappa}\xi^\kappa{}_{;\nu} = 0$$

⇒ To search for Killing vector fields, i.e., to find out if (M,g) has symmetries, is to search for vector fields ξ that obey:



Assume the spacetime $(\mathcal{M},g)$ has a symmetry, described by some Killing vector field ξ.

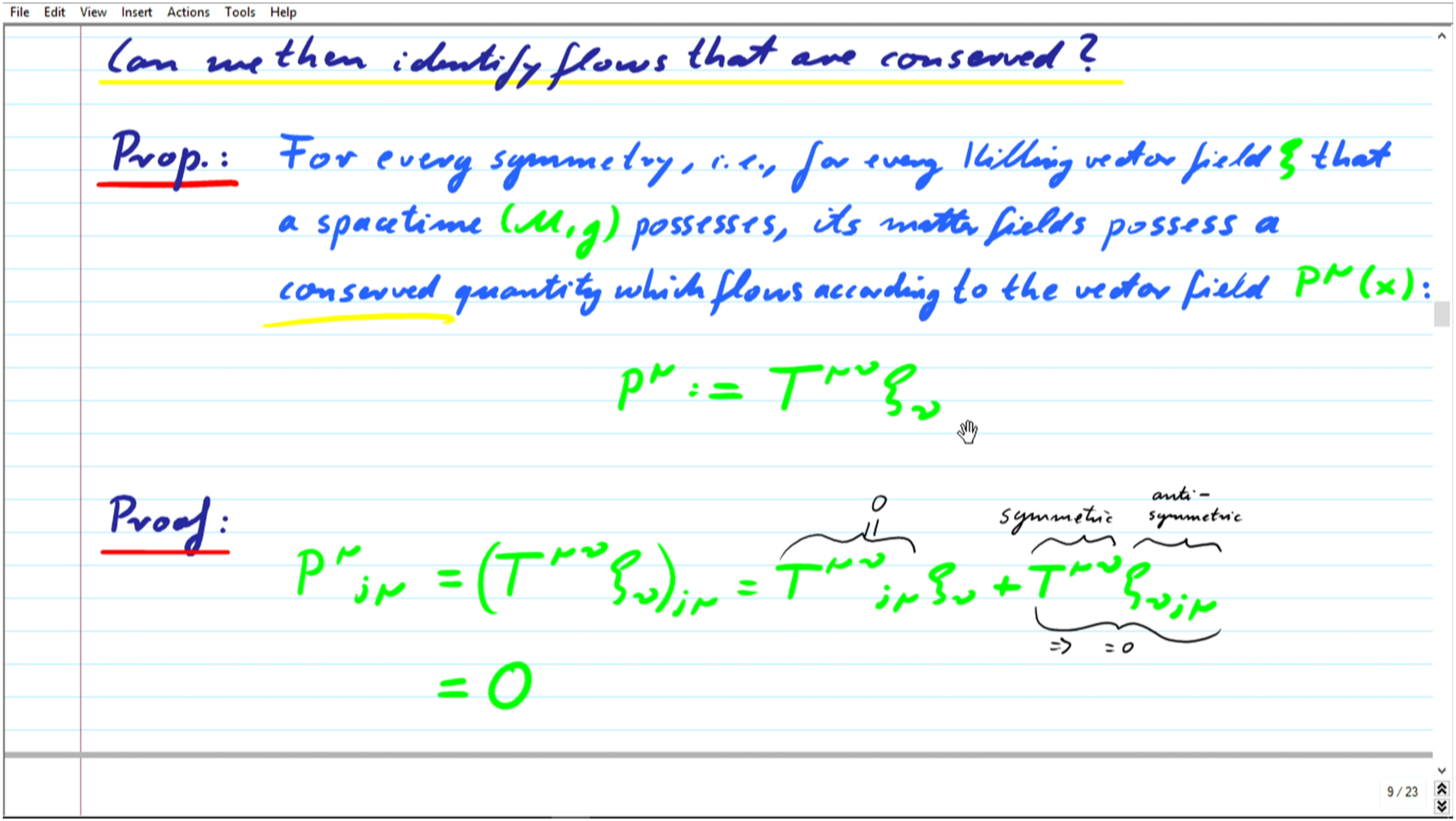
Can we then identify flows that are conserved?

Prop.: For every symmetry, i.e., for every Killing vector field ξ that a spacetime $(\mathcal{M},g)$ possesses, its matter fields possess a conserved quantity which flows according to the vector field $P^\mu(x)$:

$$P^\mu := T^{\mu\nu}\xi_\nu$$

Proof:

$$P^\mu{}_{;\mu} = \left(T^{\mu\nu}\xi_\nu\right)_{;\mu} = \overbrace{T^{\mu\nu}{}_{;\mu}}^{\overset{0}{\shortparallel}}\xi_\nu + \underbrace{\overbrace{T^{\mu\nu}}^{\text{symmetric}}\overbrace{\xi_{\nu;\mu}}^{\text{anti-symmetric}}}_{}$$



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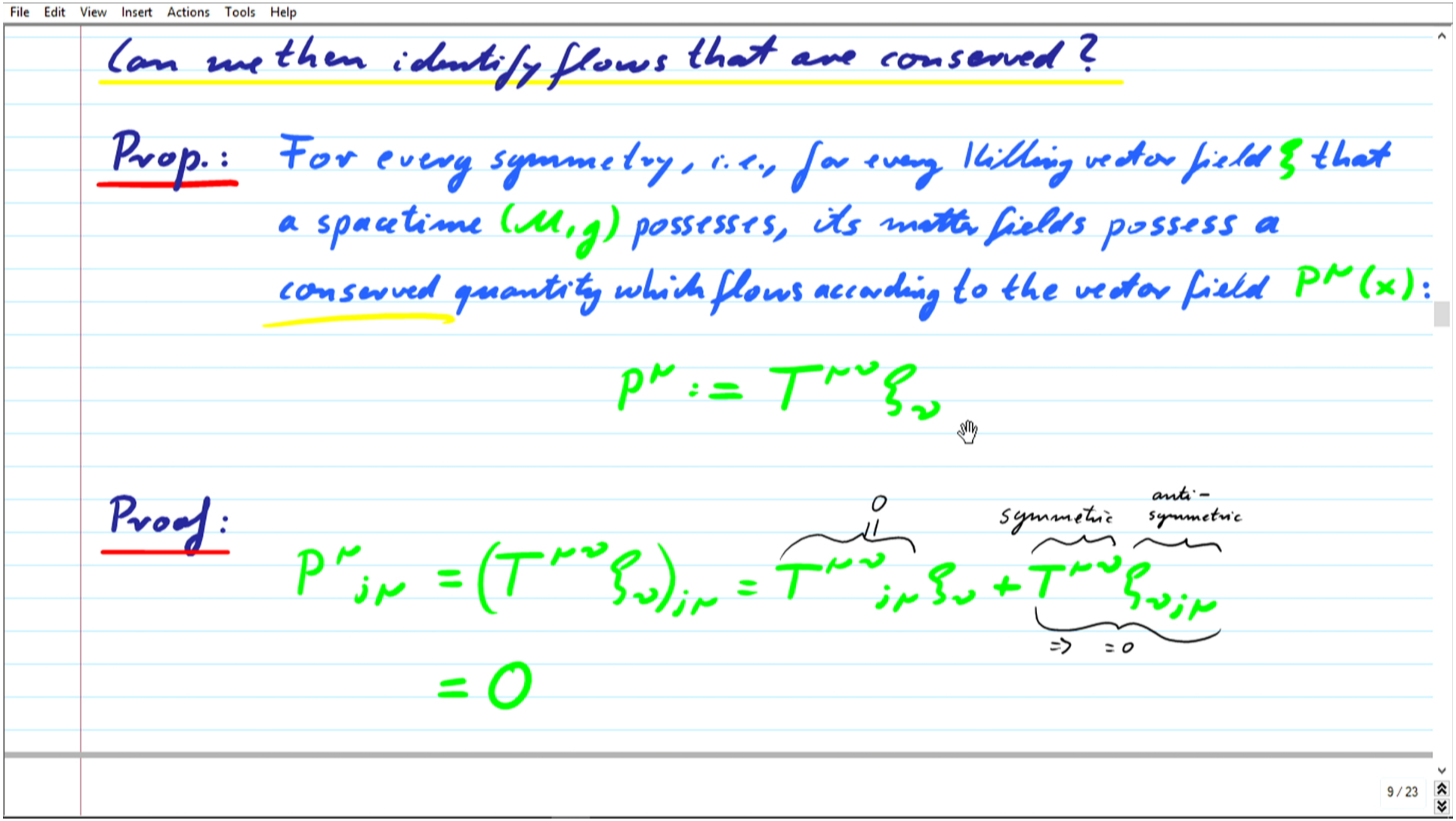

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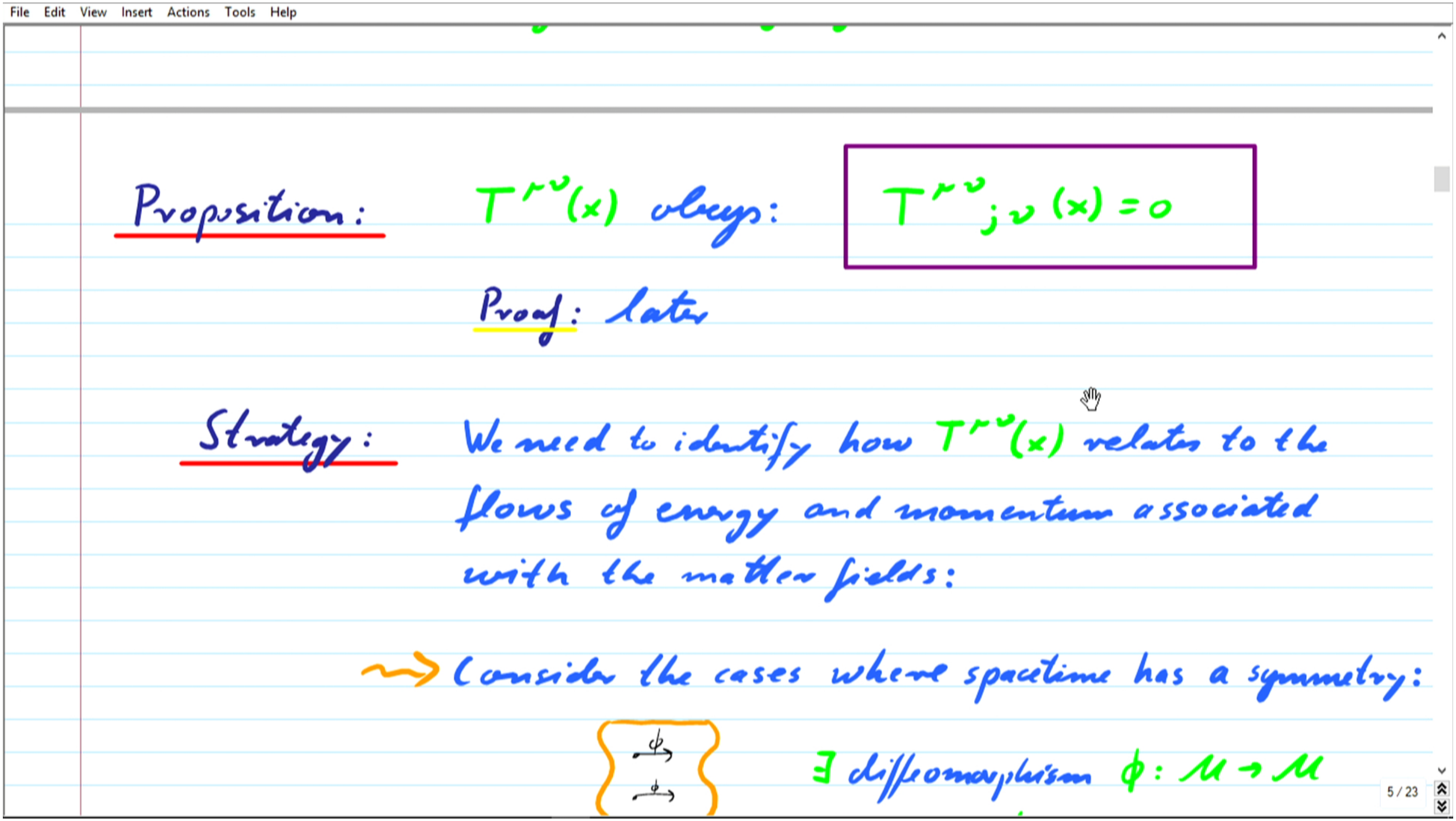


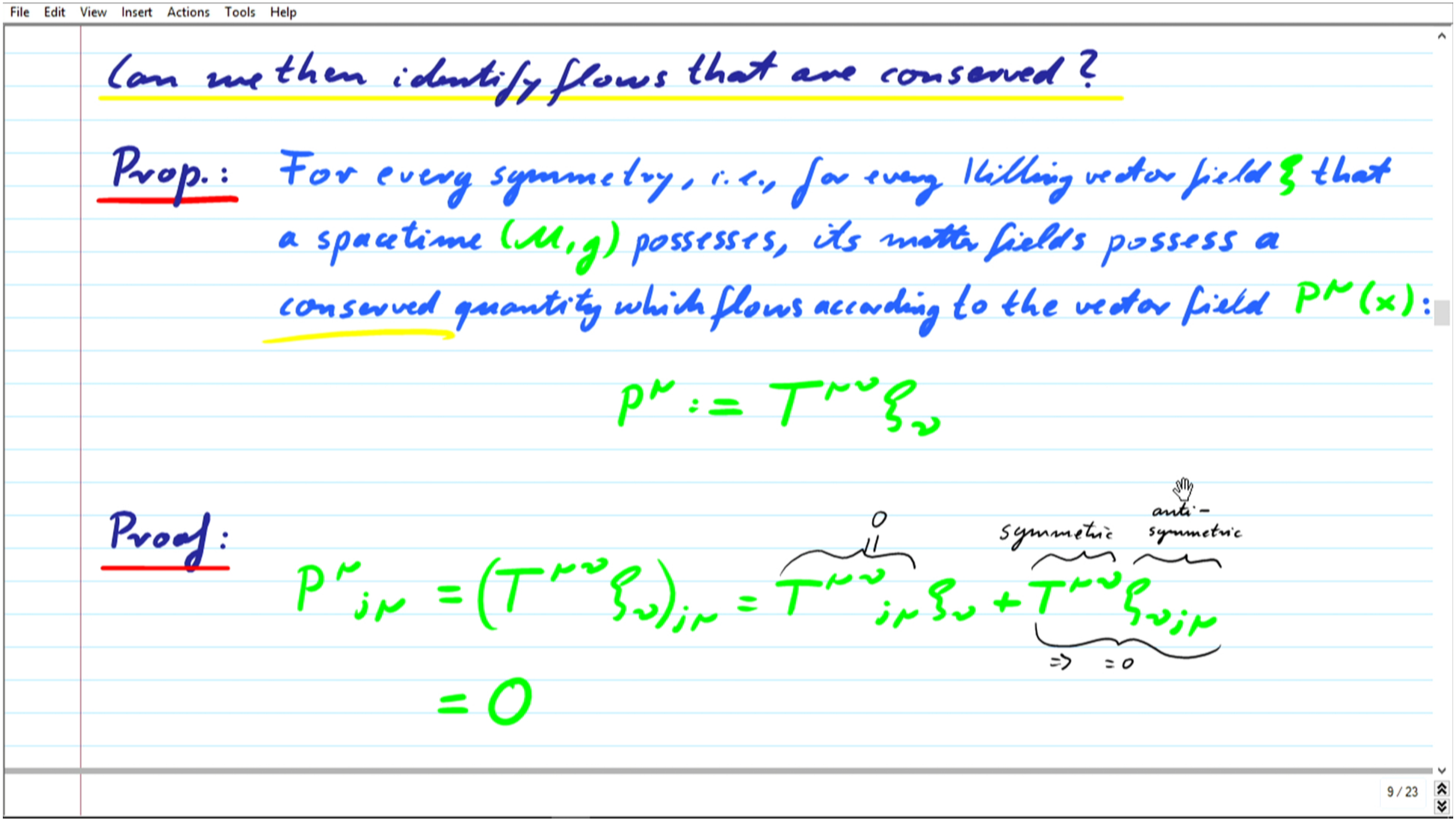
Proposition: $T^{\mu\nu}(x)$ obeys: $\boxed{T^{\mu\nu}{}_{;\nu}(x) = 0}$

Proof: later

Strategy: We need to identify how $T^{\mu\nu}(x)$ relates to the flows of energy and momentum associated with the matter fields:

⇝ Consider the cases where spacetime has a symmetry:

$\left\{ \begin{matrix} \xrightarrow{\phi} \\ \xrightarrow{\phi} \end{matrix} \right\}$ $\exists$ diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$



Can we then identify flows that are conserved?

Prop.: For every symmetry, i.e., for every Killing vector field ξ that a spacetime $(\mathcal{M}, g)$ possesses, its matter fields possess a conserved quantity which flows according to the vector field $P^\mu(x)$:

$$P^\mu := T^{\mu\nu}\xi_\nu$$

Proof:

$$P^\mu{}_{;\mu} = (T^{\mu\nu}\xi_\nu)_{;\mu} = \overbrace{T^{\mu\nu}{}_{;\mu}}^{\overset{0}{\parallel}}\xi_\nu + \underbrace{\overbrace{T^{\mu\nu}}^{\text{symmetric}}\overbrace{\xi_{\nu;\mu}}^{\text{anti-symmetric}}}_{\Rightarrow\ =0}$$

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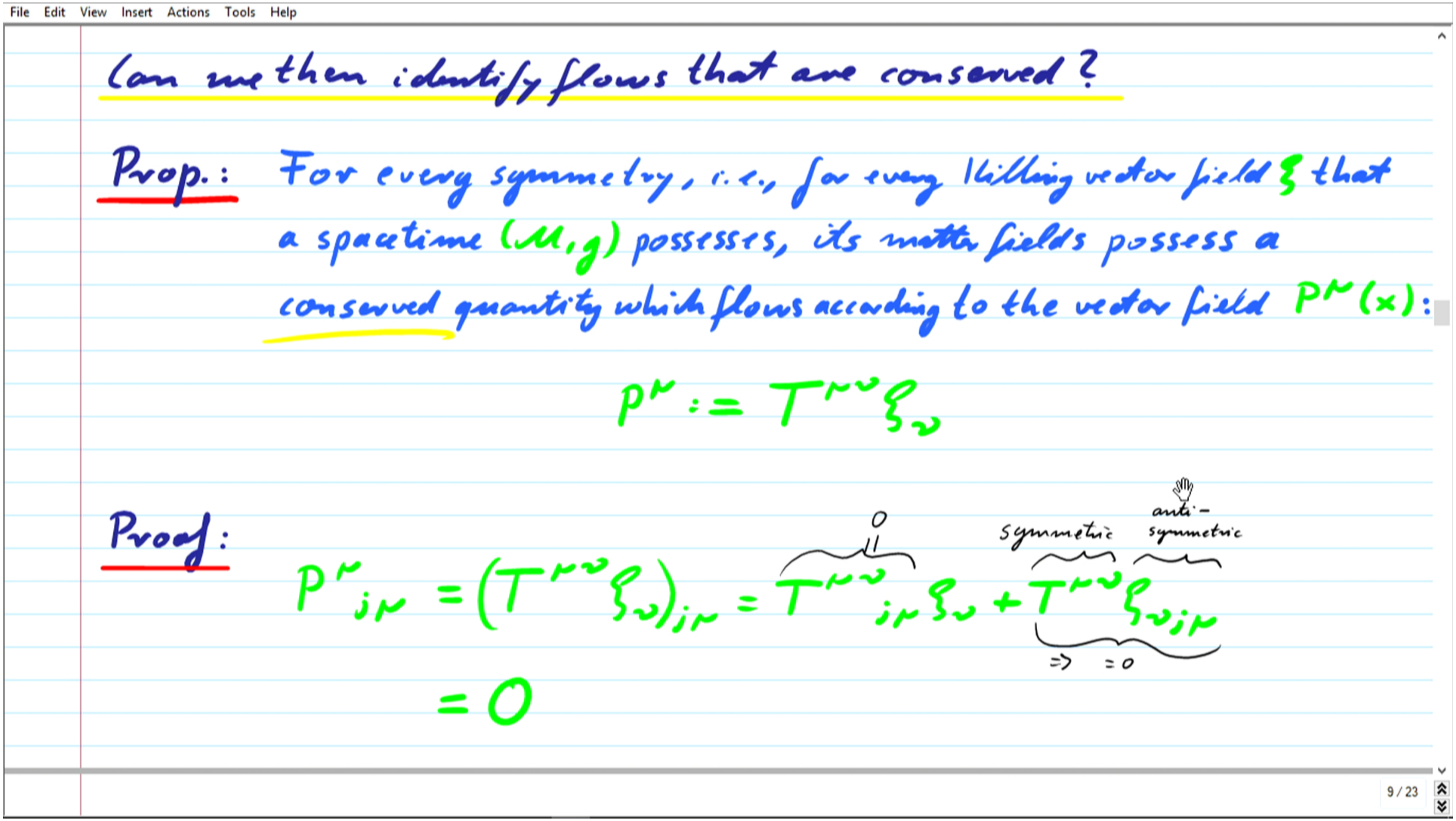

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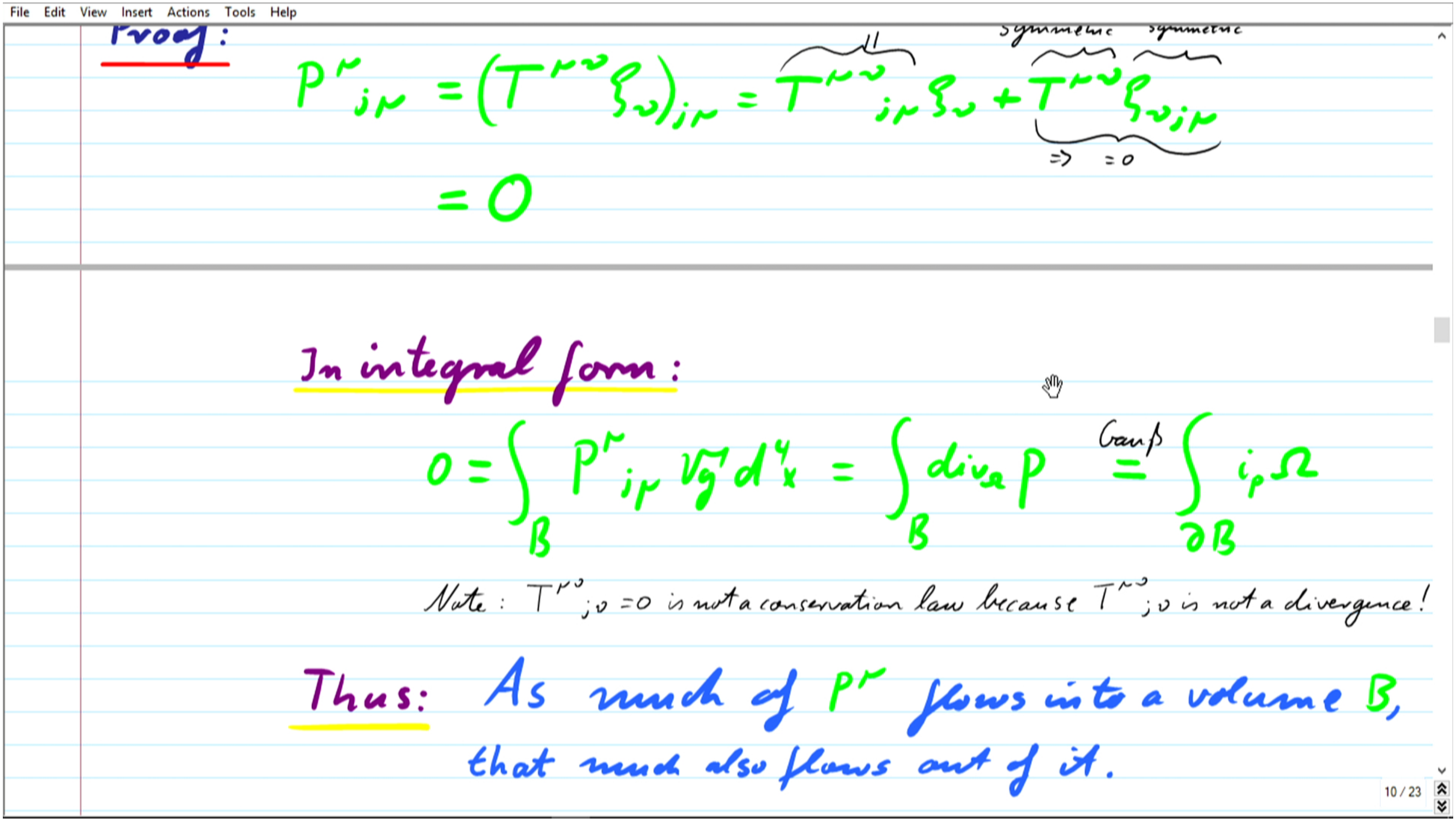

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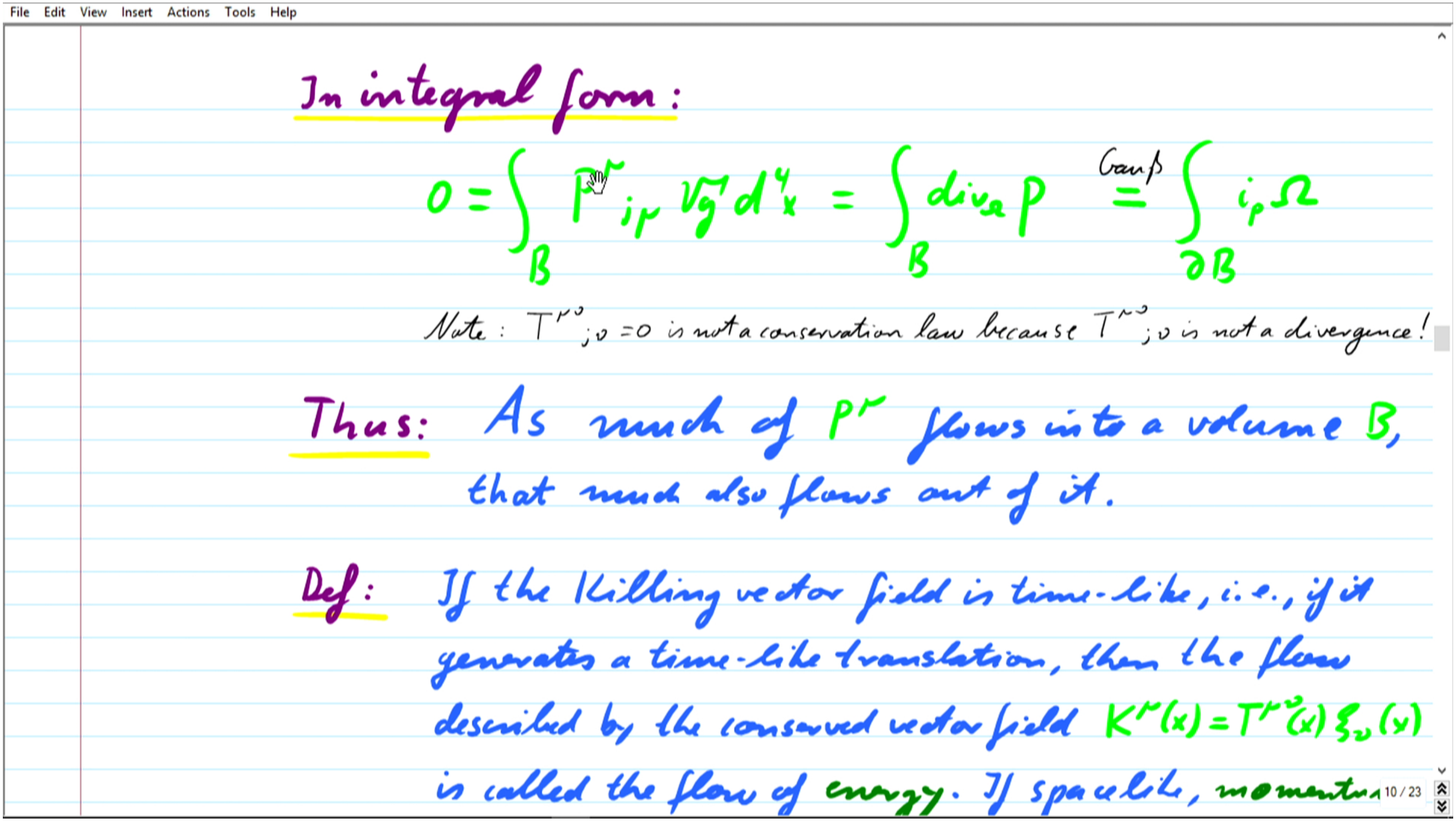
$$= 0$$

In integral form:

$$0 = \int_B P^\mu{}_{;\mu}\sqrt{g}\, d^4x = \int_B \mathrm{div}_\Omega P \overset{\text{Gauß}}{=} \int_{\partial B} i_P \Omega$$

Note: $T^{\mu\nu}{}_{;\nu} = 0$ is not a conservation law because $T^{\mu\nu}{}_{;\nu}$ is not a divergence!

Thus: As much of P^μ flows into a volume B, that much also flows out of it.



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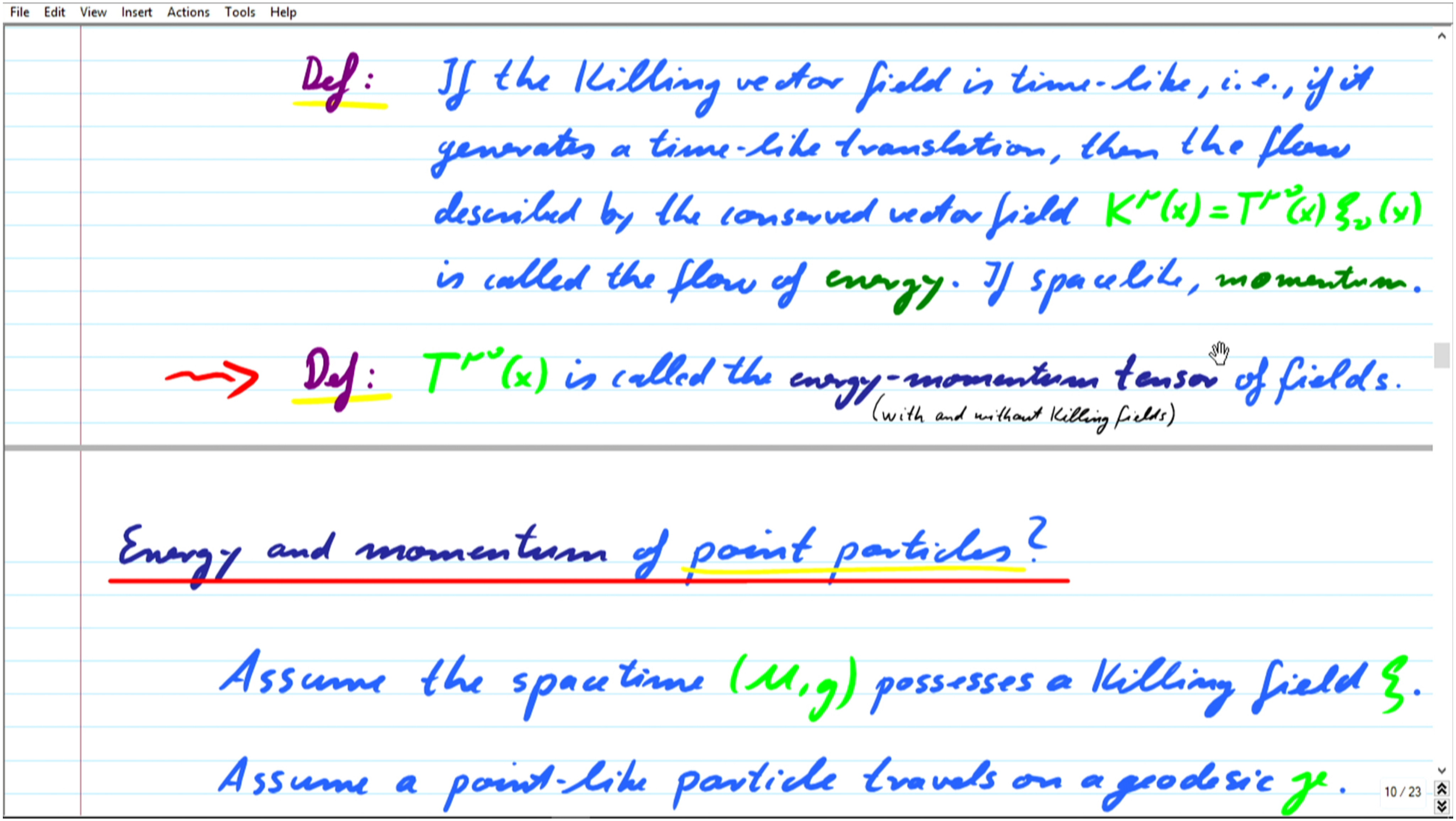
Def: If the Killing vector field is time-like, i.e., if it generates a time-like translation, then the flow described by the conserved vector field $K^\mu(x)=T^{\mu\nu}(x)\xi_\nu(x)$ is called the flow of energy. If spacelike, momentu



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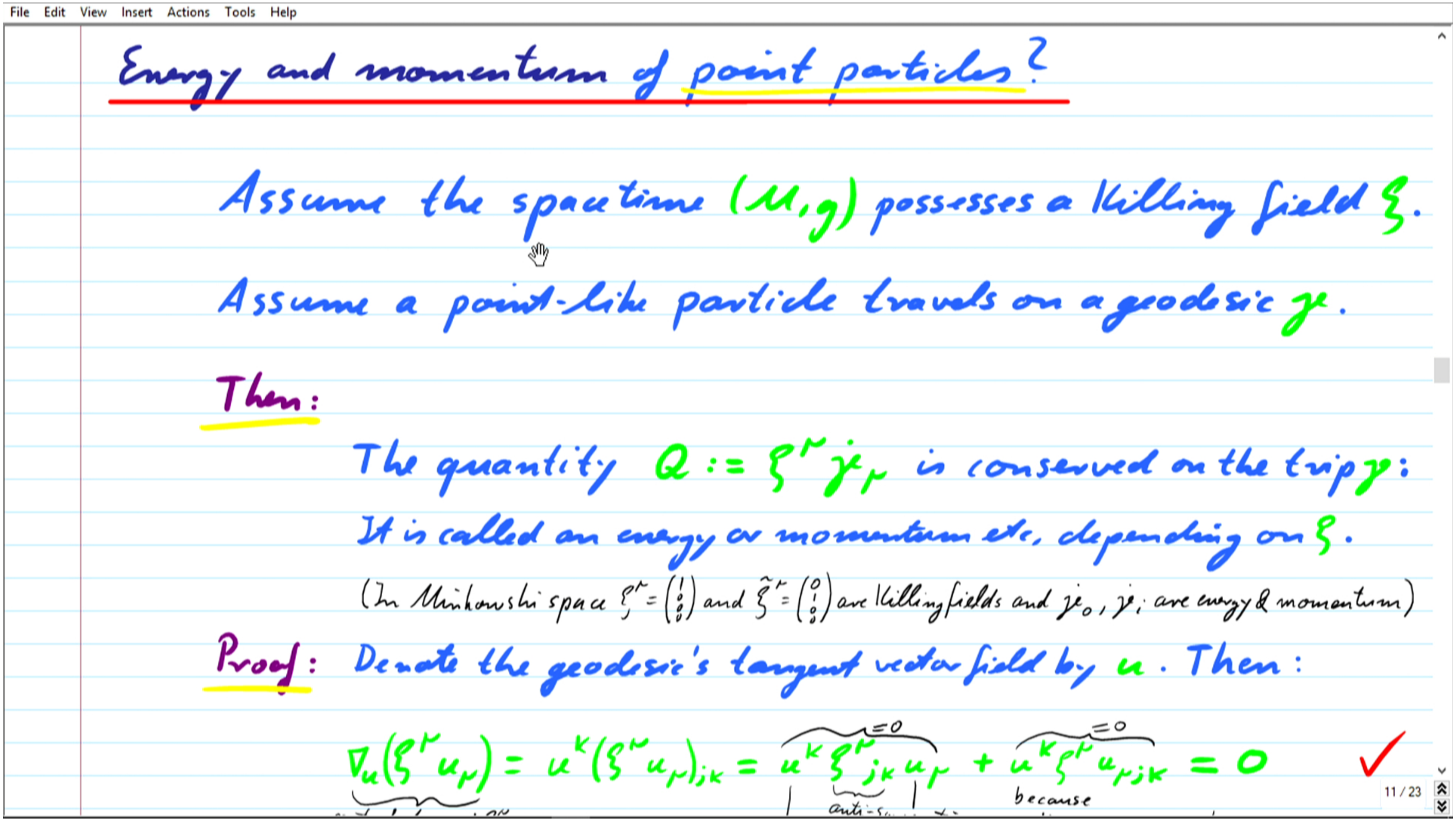
$\rightsquigarrow$ **Def:** $T^{\mu\nu}(x)$ is called the energy-momentum tensor of fields. (with and without Killing fields)

Energy and momentum of point particles?

Assume the spacetime $(\mathcal{M}, g)$ possesses a Killing field ξ.

Assume a point-like particle travels on a geodesic γ.



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Then:

The quantity $Q := \xi^\mu \dot{\gamma}_\mu$ is conserved on the traj γ:

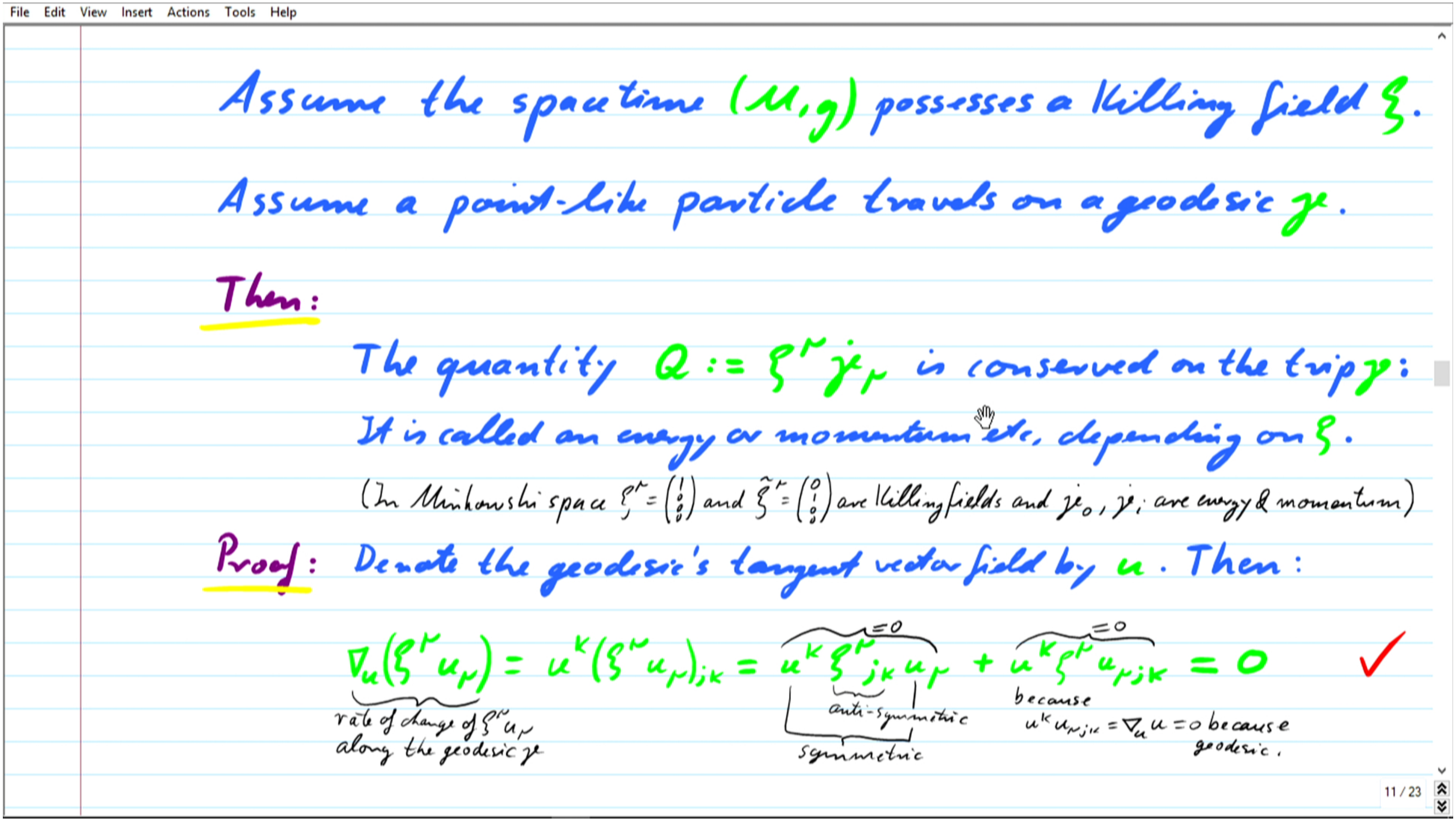
It is called an energy or momentum etc, depending on ξ.

(In Minkowski space $\xi^\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $\tilde{\xi}^\mu = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are Killing fields and $\dot{\gamma}_0$, $\dot{\gamma}_i$ are energy & momentum)

Proof: Denote the geodesic's tangent vector field by u. Then:

$$\underbrace{\nabla_u(\xi^\mu u_\mu)} = u^\kappa(\xi^\mu u_\mu)_{;\kappa} = \overbrace{u^\kappa \underbrace{\xi^\mu{}_{;\kappa}}_{\text{anti-s}} u_\mu}^{=0} + \overbrace{u^\kappa \xi^\mu u_{\mu;\kappa}}^{=0} = 0 \quad ✓$$

because



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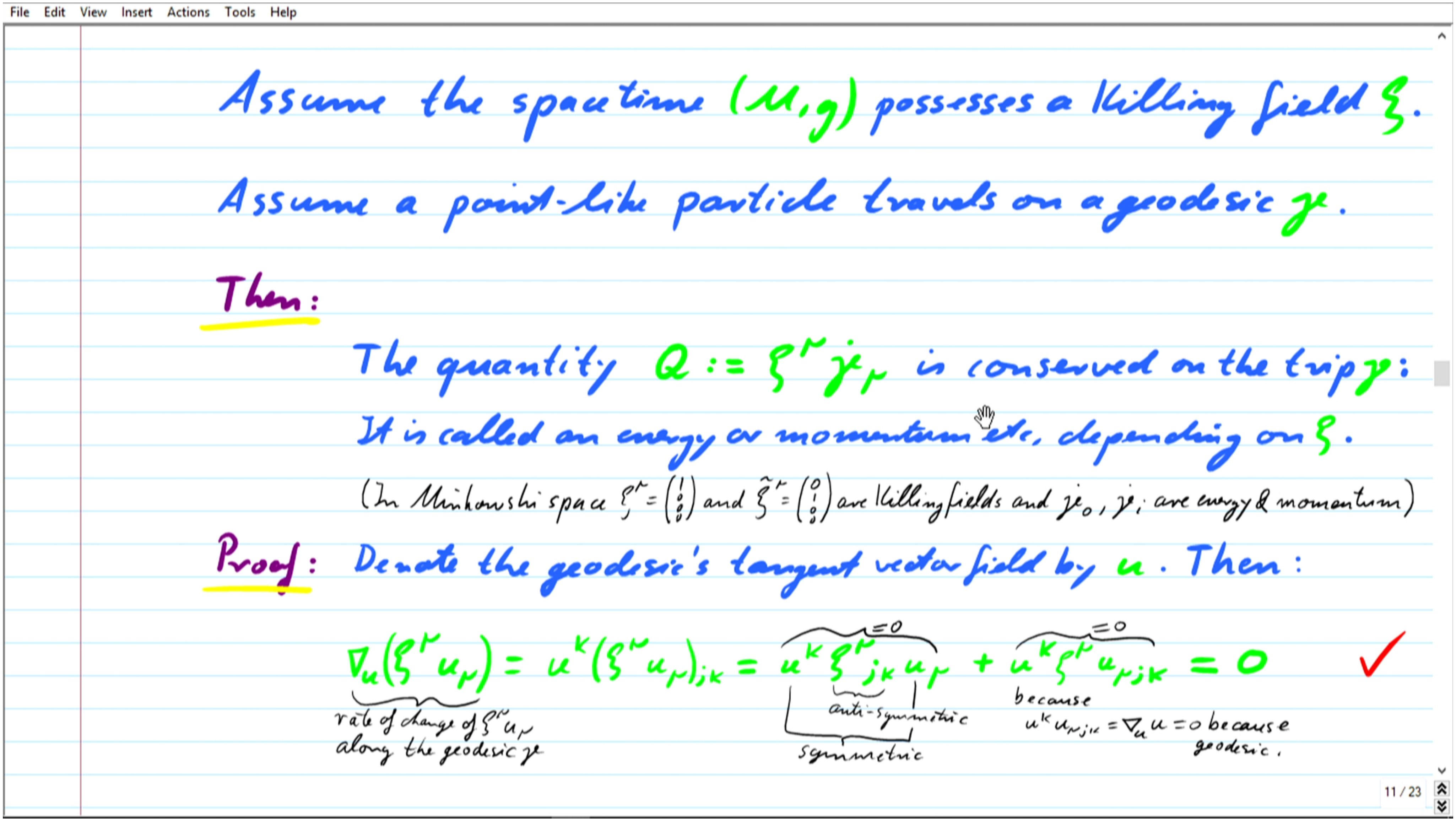
anti-symmetric

symmetric

because $u^\kappa u_{\mu;\kappa} = \nabla_u u = 0$ because geodesic.



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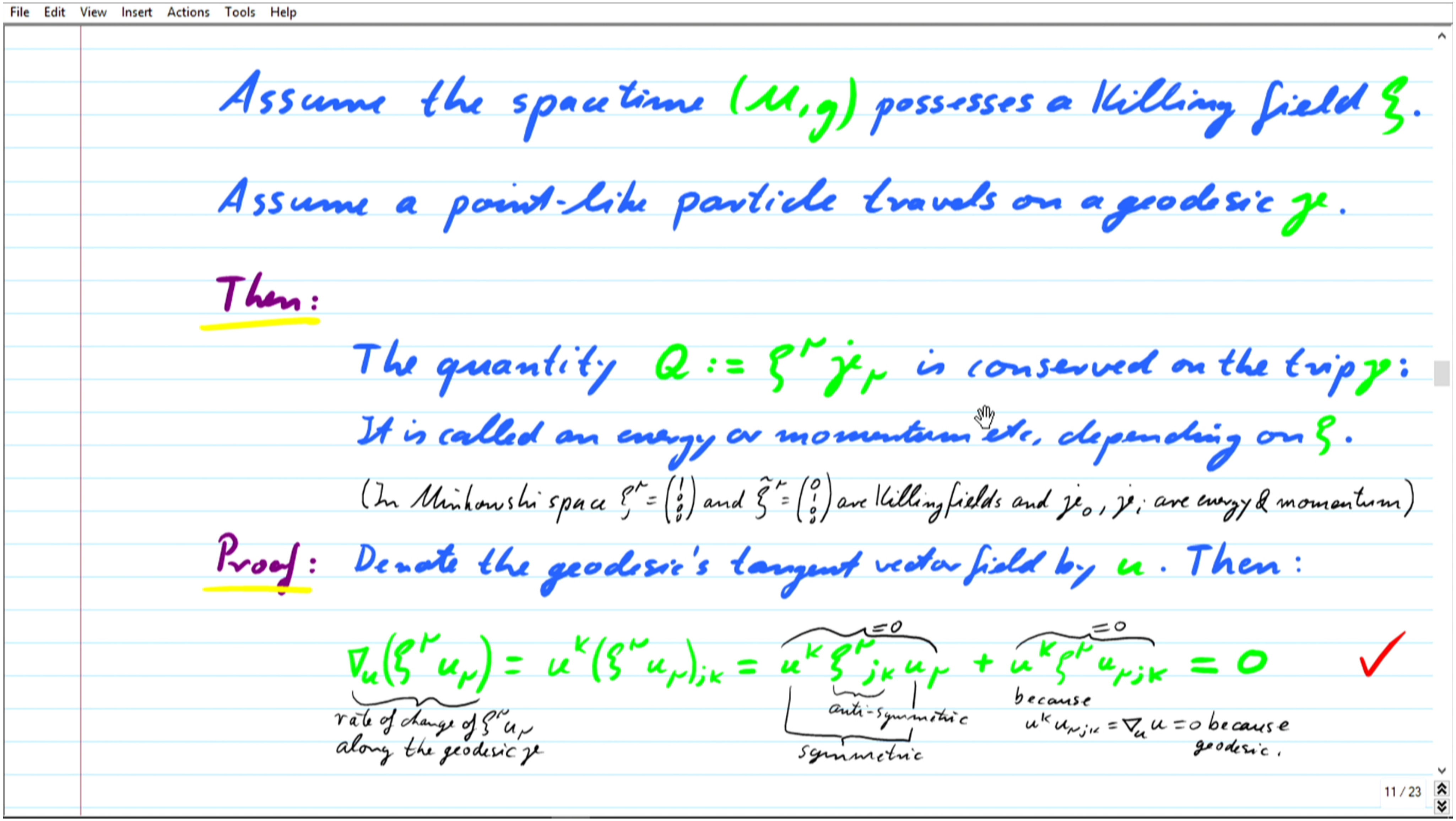

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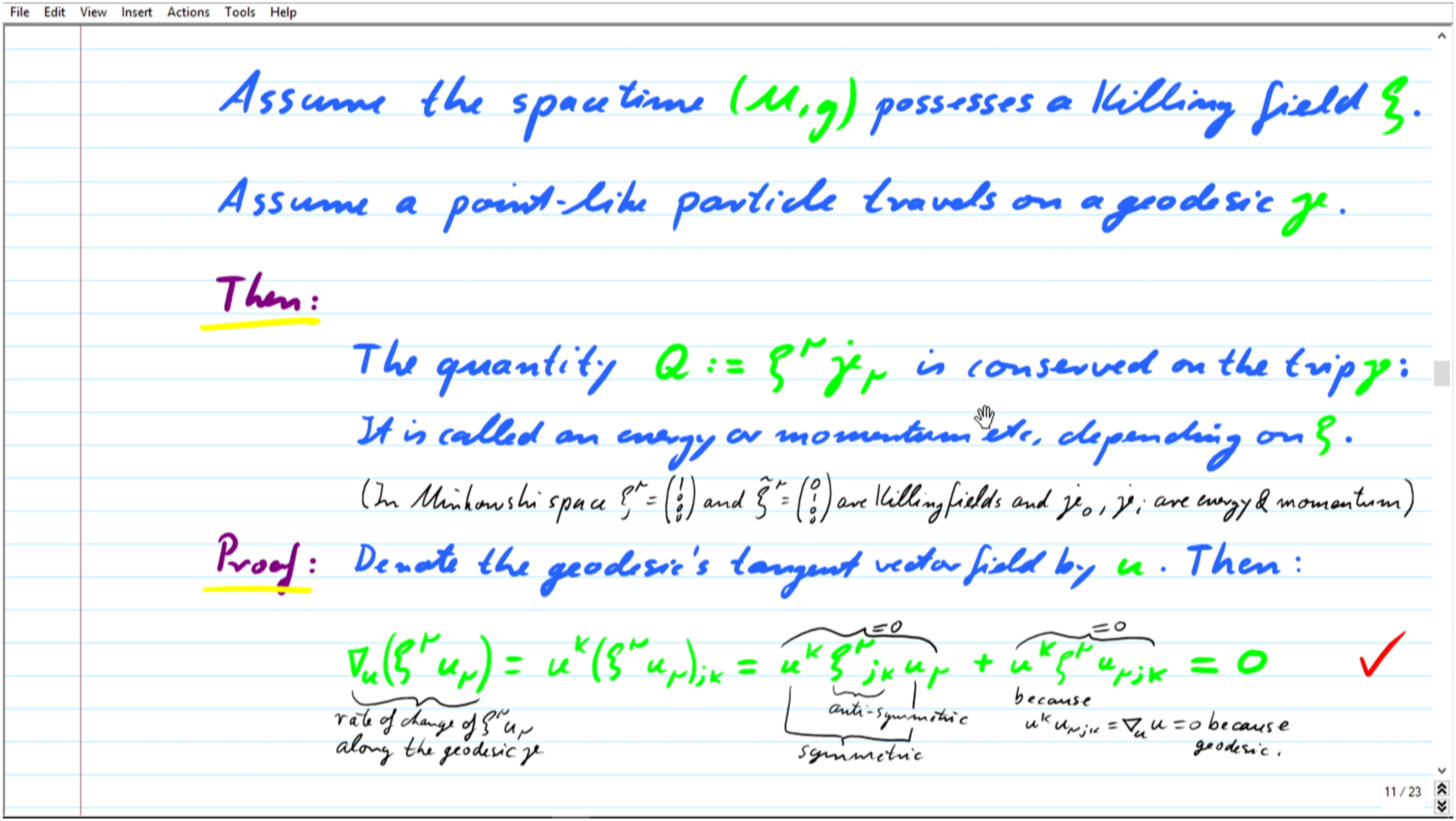

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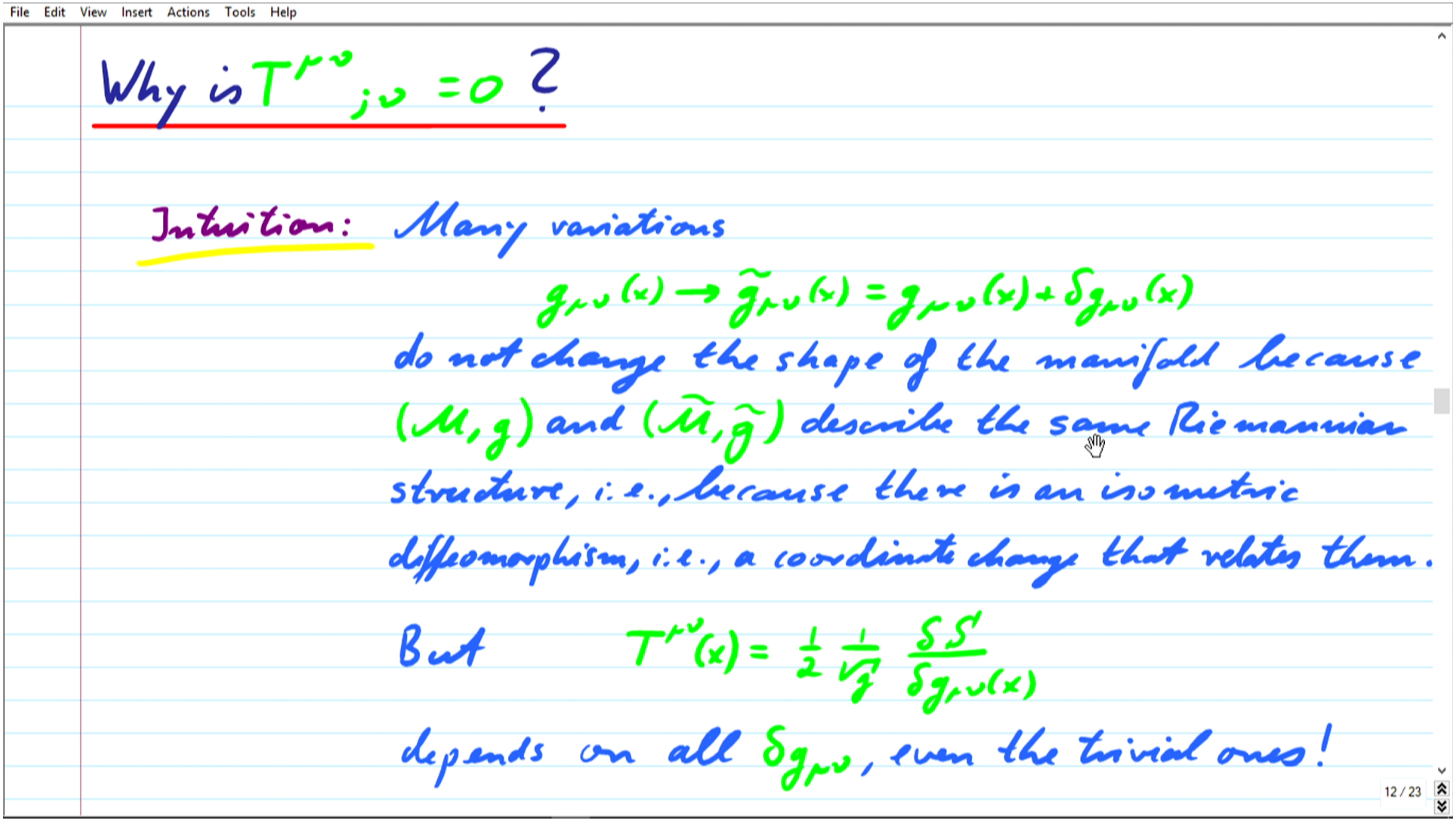


Why is $T^{\mu\nu}{}_{;\nu} = 0$?

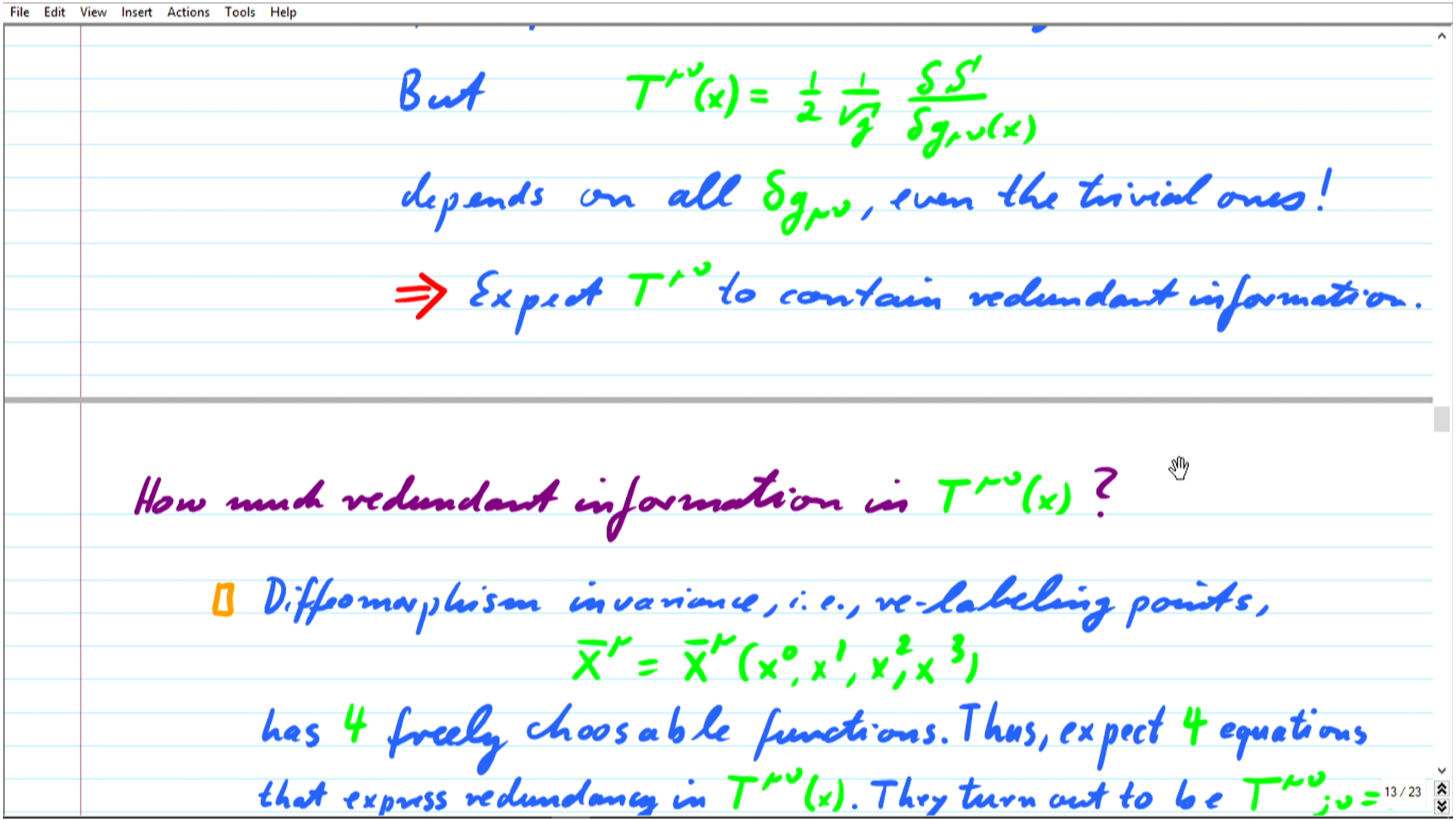
Intuition: Many variations

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$$

do not change the shape of the manifold because $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ describe the same Riemannian structure, i.e., because there is an isometric diffeomorphism, i.e., a coordinate change that relates them.

But
$$T^{\mu\nu}(x) = \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}$$

depends on all $\delta g_{\mu\nu}$, even the trivial ones!



But $T^{\mu\nu}(x) = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}$

depends on all $\delta g_{\mu\nu}$, even the trivial ones!

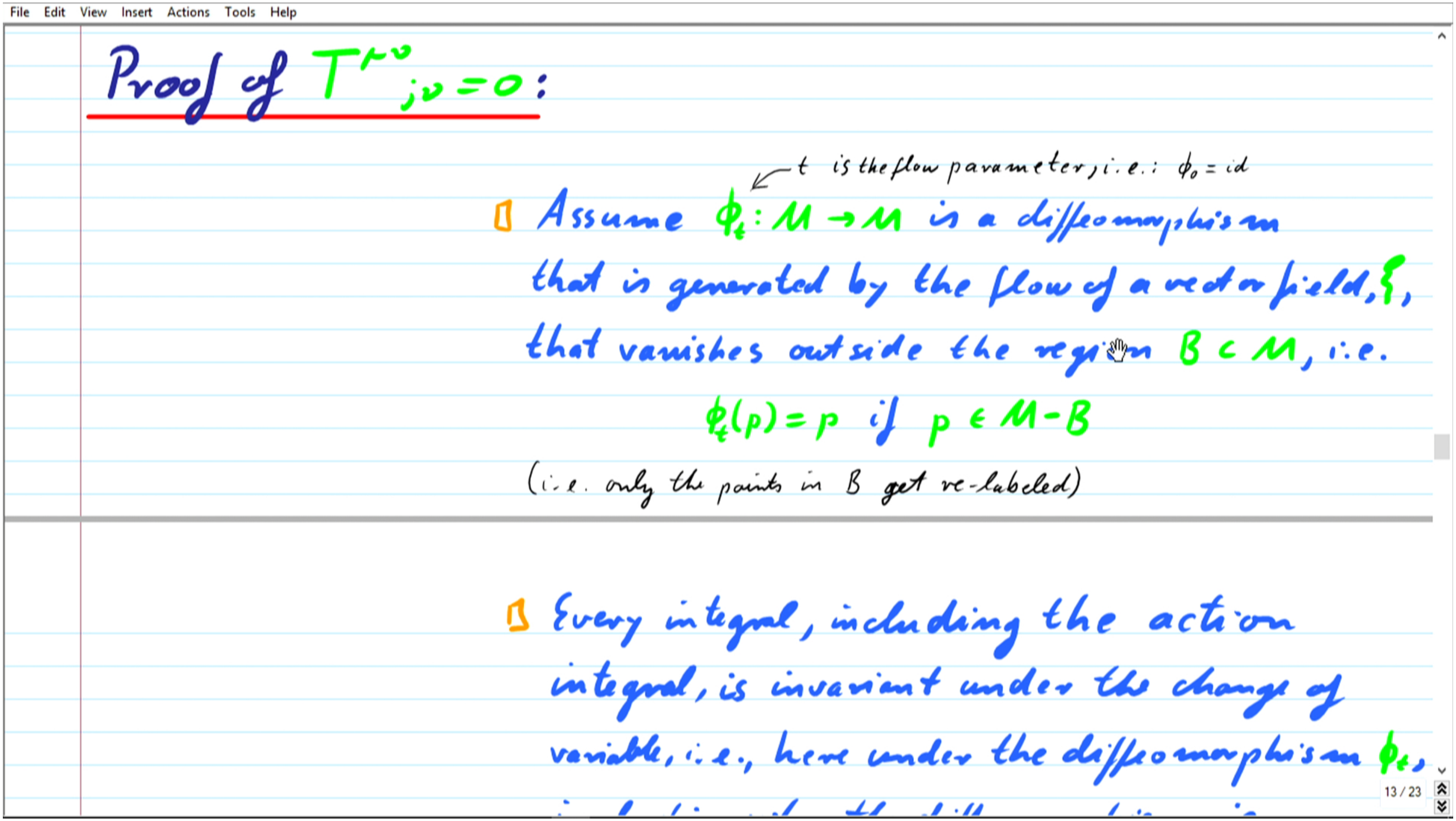
$\Rightarrow$ Expect $T^{\mu\nu}$ to contain redundant information.

How much redundant information in $T^{\mu\nu}(x)$?

- Diffeomorphism invariance, i.e., re-labeling points,

$$\bar{x}^{\mu} = \bar{x}^{\mu}(x^0, x^1, x^2, x^3)$$

has 4 freely choosable functions. Thus, expect 4 equations that express redundancy in $T^{\mu\nu}(x)$. They turn out to be $T^{\mu\nu}{}_{;\nu} =$



Proof of $T^{\mu\nu}{}_{;\nu}=0$:

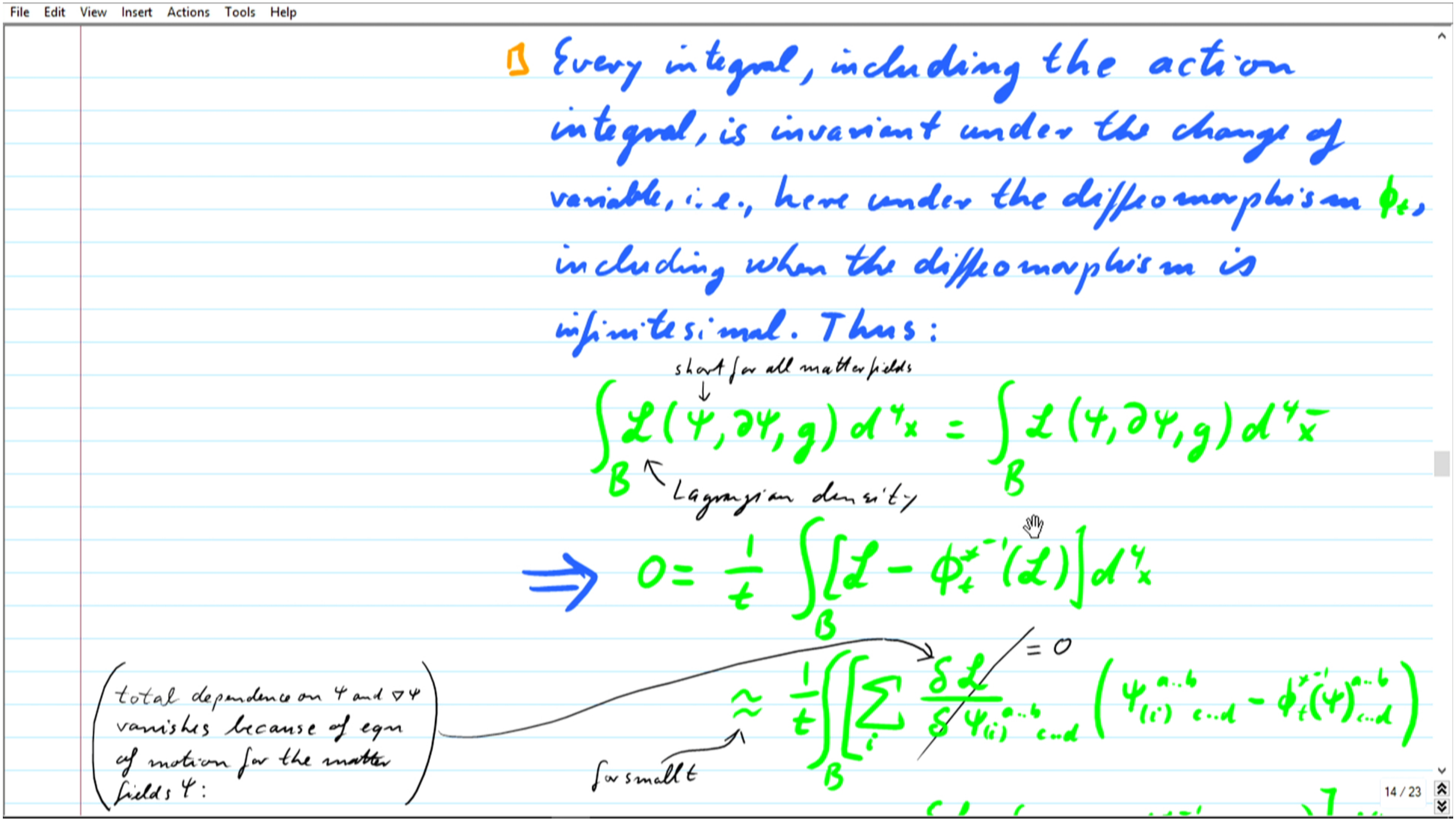
t is the flow parameter, i.e.: $\phi_0 = id$

- Assume $\phi_t : M \to M$ is a diffeomorphism that is generated by the flow of a vector field, ξ, that vanishes outside the region $B \subset M$, i.e.

$$\phi_t(p) = p \text{ if } p \in M - B$$

(i.e. only the points in B get re-labeled)

- Every integral, including the action integral, is invariant under the change of variable, i.e., here under the diffeomorphism ϕ_t,



Every integral, including the action integral, is invariant under the change of variable, i.e., here under the diffeomorphism ϕ_t, including when the diffeomorphism is infinitesimal. Thus:

short for all matter fields

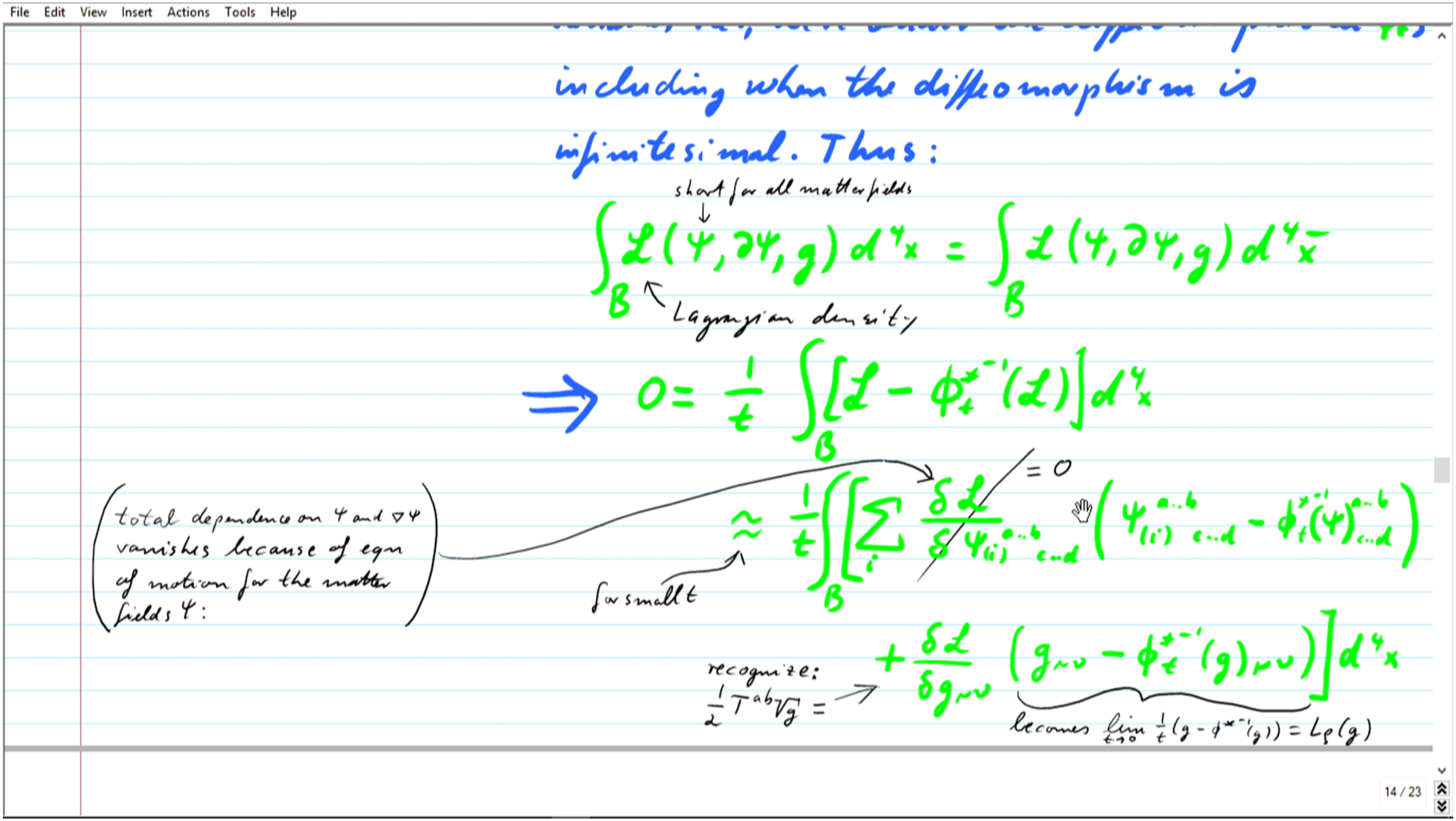
$$\int_B \mathcal{L}(\overset{\downarrow}{\psi},\partial\psi,g)\,d^4x = \int_B \mathcal{L}(\psi,\partial\psi,g)\,d^4\bar{x}$$

Lagrangian density

$$\Rightarrow\quad 0 = \frac{1}{t}\int_B \left[\mathcal{L} - \phi_t^{*-1}(\mathcal{L})\right] d^4x$$

$$\underset{\text{for small } t}{\approx} \frac{1}{t}\int_B \left[\sum_i \cancel{\frac{\delta\mathcal{L}}{\delta\psi_{(i)}{}^{a..b}{}_{c..d}}}^{\,=0} \left(\psi_{(i)}{}^{a..b}{}_{c..d} - \phi_t^{*-1}(\psi)^{a..b}{}_{c..d}\right)\right.$$

(total dependence on ψ and $\nabla\psi$ vanishes because of eqn of motion for the matter fields ψ:)



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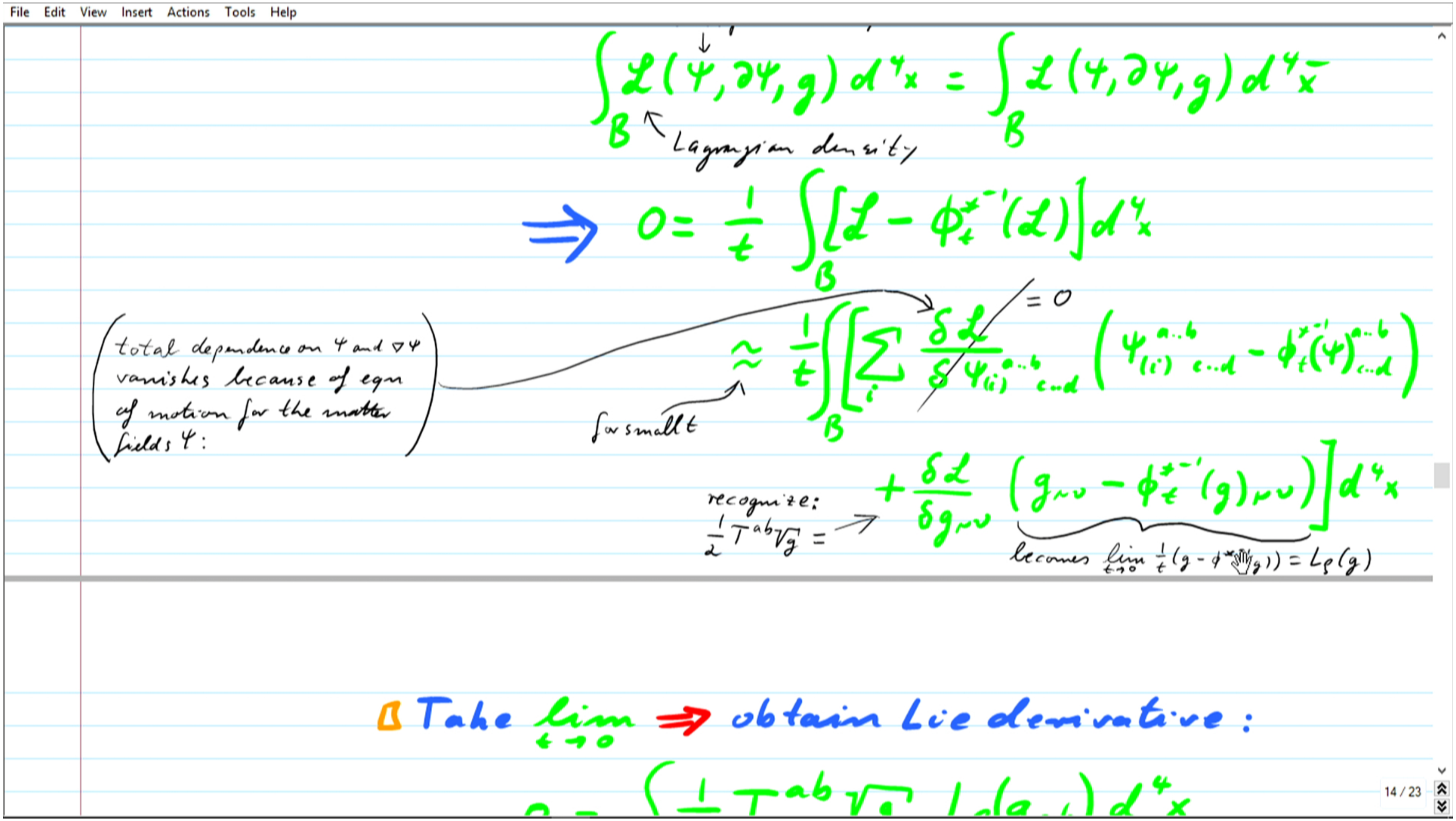
(total dependence on Ψ and $\nabla\Psi$ vanishes because of eqn of motion for the matter fields Ψ:)

for small t

$$\approx \frac{1}{t}\int_B \Big[\sum_i \cancel{\frac{\delta\mathcal{L}}{\delta\Psi_{(i)}{}^{a..b}{}_{c..d}}} = 0 \left(\Psi_{(i)}{}^{a..b}{}_{c..d} - \phi_t^{*-1}(\Psi)^{a..b}{}_{c..d}\right)$$

recognize: $\frac{1}{2}T^{ab}\sqrt{g} =$

$$+ \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} \underbrace{\left(g_{\mu\nu} - \phi_t^{*-1}(g)_{\mu\nu}\right)}_{\text{becomes } \lim_{t\to 0}\frac{1}{t}(g - \phi^{*-1}(g)) = L_\xi(g)}\Big] d^4x$$



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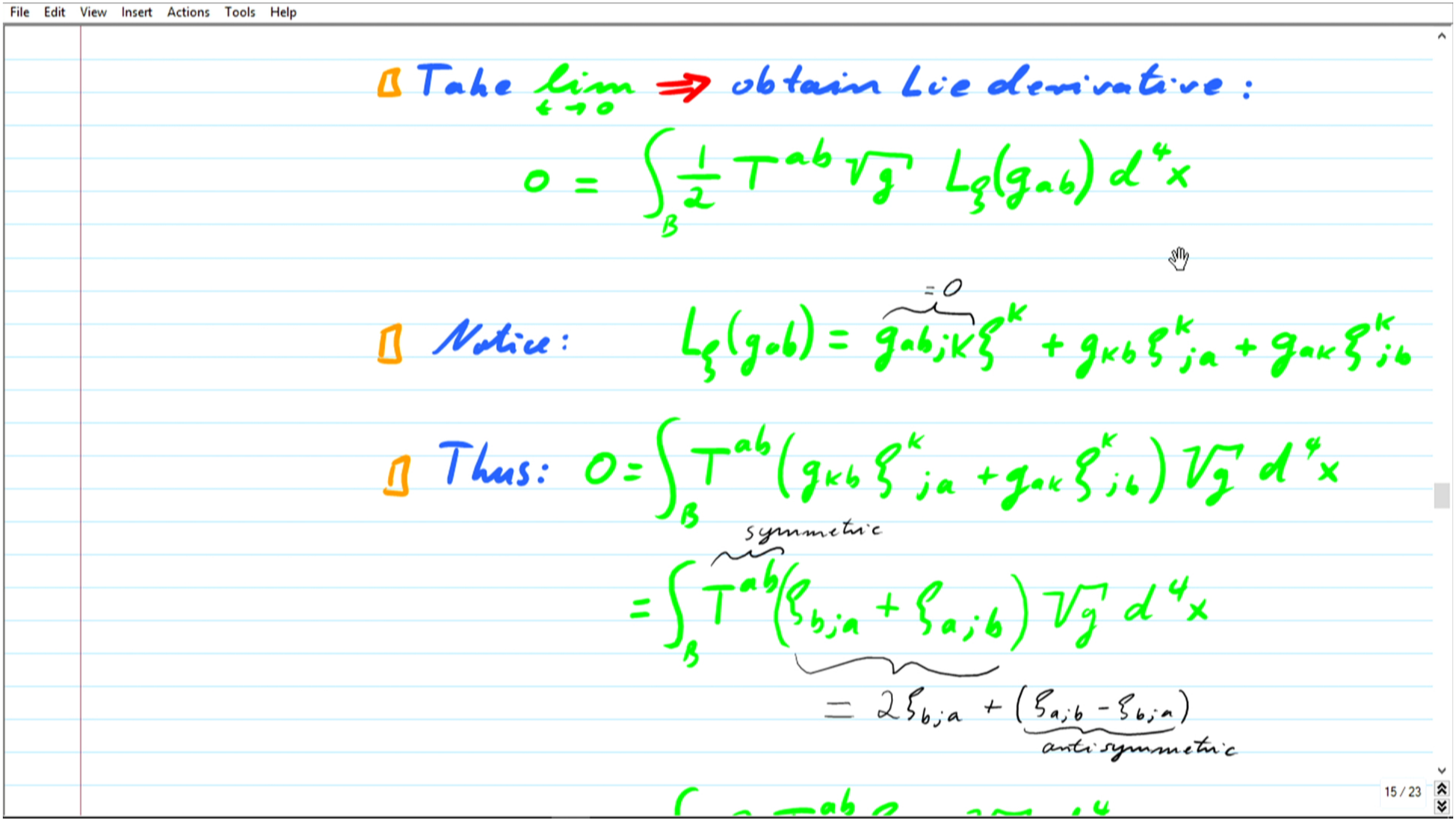
(total dependence on ψ and $\nabla\psi$ vanishes because of eqn of motion for the matter fields ψ:)

recognize: $\frac{1}{2}T^{ab}\sqrt{g} =$ →

□ Take $\lim_{t\to 0}$ ⇒ obtain Lie derivative :

$$0 = \int \frac{1}{\ } T^{ab}\sqrt{g}\; L(g)\, d^4x$$



- Take $\lim_{\epsilon \to 0}$ ⇒ obtain Lie derivative:

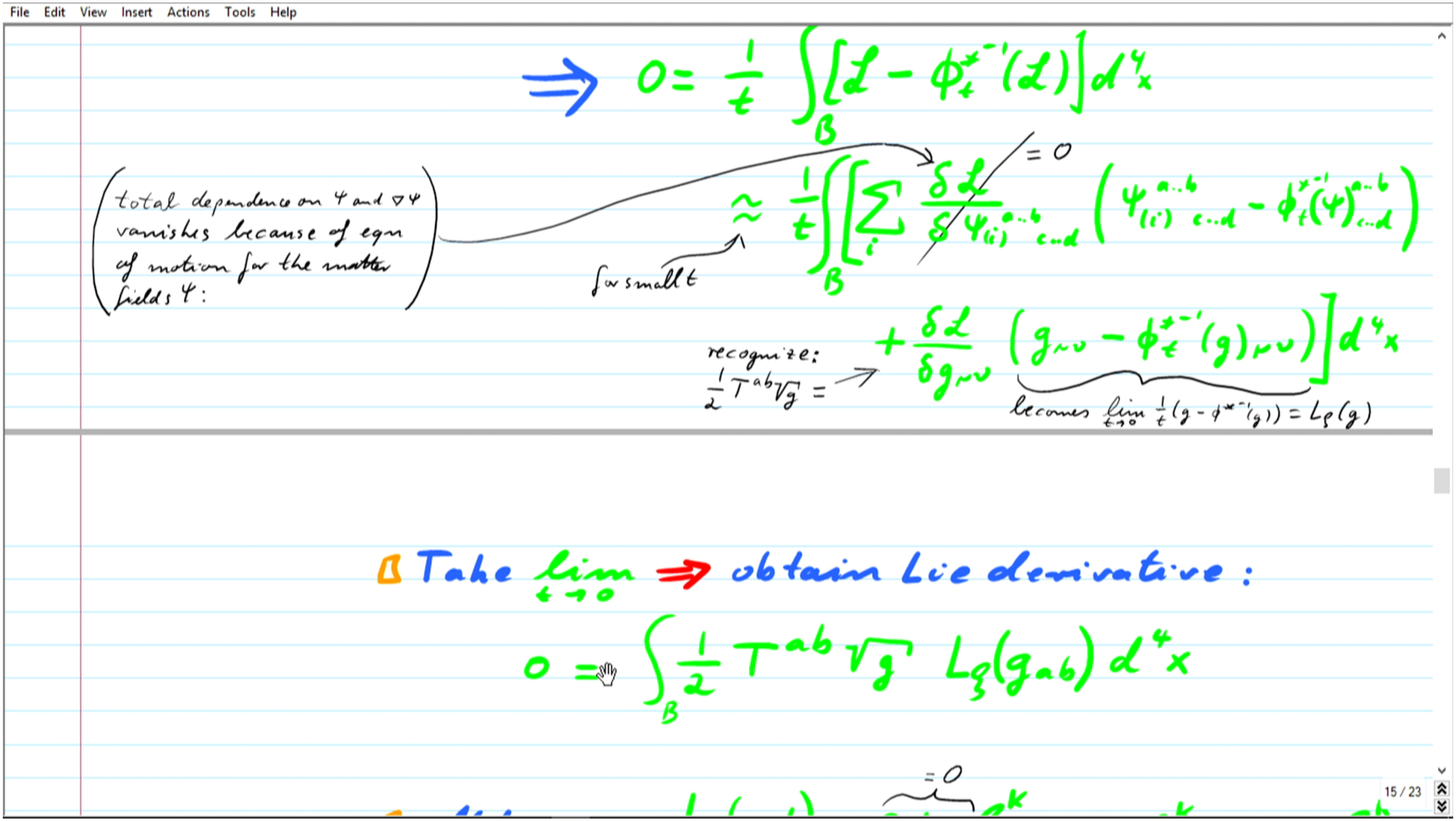
$$0 = \int_B \frac{1}{2} T^{ab} \sqrt{g}\; L_\xi(g_{ab})\, d^4x$$

- Notice: $L_\xi(g_{ab}) = \overbrace{g_{ab;k}}^{=0}\xi^k + g_{kb}\xi^k{}_{;a} + g_{ak}\xi^k{}_{;b}$

- Thus: $0 = \int_B T^{ab}\left(g_{kb}\xi^k{}_{;a} + g_{ak}\xi^k{}_{;b}\right)\sqrt{g}\; d^4x$

$$= \int_B \overbrace{T^{ab}}^{\text{symmetric}} \underbrace{\left(\xi_{b;a} + \xi_{a;b}\right)} \sqrt{g}\; d^4x$$

$$= 2\xi_{b;a} + \underbrace{\left(\xi_{a;b} - \xi_{b;a}\right)}_{\text{antisymmetric}}$$



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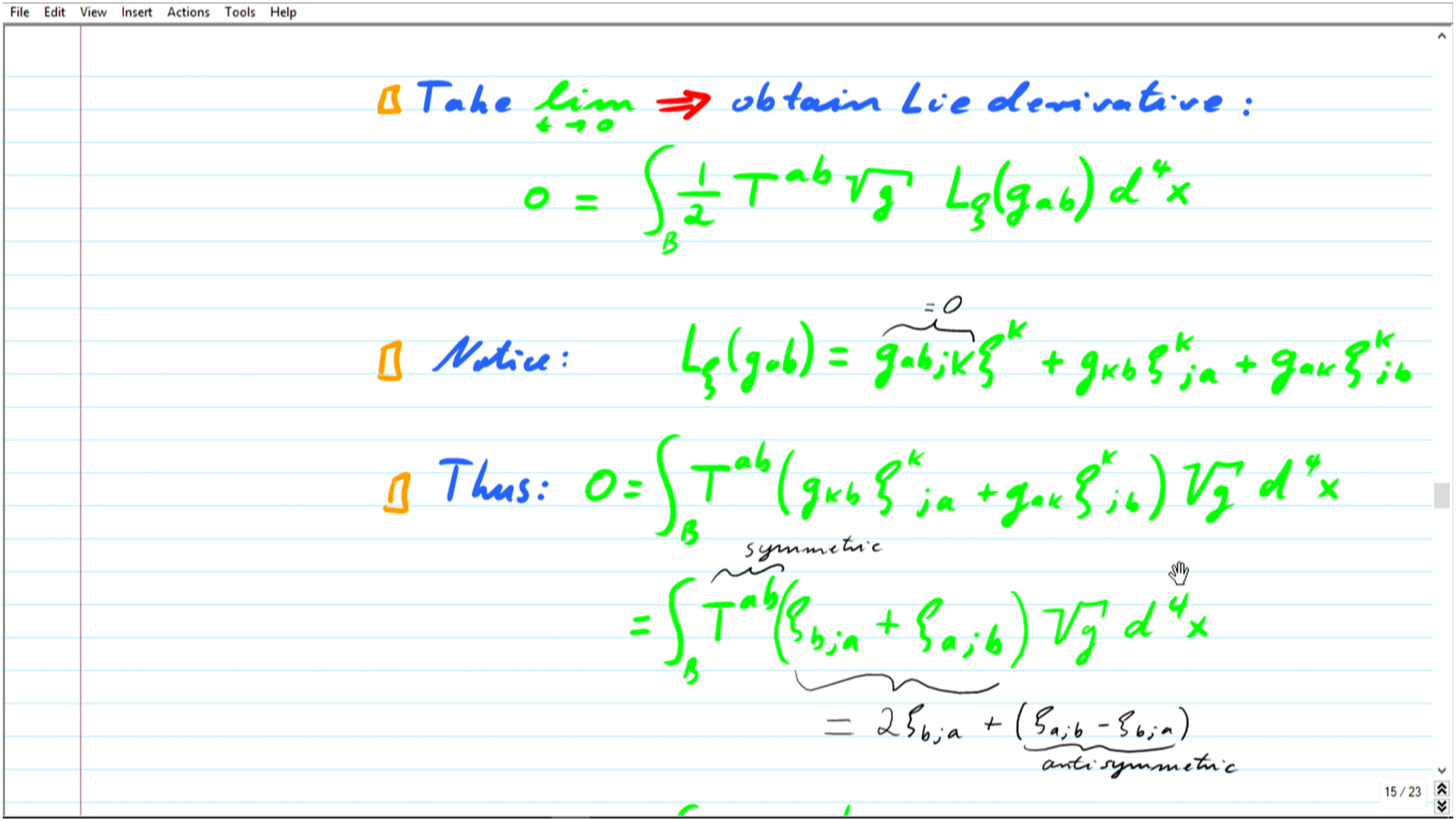

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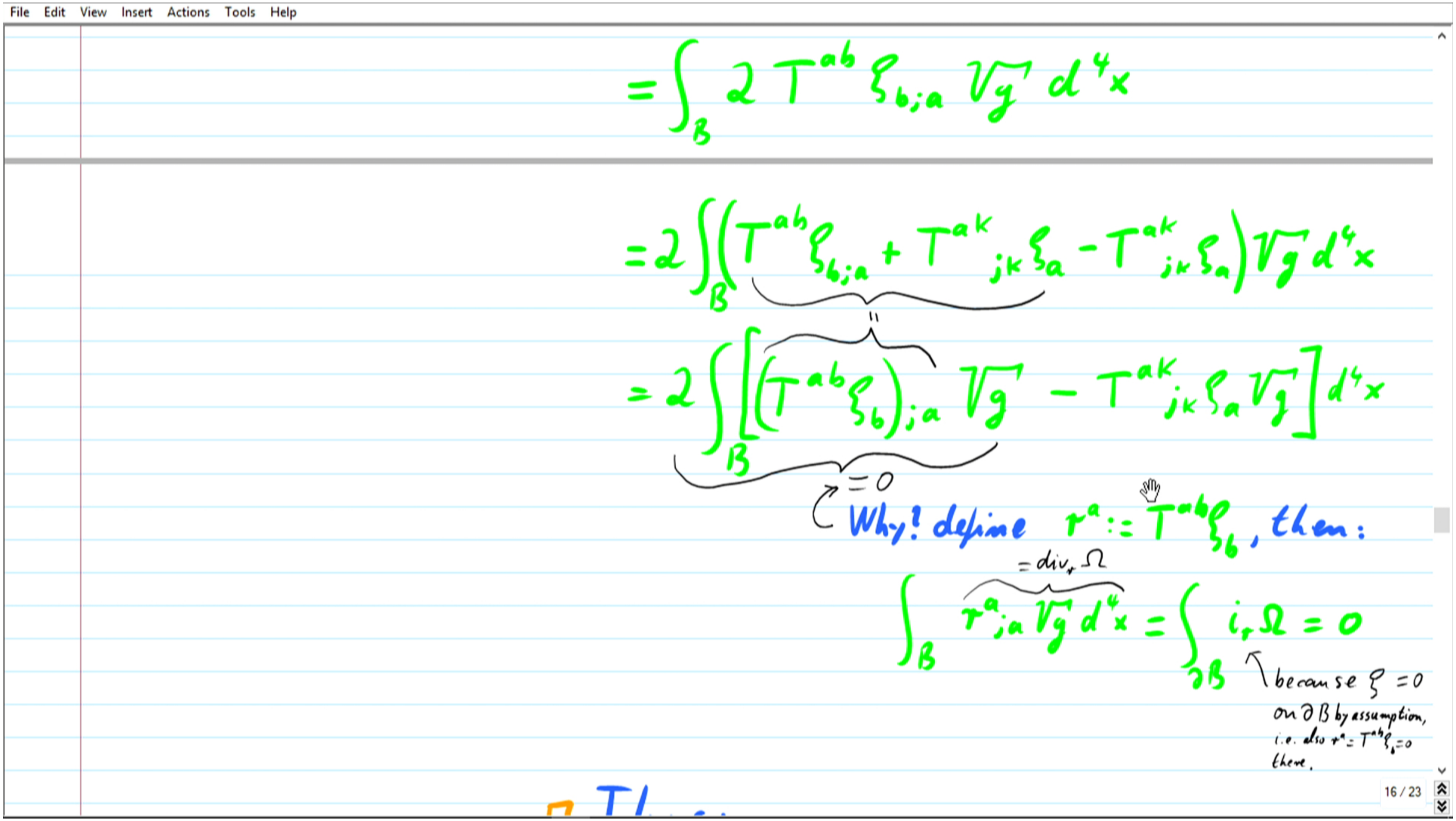


$$= \int_B 2\, T^{ab}\,\xi_{b;a}\,\sqrt{g}\; d^4x$$

$$= 2\int_B \Big(\underbrace{T^{ab}\xi_{b;a} + T^{ak}{}_{;k}\xi_a} - T^{ak}{}_{;k}\xi_a\Big)\sqrt{g}\; d^4x$$

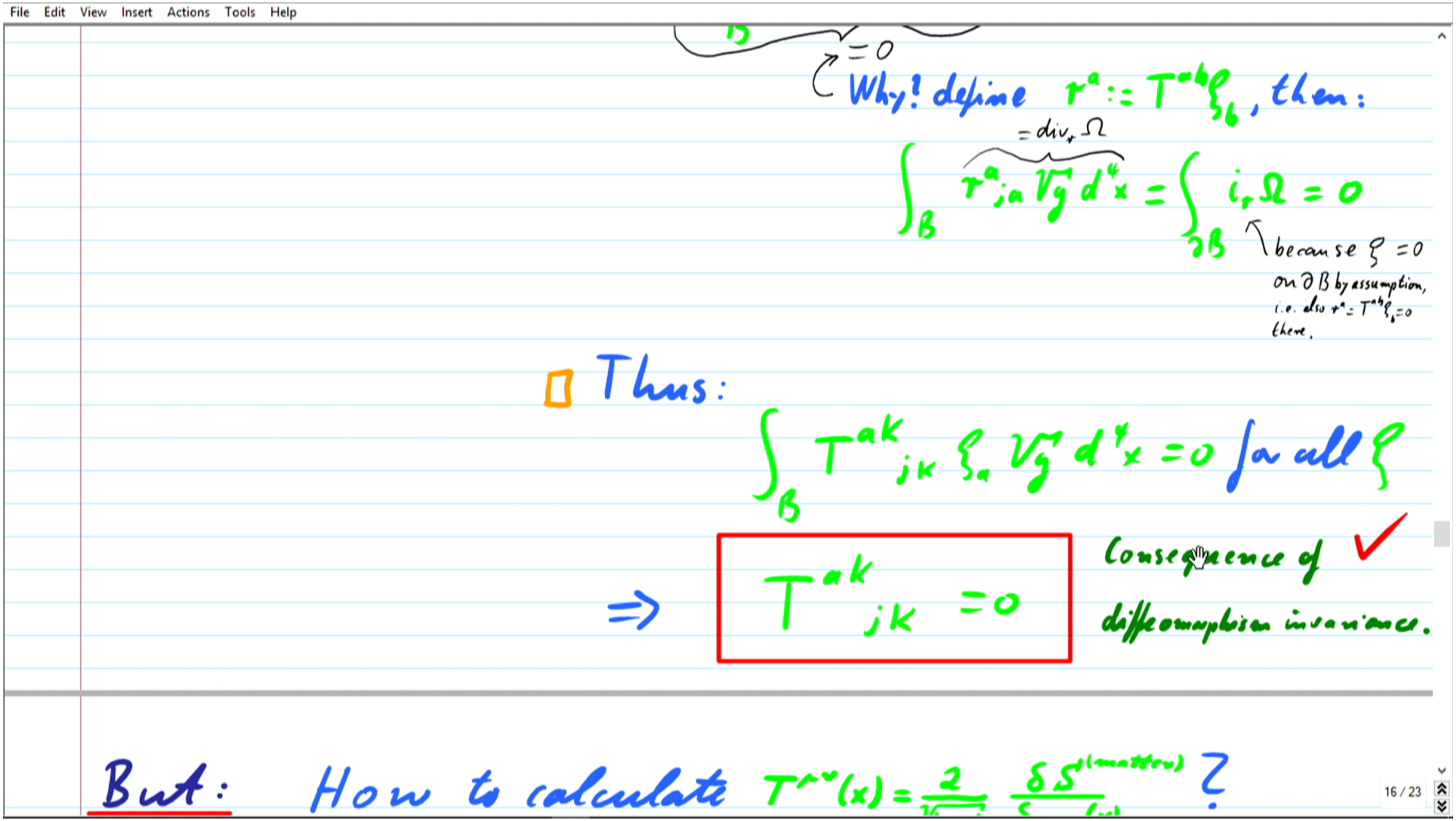
$$\|$$

$$= 2\underbrace{\int_B \Big[\overbrace{(T^{ab}\xi_b)_{;a}}\sqrt{g}}_{=0} - T^{ak}{}_{;k}\xi_a\sqrt{g}\Big] d^4x$$

Why? define $r^a := T^{ab}\xi_b$, then:

$$\int_B \overbrace{r^a{}_{;a}\sqrt{g}\; d^4x}^{=\mathrm{div}_r\,\Omega} = \int_{\partial B} i_r\Omega = 0$$

because $\xi = 0$ on ∂B by assumption, i.e. also $r^a = T^{ab}\xi_b = 0$ there.



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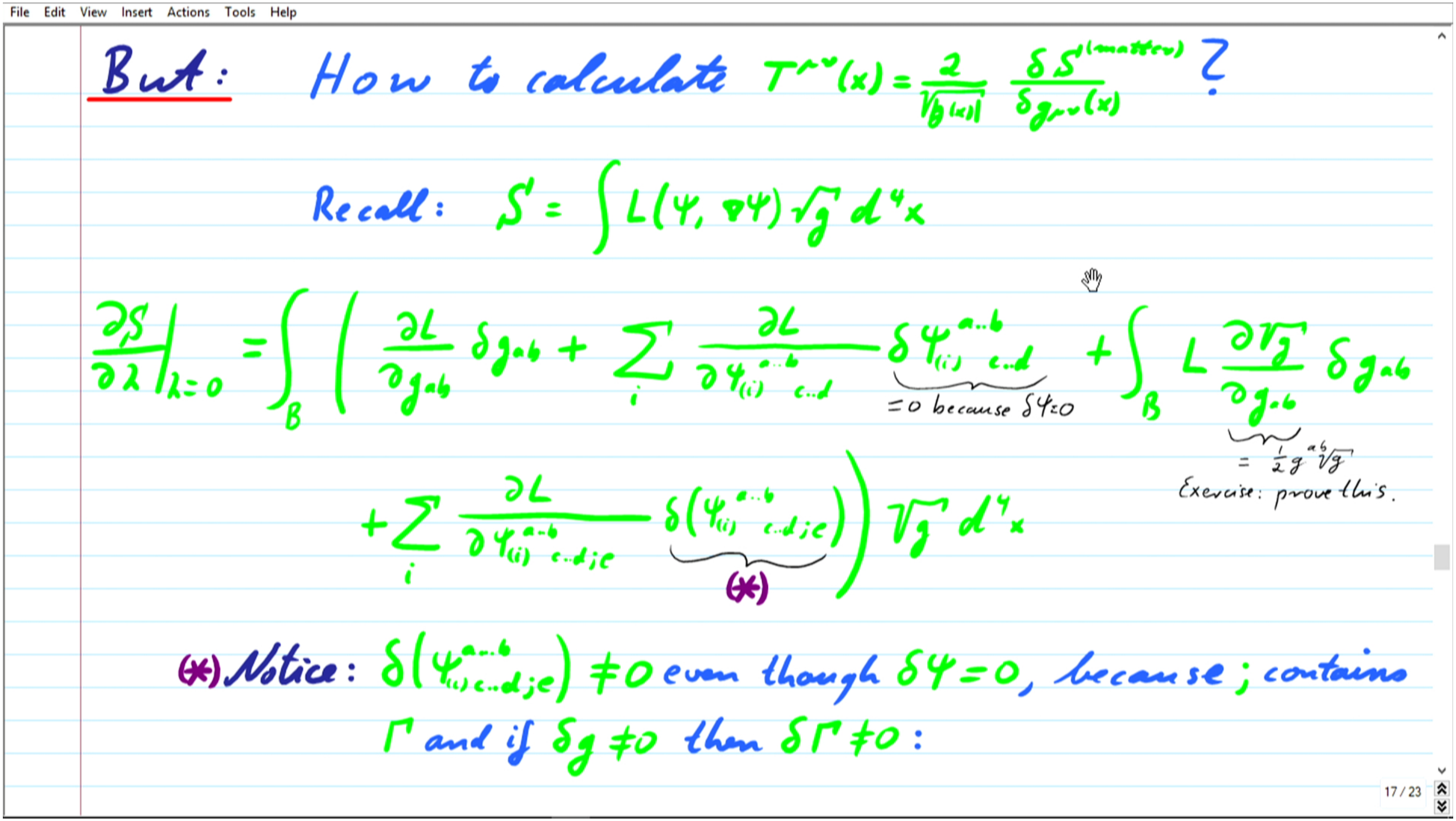
☐ Thus:

$$\int_B T^{ak}{}_{;k}\,\xi_a\sqrt{g}\,d^4x = 0 \text{ for all } \xi$$

$$\Rightarrow \boxed{T^{ak}{}_{;k} = 0}$$

Consequence of diffeomorphism invariance. ✓

But: How to calculate $T^{\mu\nu}(x) = \frac{2}{\sqrt{}}\,\frac{\delta S^{(\mathrm{matter})}}{\delta }$?



But: How to calculate $T^{\mu\nu}(x) = \frac{2}{\sqrt{|g(x)|}} \frac{\delta S^{(matter)}}{\delta g_{\mu\nu}(x)}$?

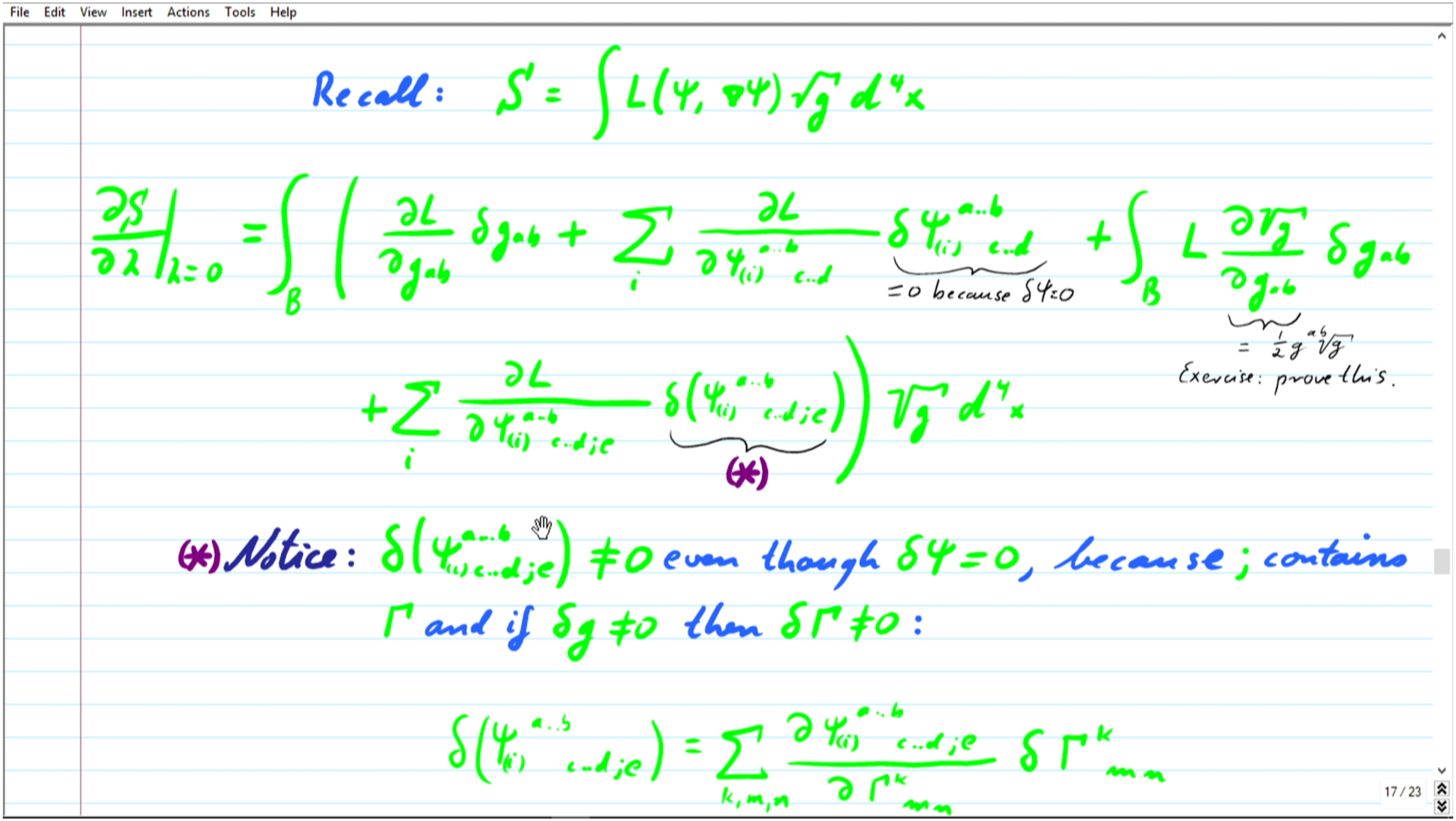
Recall: $S = \int L(\psi, \nabla\psi)\sqrt{g}\, d^4x$

$$\frac{\partial S}{\partial \lambda}\Big|_{\lambda=0} = \int_B \Big(\frac{\partial L}{\partial g_{ab}} \delta g_{ab} + \sum_i \frac{\partial L}{\partial \psi_{(i)}{}^{a..b}{}_{c..d}} \underbrace{\delta \psi_{(i)}{}^{a..b}{}_{c..d}}_{=0 \text{ because } \delta\psi=0} + \int_B L \underbrace{\frac{\partial \sqrt{g}}{\partial g_{ab}}}_{=\frac{1}{2} g^{ab}\sqrt{g}} \delta g_{ab}$$

Exercise: prove this.

$$+ \sum_i \frac{\partial L}{\partial \psi_{(i)}{}^{a..b}{}_{c..d;e}} \underbrace{\delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right)}_{(*)} \Big) \sqrt{g}\, d^4x$$

(*) **Notice:** $\delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right) \neq 0$ even though $\delta\psi = 0$, because ; contains Γ and if $\delta g \neq 0$ then $\delta\Gamma \neq 0$:



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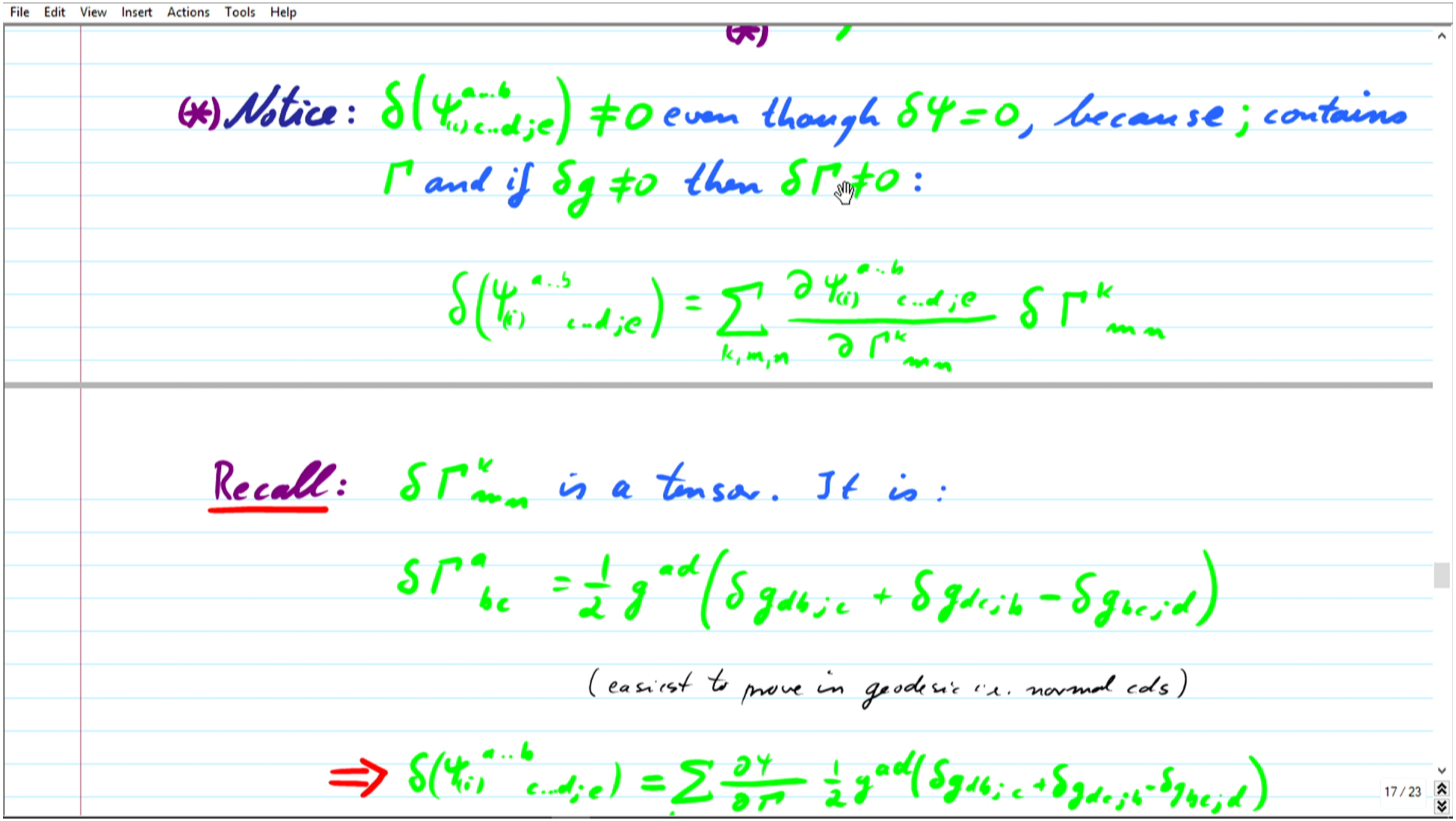
$$\delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right) = \sum_{k,m,n} \frac{\partial \psi_{(i)}{}^{a..b}{}_{c..d;e}}{\partial \Gamma^{k}{}_{mn}} \delta \Gamma^{k}{}_{mn}$$



(*) **Notice:** $\delta(\Psi_{(i)}{}^{a..b}{}_{c..d;e}) \neq 0$ even though $\delta\Psi = 0$, because ; contains Γ and if $\delta g \neq 0$ then $\delta\Gamma \neq 0$:

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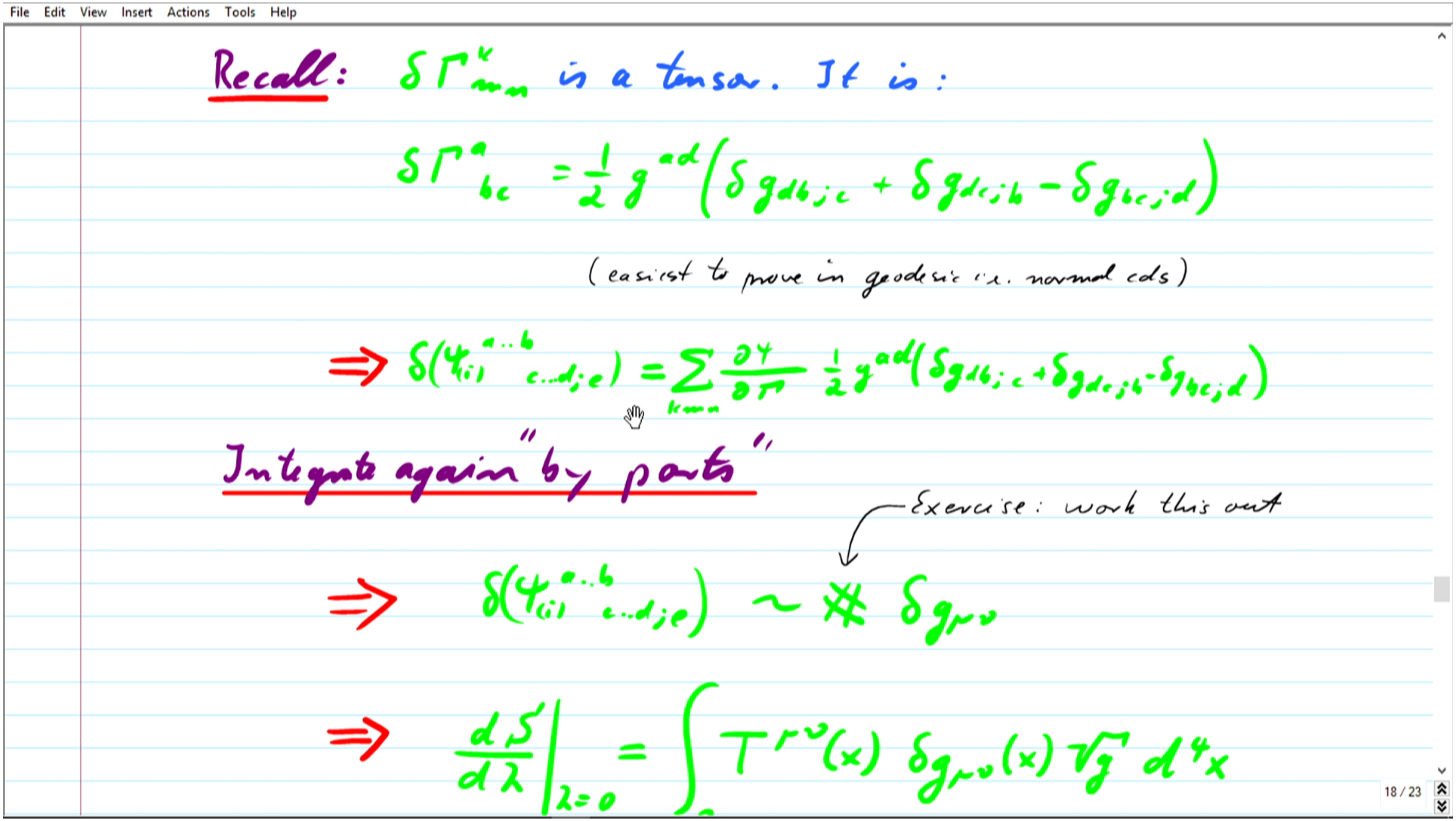
Recall: $\delta\Gamma^{k}{}_{mn}$ is a tensor. It is:

$$\delta\Gamma^{a}{}_{bc} = \frac{1}{2} g^{ad}\left(\delta g_{db;c} + \delta g_{dc;b} - \delta g_{bc;d}\right)$$

(easiest to prove in geodesic i.e. normal cds)

$$\Rightarrow \delta(\Psi_{(i)}{}^{a..b}{}_{c..d;e}) = \sum \frac{\partial\Psi}{\partial\Gamma} \frac{1}{2} g^{ad}\left(\delta g_{db;c} + \delta g_{dc;b} - \delta g_{bc;d}\right)$$



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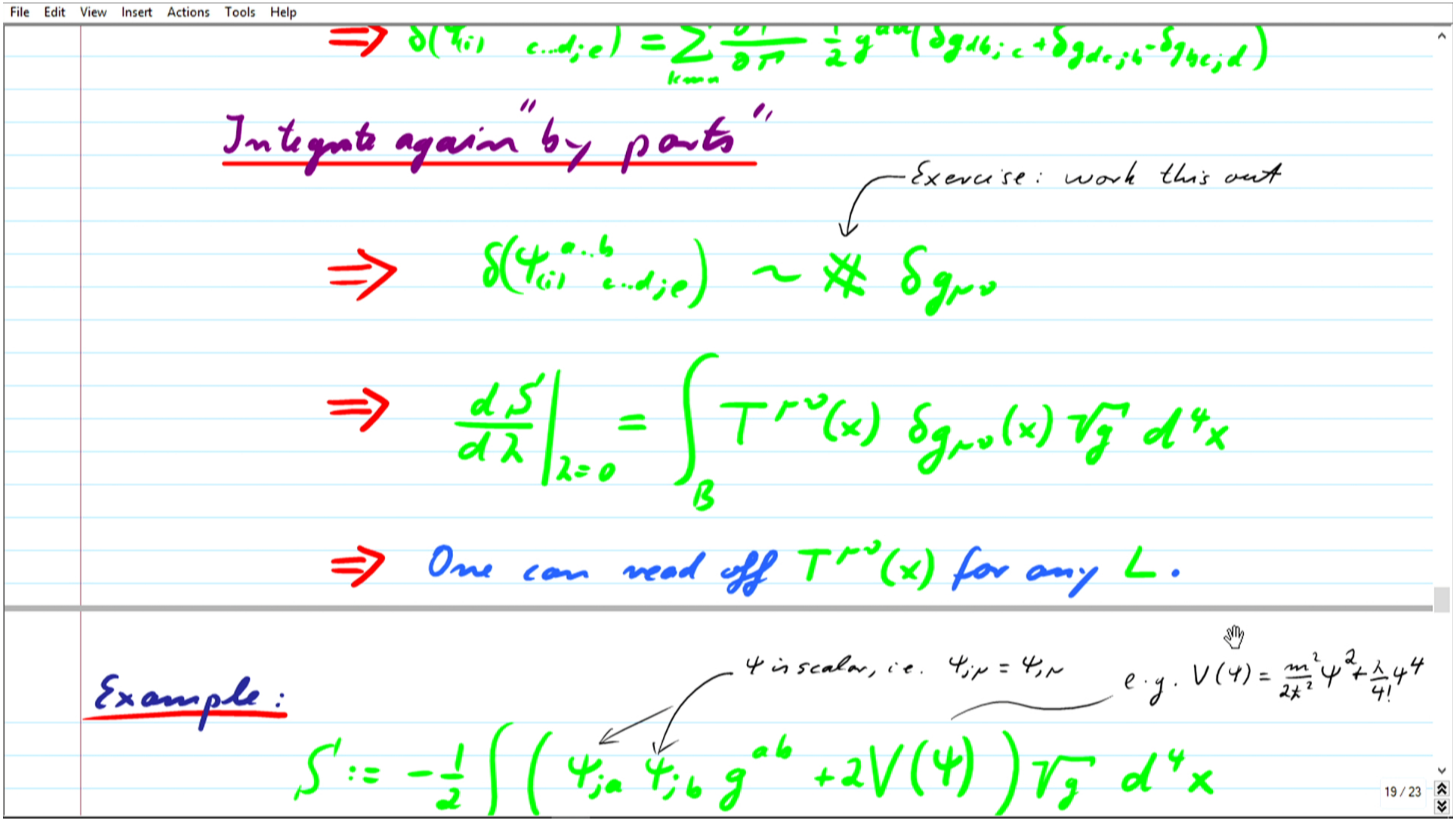
$$\Rightarrow \delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right) = \sum_{kmn} \frac{\partial\psi}{\partial\Gamma} \frac{1}{2} g^{ad}\left(\delta g_{db;c} + \delta g_{dc;b} - \delta g_{bc;d}\right)$$

Integrate again "by parts"

Exercise: work this out

$$\Rightarrow \delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right) \sim \# \, \delta g_{\mu\nu}$$

$$\Rightarrow \frac{dS}{d\lambda}\Big|_{\lambda=0} = \int T^{\mu\nu}(x)\, \delta g_{\mu\nu}(x) \sqrt{g}\, d^4x$$



$$\Rightarrow \delta(\psi_{(i)}{}_{c\ldots d;e}) = \sum_{k\ldots n} \frac{\partial}{\partial T} \frac{1}{2} g^{a}(\delta g_{db;c} + \delta g_{dc;b} - \delta g_{bc;d})$$

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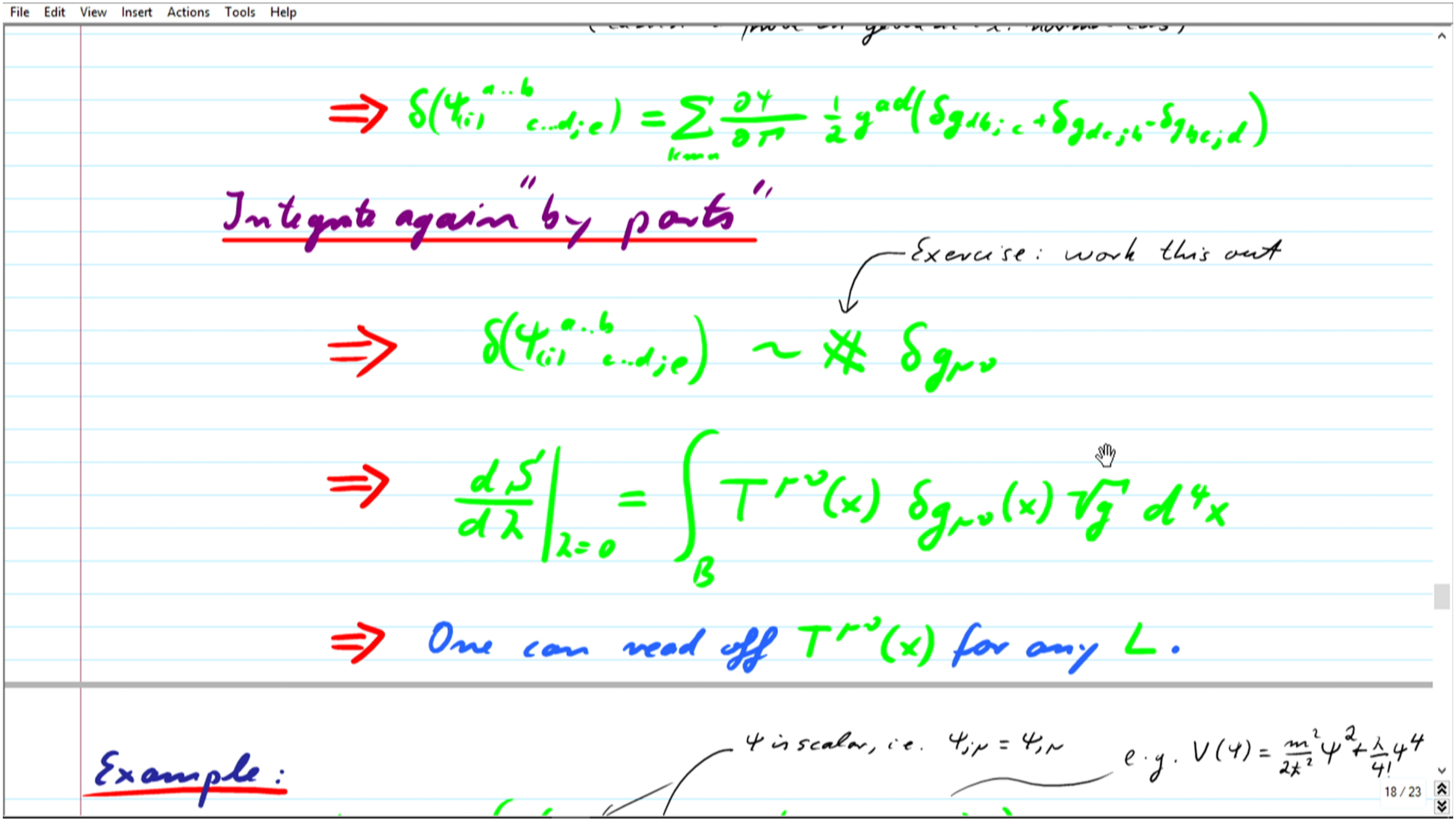
$\Rightarrow$ One can read off $T^{\mu\nu}(x)$ for any L.

Example:

ψ is scalar, i.e. $\psi_{;\mu} = \psi_{,\mu}$

e.g. $V(\psi) = \frac{m^2}{2\hbar^2}\psi^2 + \frac{\lambda}{4!}\psi^4$

$$S := -\frac{1}{2} \int \left(\psi_{;a} \psi_{;b} \, g^{ab} + 2V(\psi) \right) \sqrt{g} \, d^4x$$



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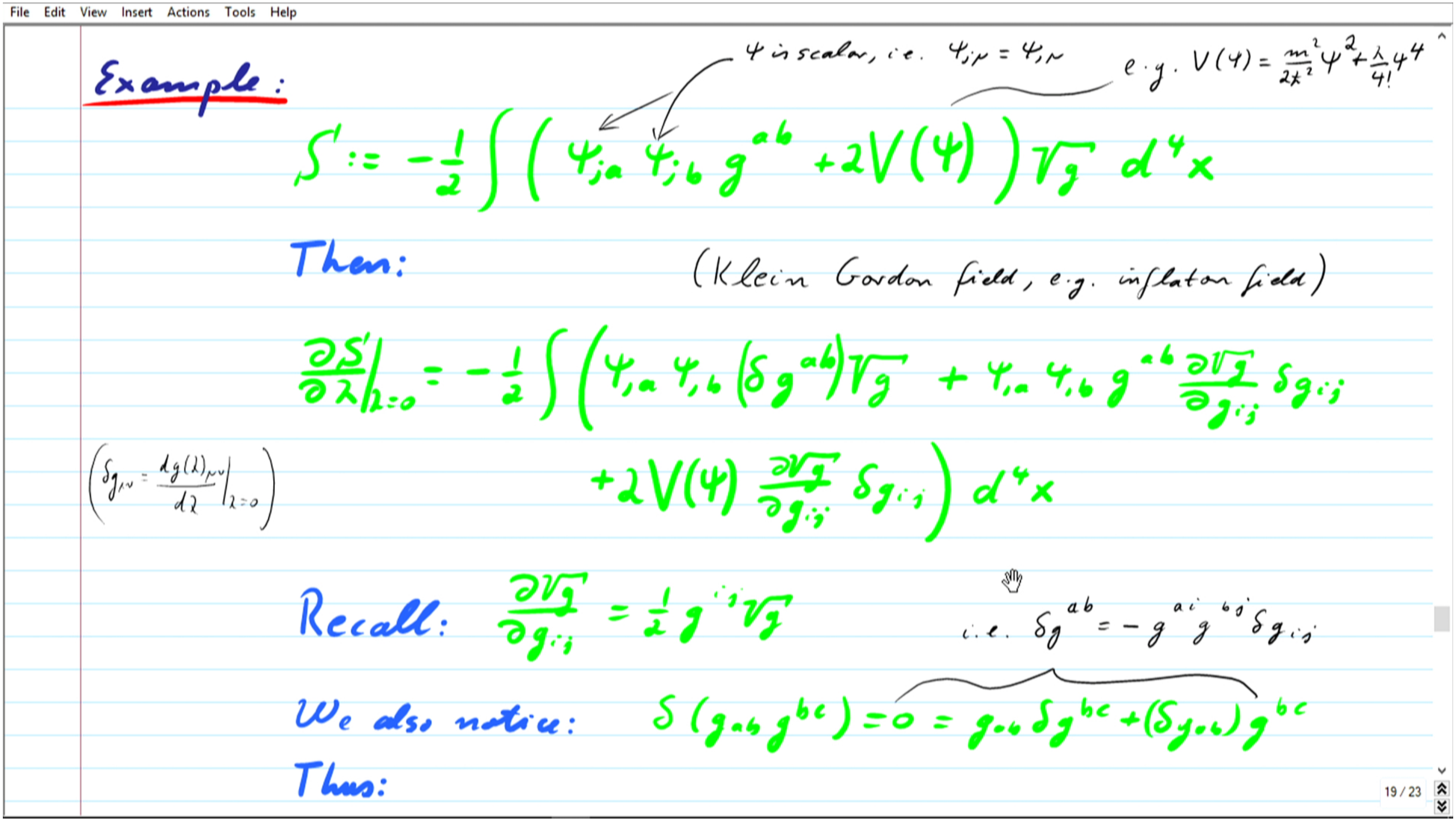

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Then:

(Klein Gordon field, e.g. inflaton field)

$$\left.\frac{\partial S'}{\partial \lambda}\right|_{\lambda=0} = -\frac{1}{2}\int \Big(\psi_{,a}\psi_{,b}\left(\delta g^{ab}\right)\sqrt{g} + \psi_{,a}\psi_{,b}\, g^{ab}\frac{\partial \sqrt{g}}{\partial g_{ij}}\,\delta g_{ij}$$

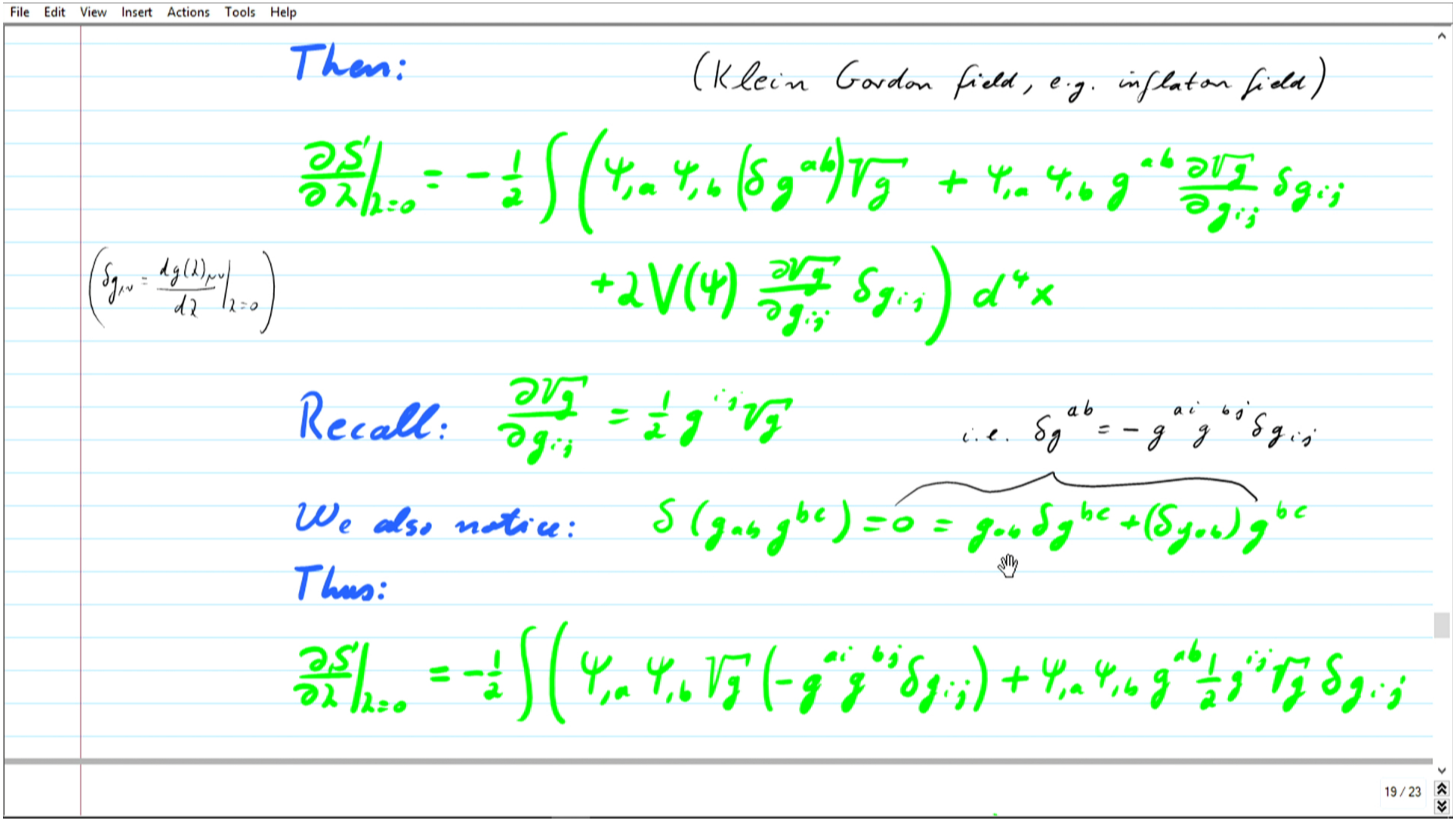
$$\left(\delta g_{\mu\nu} = \left.\frac{dg(\lambda)_{\mu\nu}}{d\lambda}\right|_{\lambda=0} \right) \qquad + 2V(\psi)\frac{\partial\sqrt{g}}{\partial g_{ij}}\,\delta g_{ij} \Big)\, d^4x$$

Recall: $\frac{\partial \sqrt{g}}{\partial g_{ij}} = \frac{1}{2} g^{ij}\sqrt{g}$

i.e. $\delta g^{ab} = -g^{ai}g^{bj}\,\delta g_{ij}$

We also notice: $\delta\left(g_{ab}\,g^{bc}\right) = \overbrace{0 = g_{ab}\,\delta g^{bc} + \left(\delta g_{ab}\right) g^{bc}}$

Thus:



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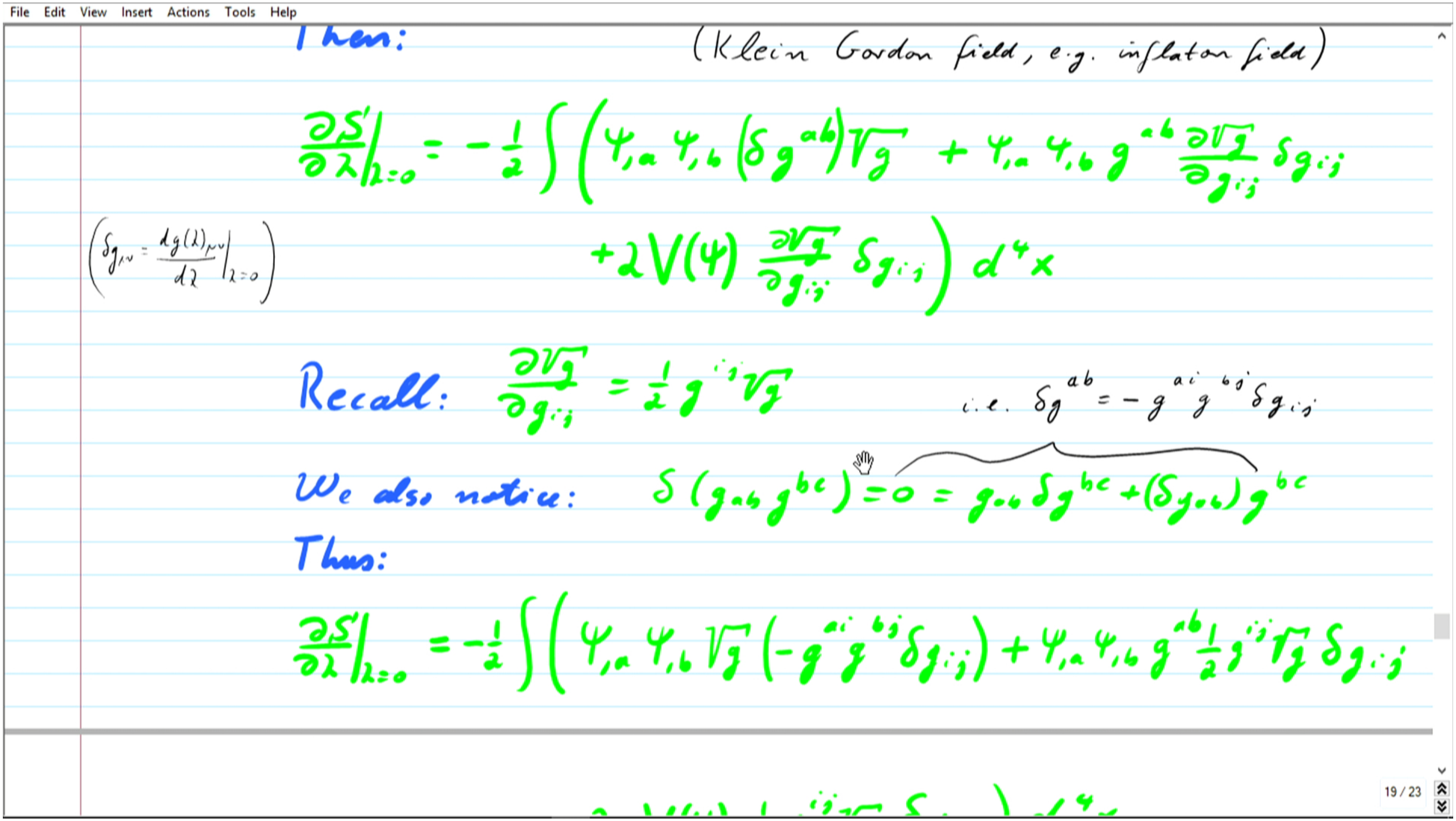

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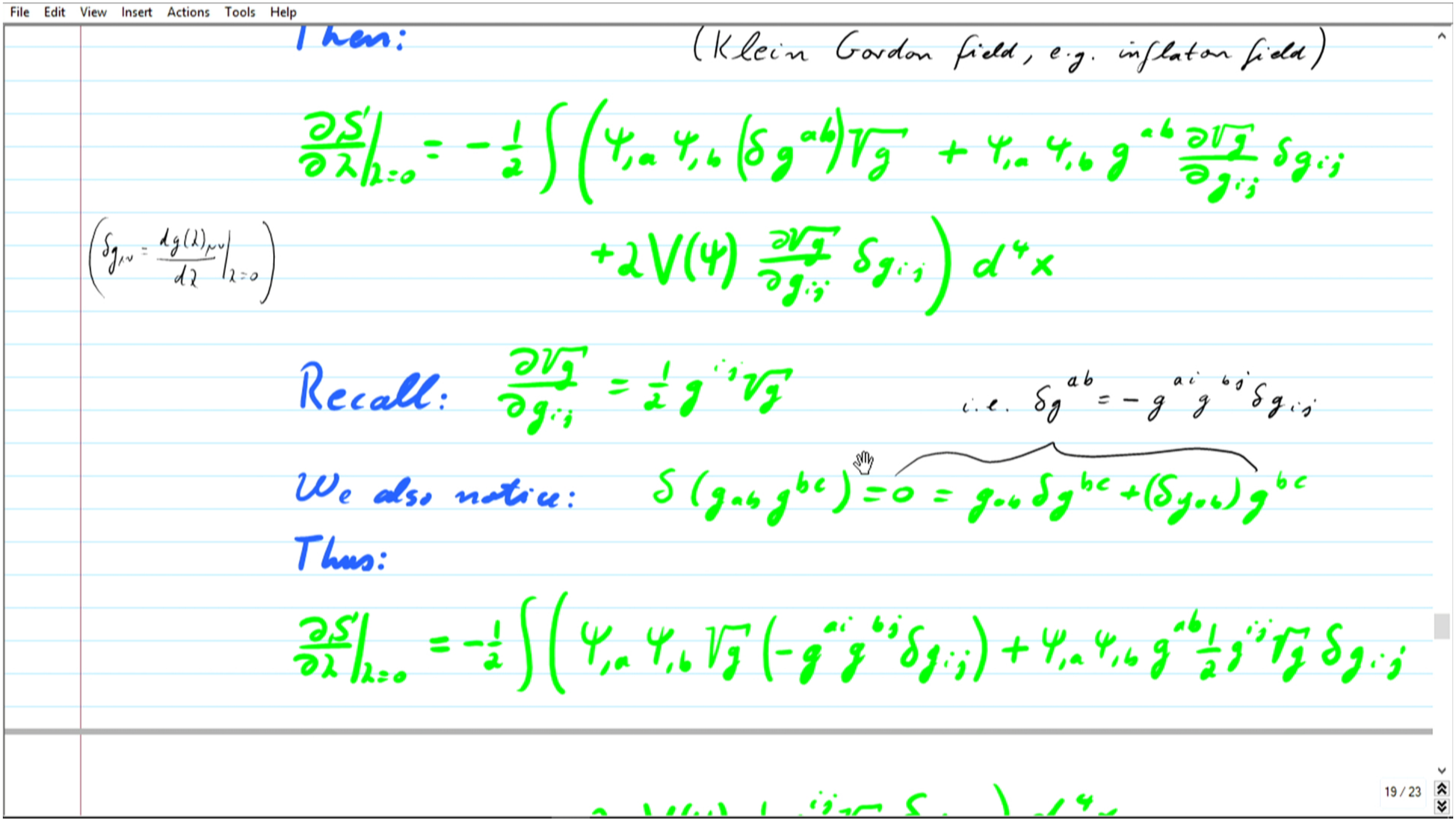

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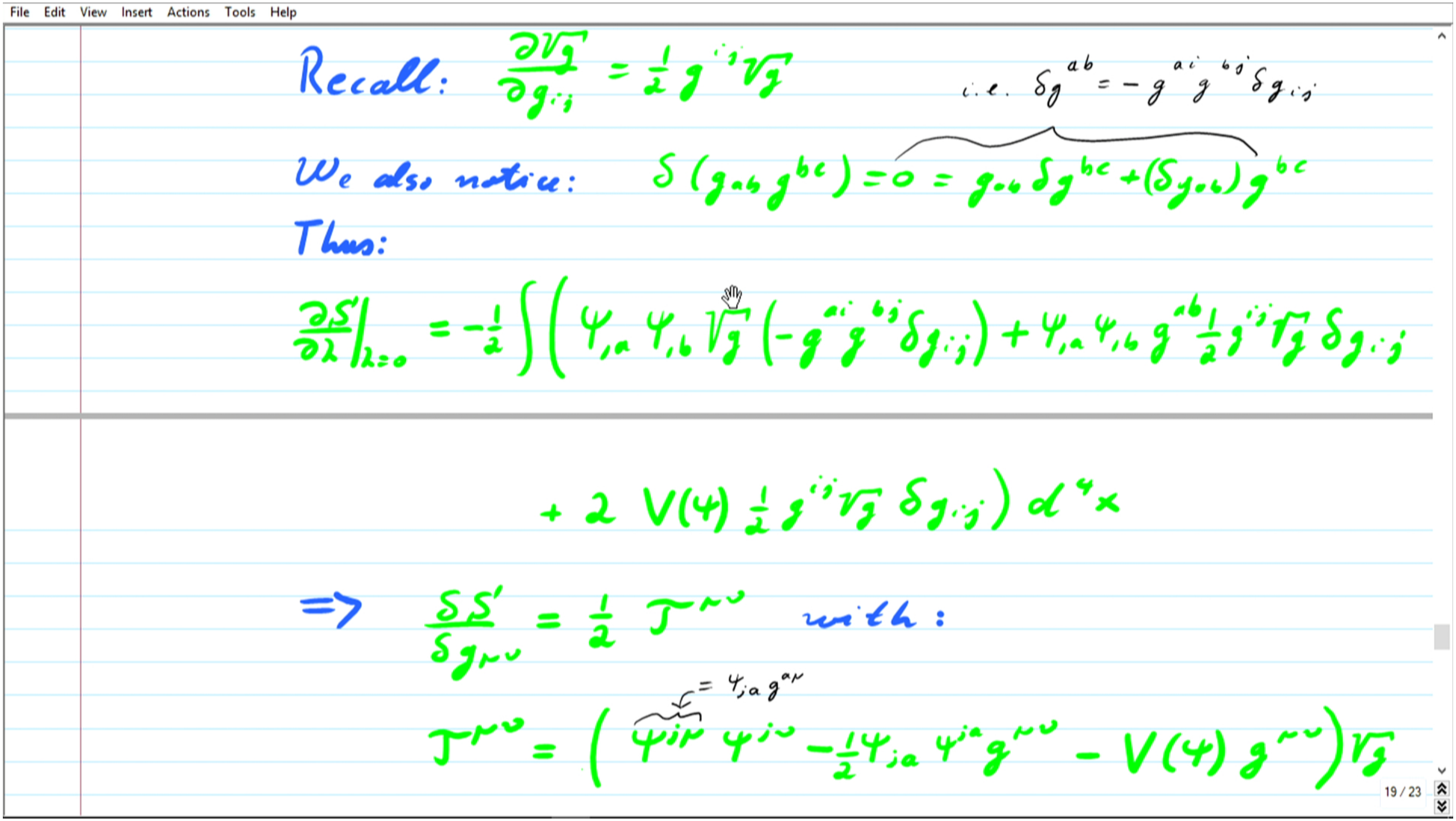

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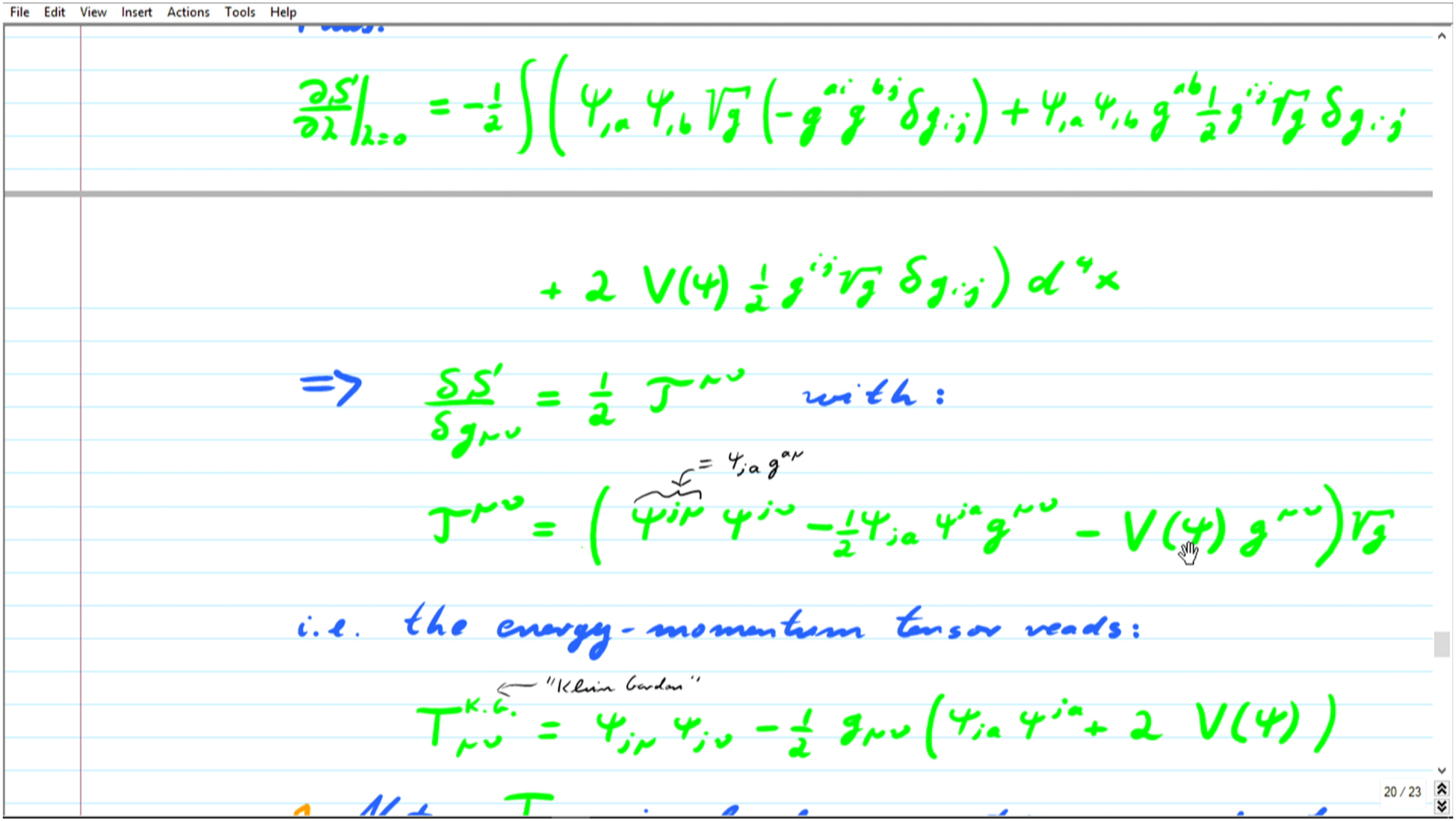
$\Rightarrow$ $\frac{\delta S'}{\delta g_{\mu\nu}} = \frac{1}{2} T^{\mu\nu}$ with:

$$T^{\mu\nu} = \Big(\overbrace{\phi^{;\mu}}^{= \phi_{;a} g^{a\mu}} \phi^{;\nu} - \frac{1}{2} \phi_{;a} \phi^{;a} g^{\mu\nu} - V(\phi) g^{\mu\nu} \Big) \sqrt{g}$$



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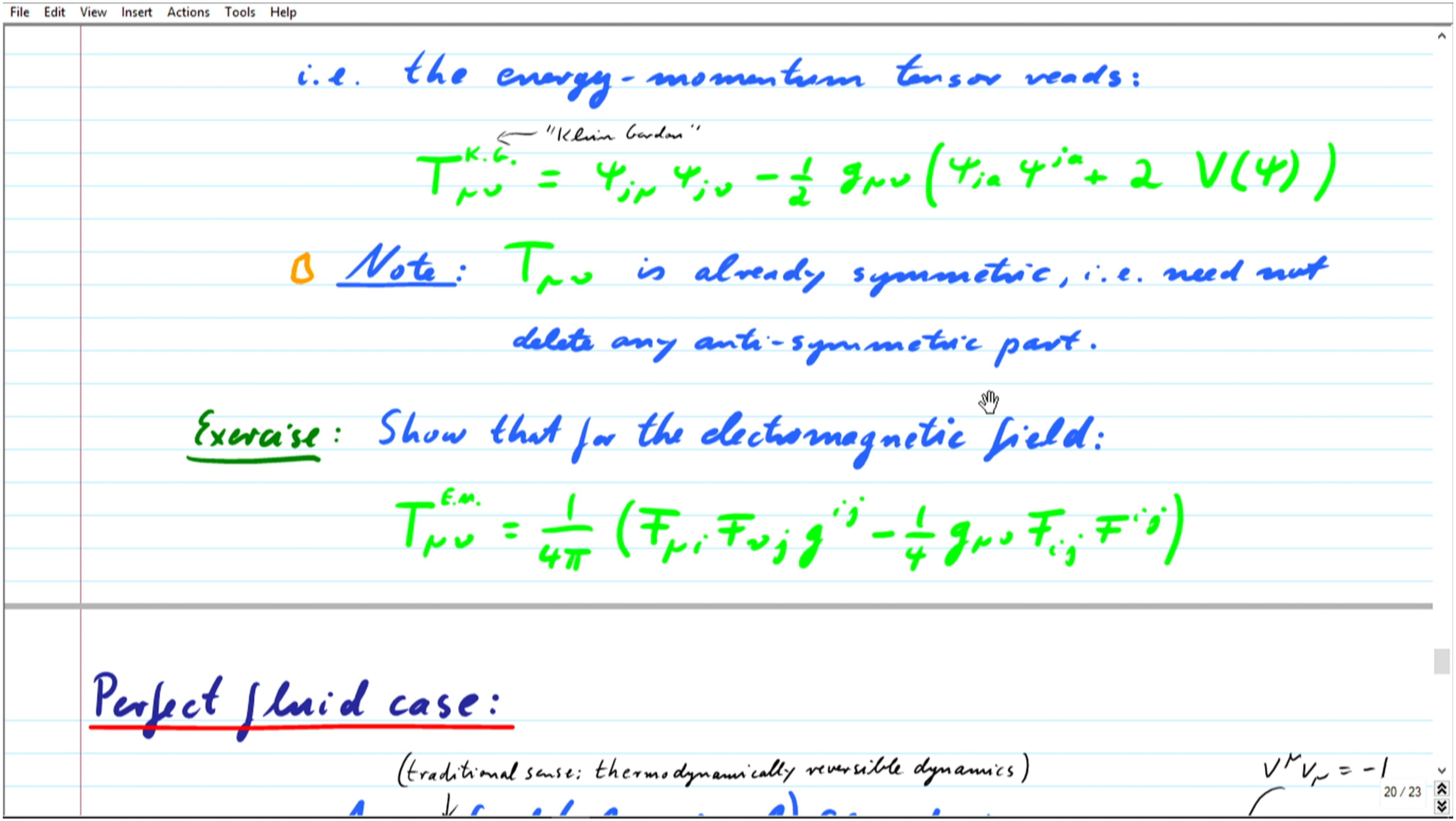
i.e. the energy-momentum tensor reads:

$$\overset{\longleftarrow \text{"Klein Gordon"}}{T^{K.G.}_{\mu\nu}} = \varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}\,g_{\mu\nu}\left(\varphi_{;a}\varphi^{;a} + 2\,V(\varphi)\right)$$



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Note: $T_{\mu\nu}$ is already symmetric, i.e. need not delete any anti-symmetric part.

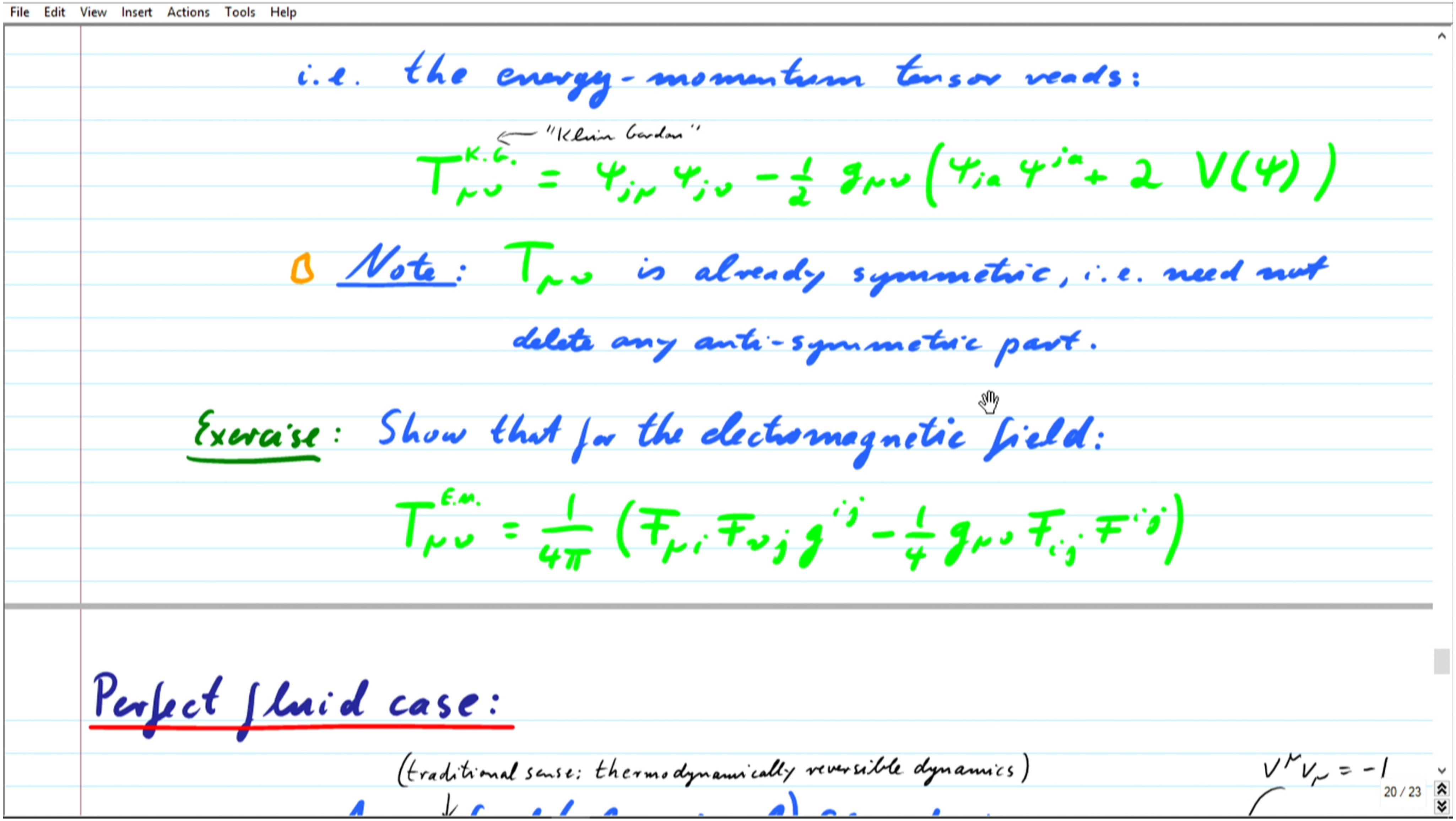
Exercise: Show that for the electromagnetic field:

$$T^{E.M.}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu i}F_{\nu j}g^{ij} - \frac{1}{4} g_{\mu\nu} F_{ij}F^{ij}\right)$$

Perfect fluid case:

(traditional sense: thermodynamically reversible dynamics)

$V^\mu V_\mu = -1$



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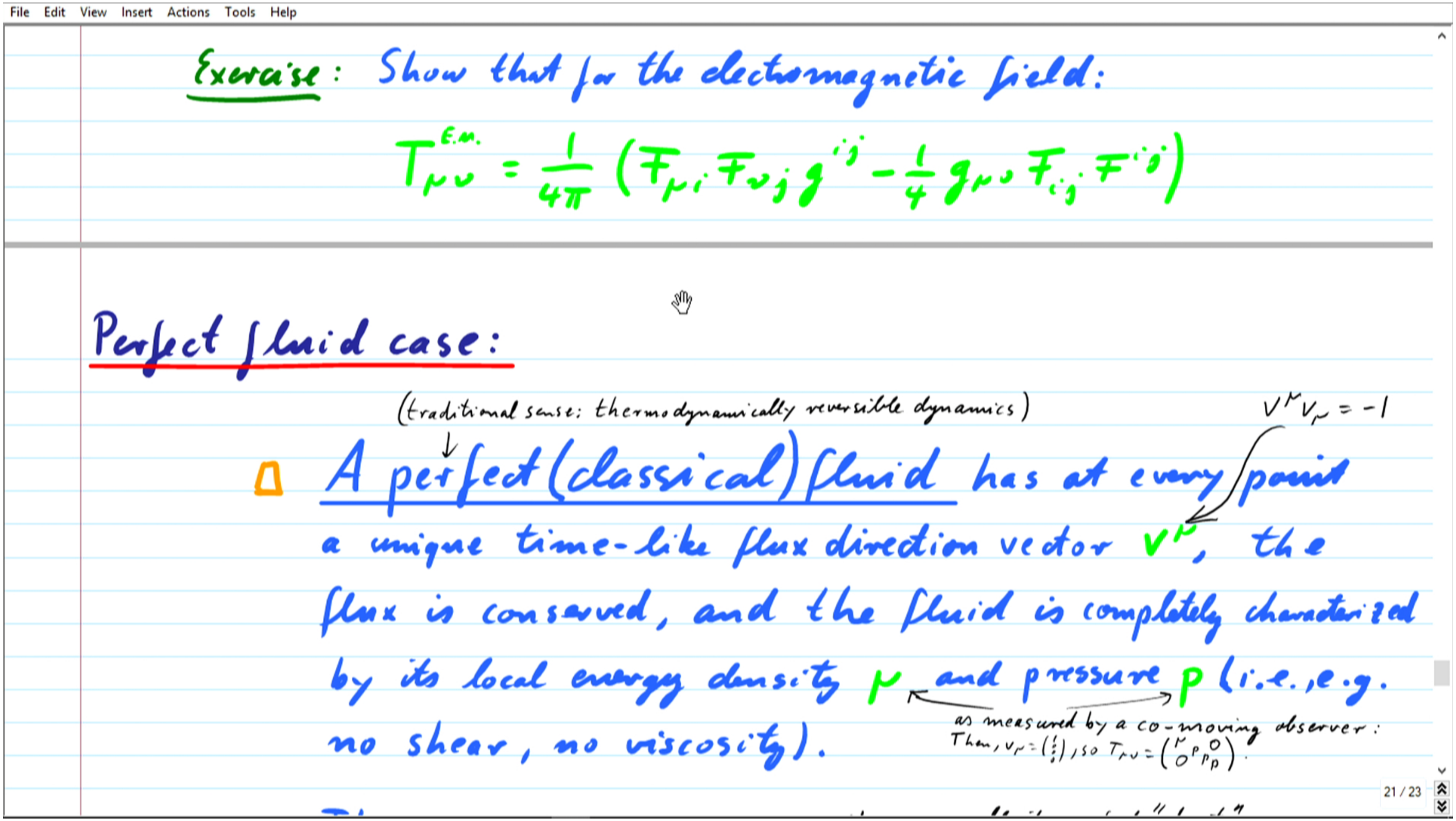

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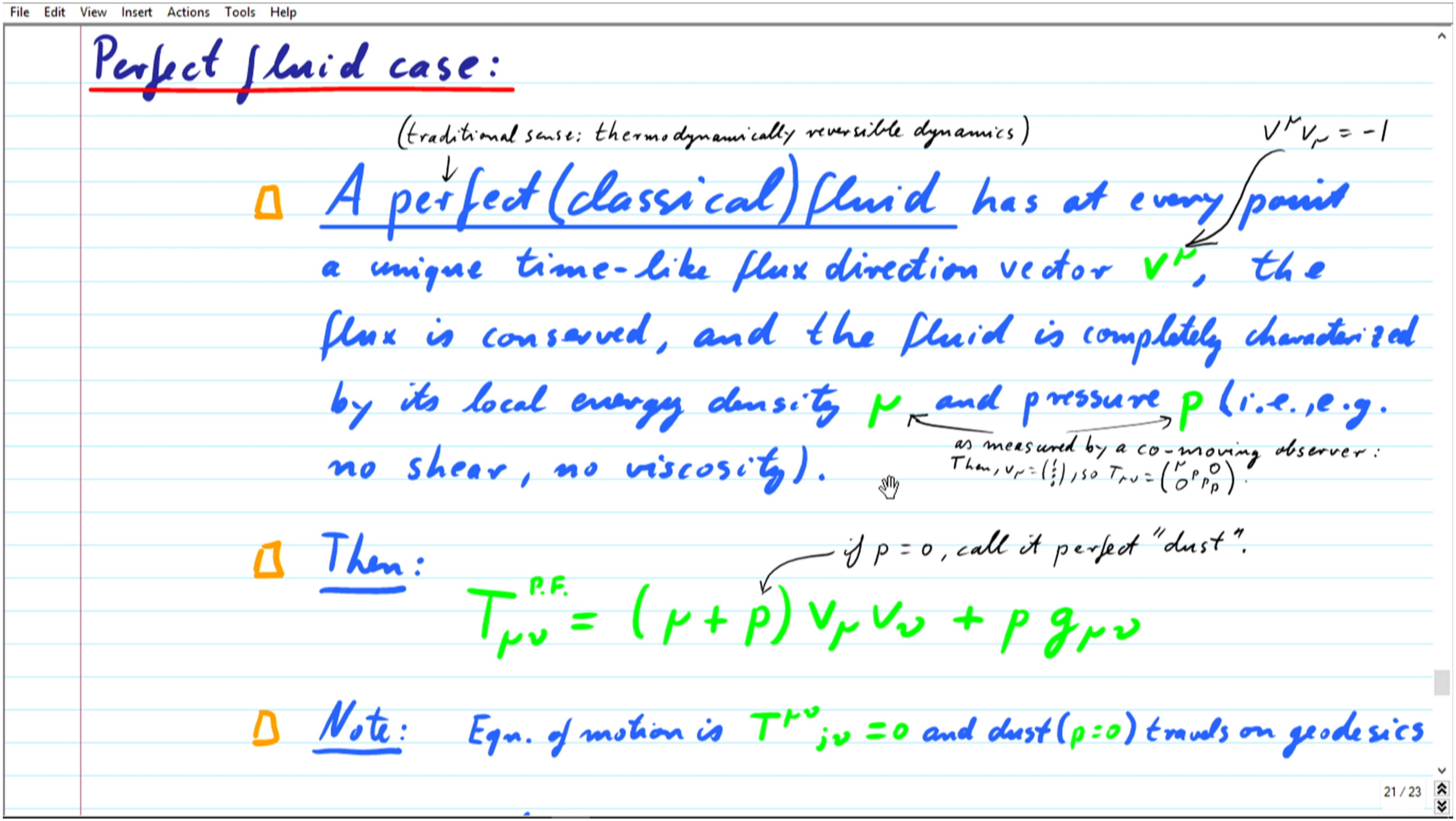
☐ A perfect (classical) fluid has at every point a unique time-like flux direction vector v^μ, the flux is conserved, and the fluid is completely characterized by its local energy density μ and pressure p (i.e., e.g. no shear, no viscosity).

as measured by a co-moving observer: Then, $v_\mu = \begin{pmatrix}1\\ \vdots\end{pmatrix}$, so $T_{\mu\nu} = \begin{pmatrix} \mu & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$.



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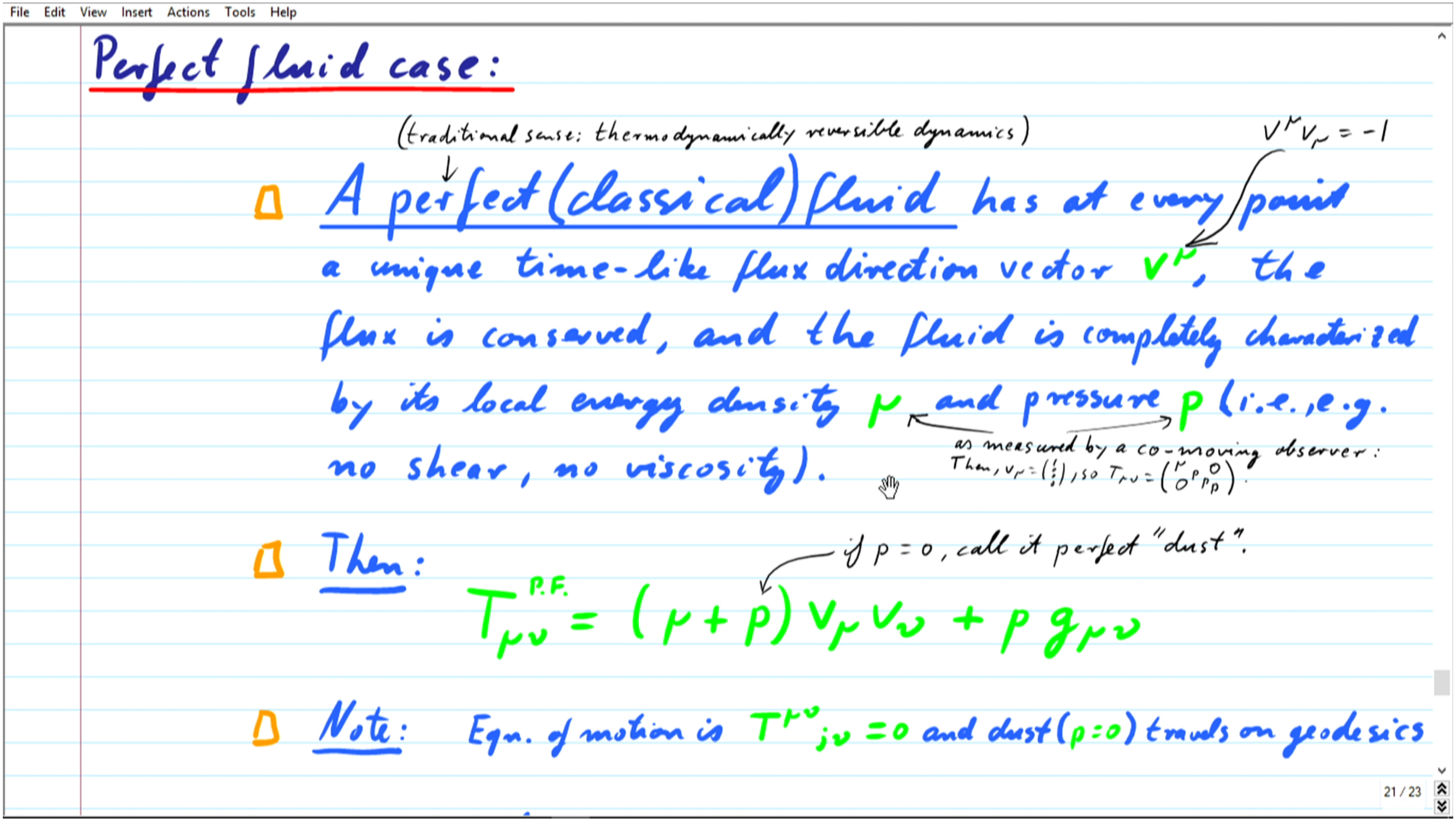
- Then:

if $p = 0$, call it perfect "dust".

$$T^{P.F.}_{\mu\nu} = (\mu + p)\, v_\mu v_\nu + p\, g_{\mu\nu}$$

- Note: Eqn. of motion is $T^{\mu\nu}{}_{;\nu} = 0$ and dust ($p=0$) travels on geodesics



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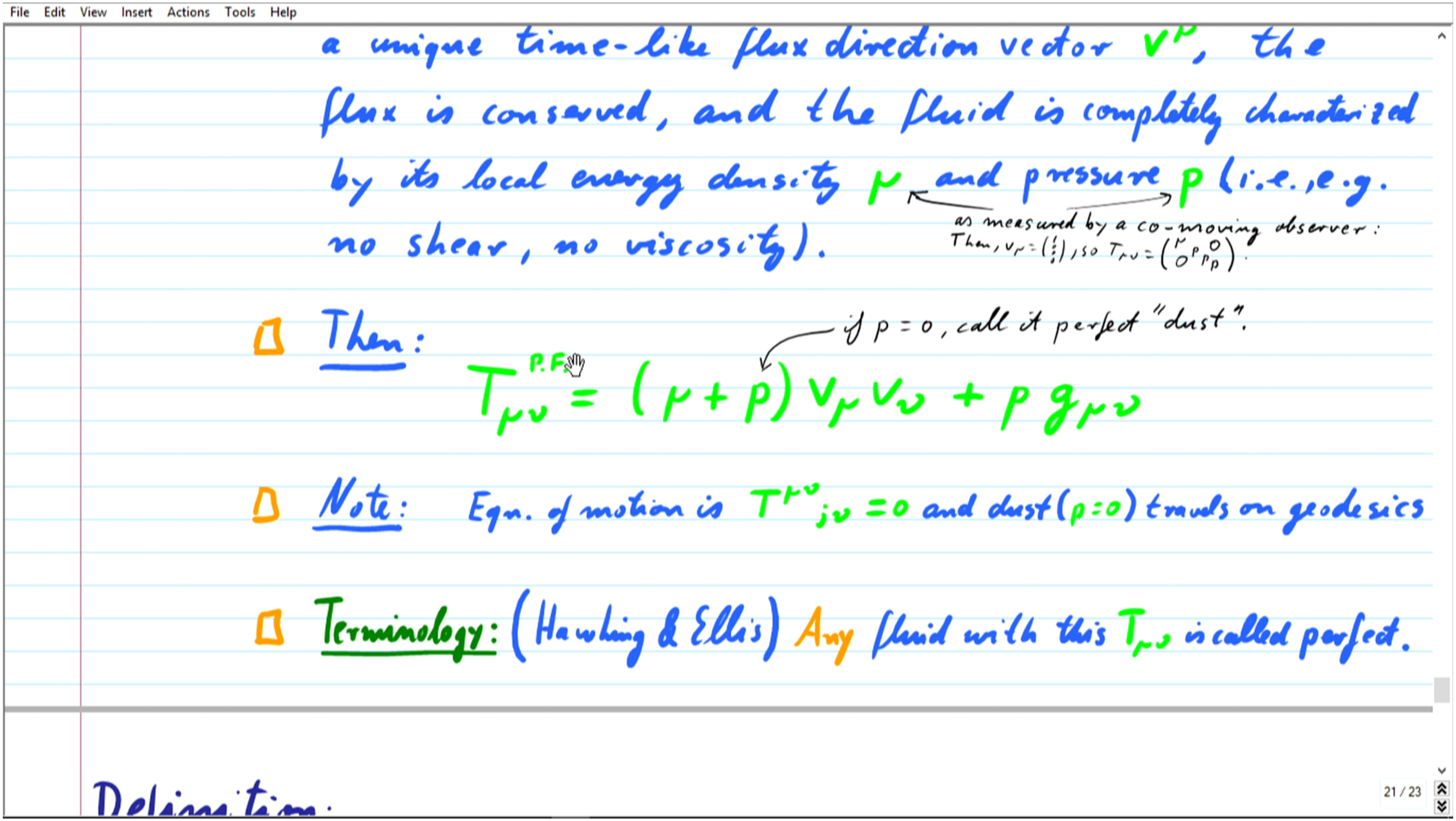

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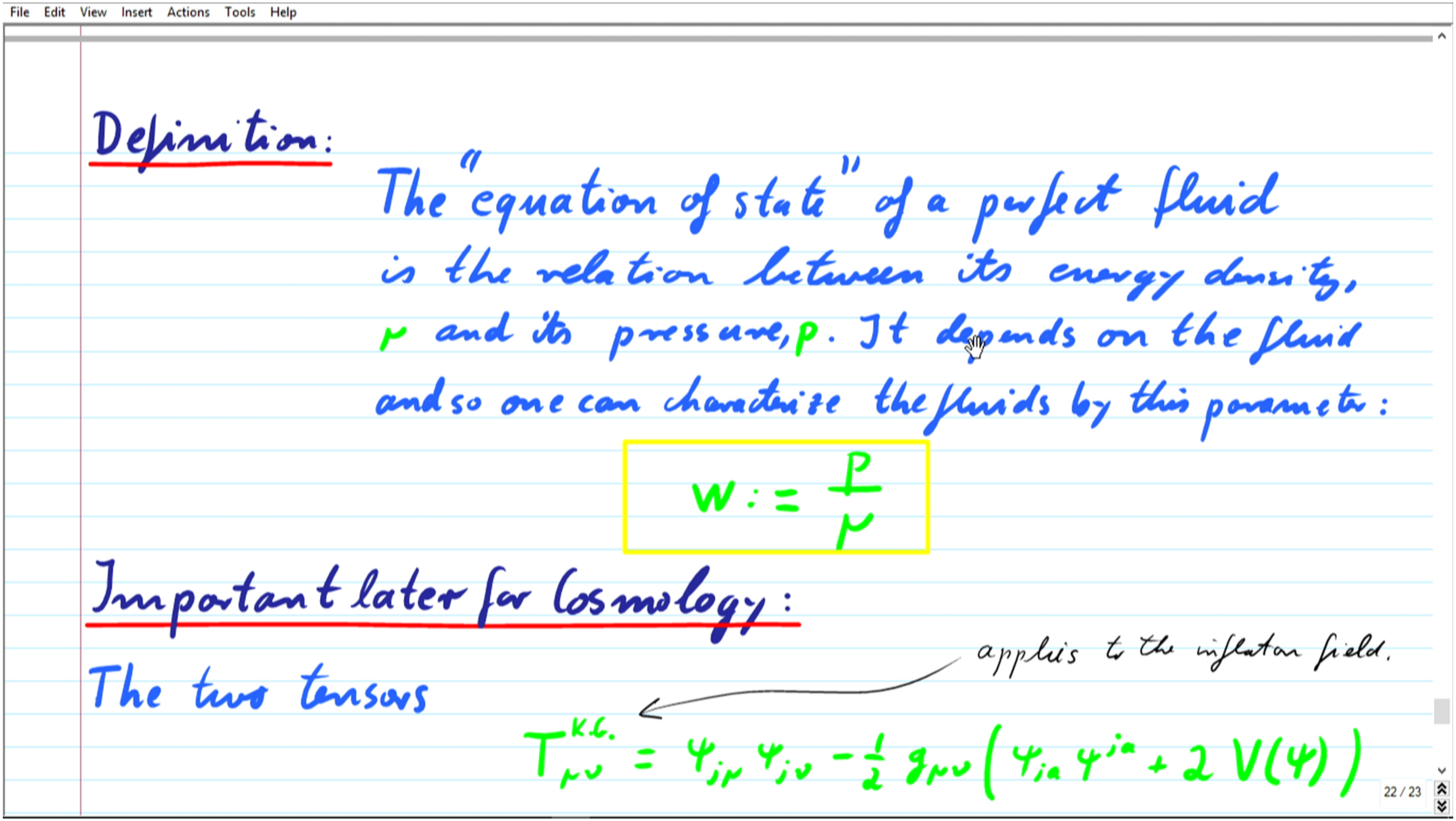
☐ **Terminology:** (Hawking & Ellis) Any fluid with this $T_{\mu\nu}$ is called perfect.

Delineation:



Definition:

The "equation of state" of a perfect fluid is the relation between its energy density, μ and its pressure, p. It depends on the fluid and so one can characterize the fluids by this parameter:

$$\boxed{w := \frac{p}{\mu}}$$

Important later for Cosmology:

The two tensors

applies to the inflaton field.

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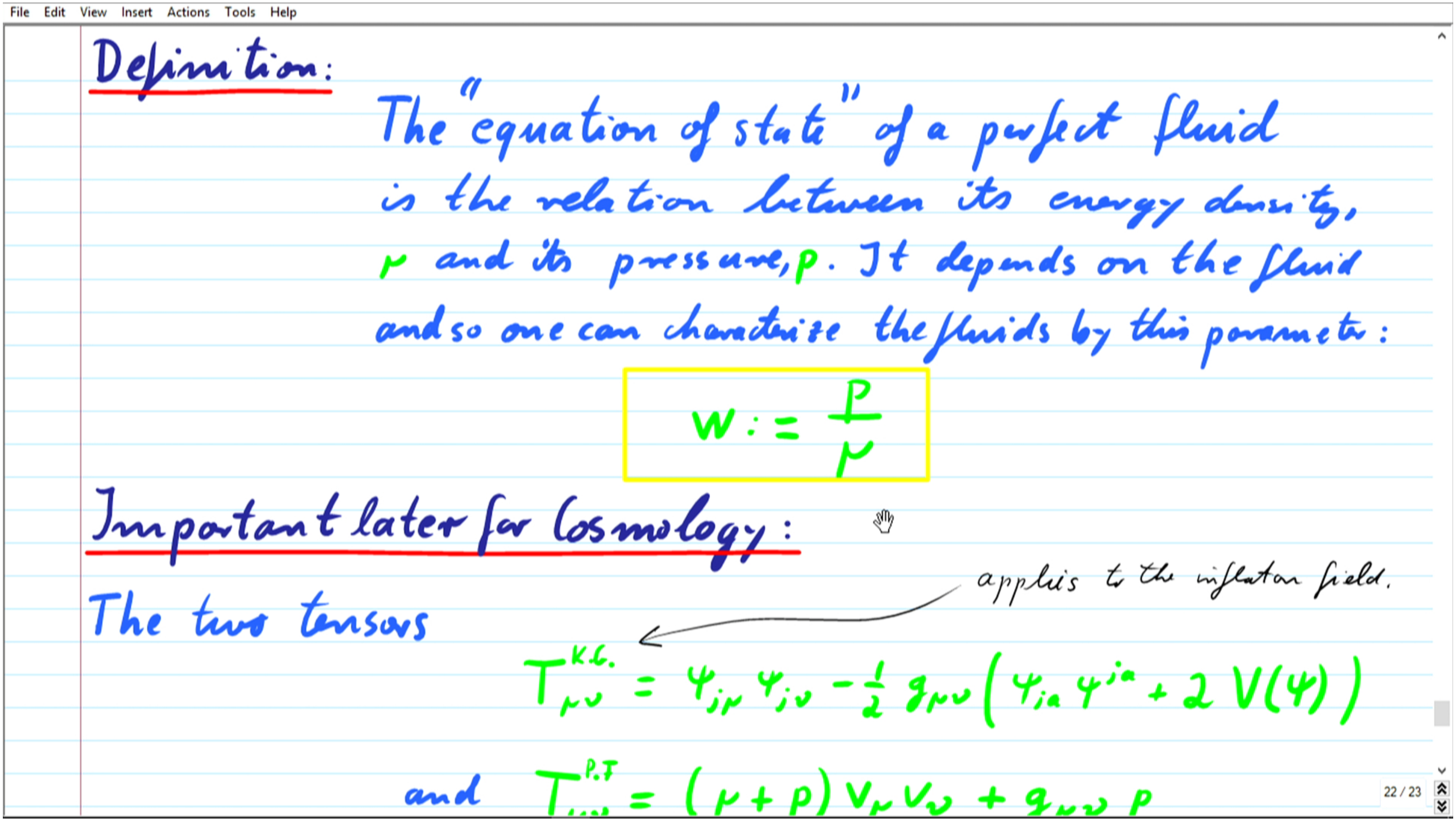

Definition:

The "equation of state" of a perfect fluid is the relation between its energy density, ρ and its pressure, p. It depends on the fluid and so one can characterize the fluids by this parameter:

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The two tensors

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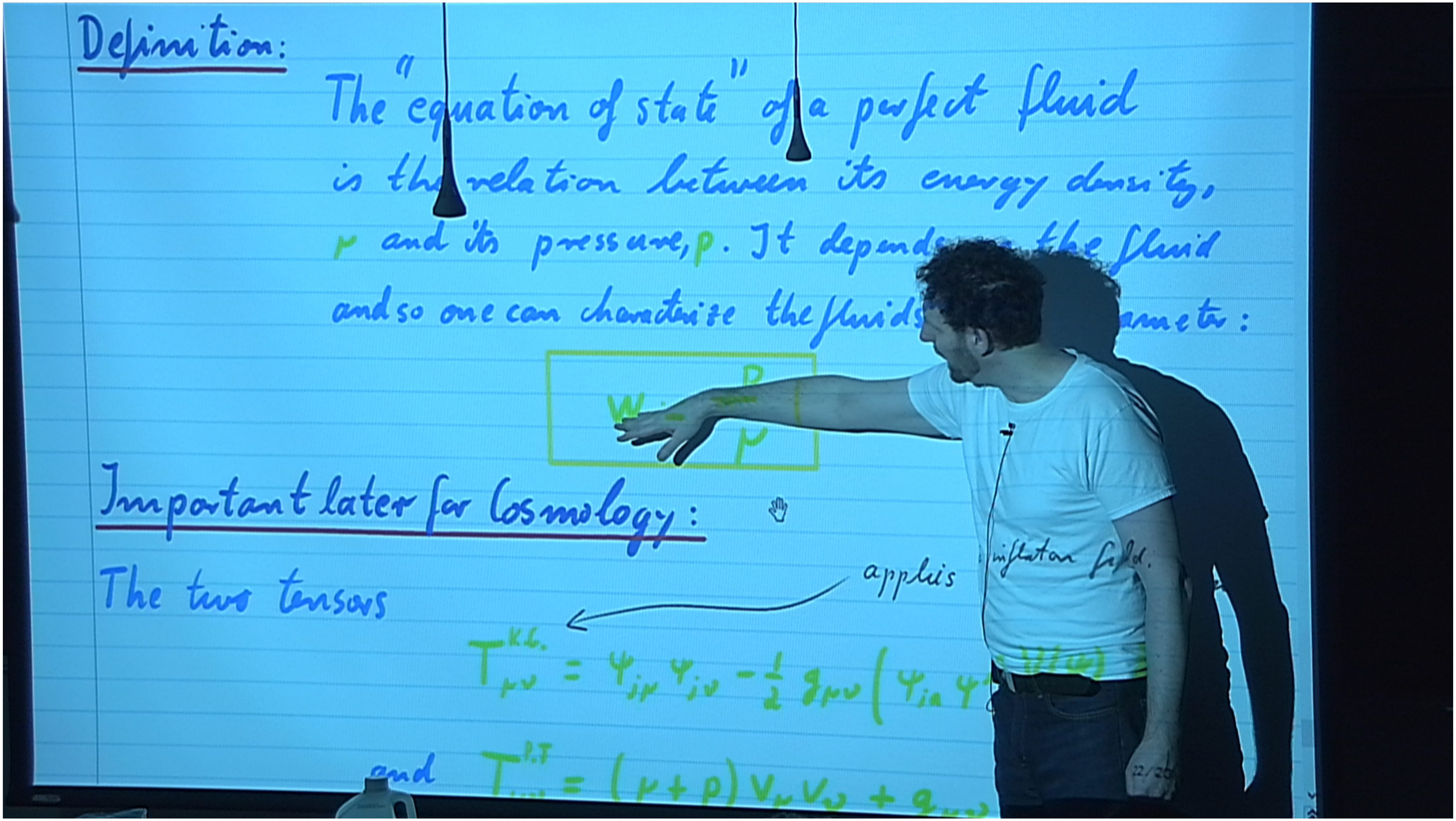
$$T^{K.G.}_{\mu\nu} = \varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2} g_{\mu\nu}\left(\varphi_{;a}\varphi^{;a} + 2V(\varphi)\right)$$

and $T^{P.F}_{\mu\nu} = (\rho + p) v_\mu v_\nu + g_{\mu\nu} p$



Definition:

The "equation of state" of a perfect fluid is the relation between its energy density, ρ and its pressure, p. It depend the fluid and so one can characterize the fluids ameter:

$$\boxed{w = \frac{p}{\rho}}$$

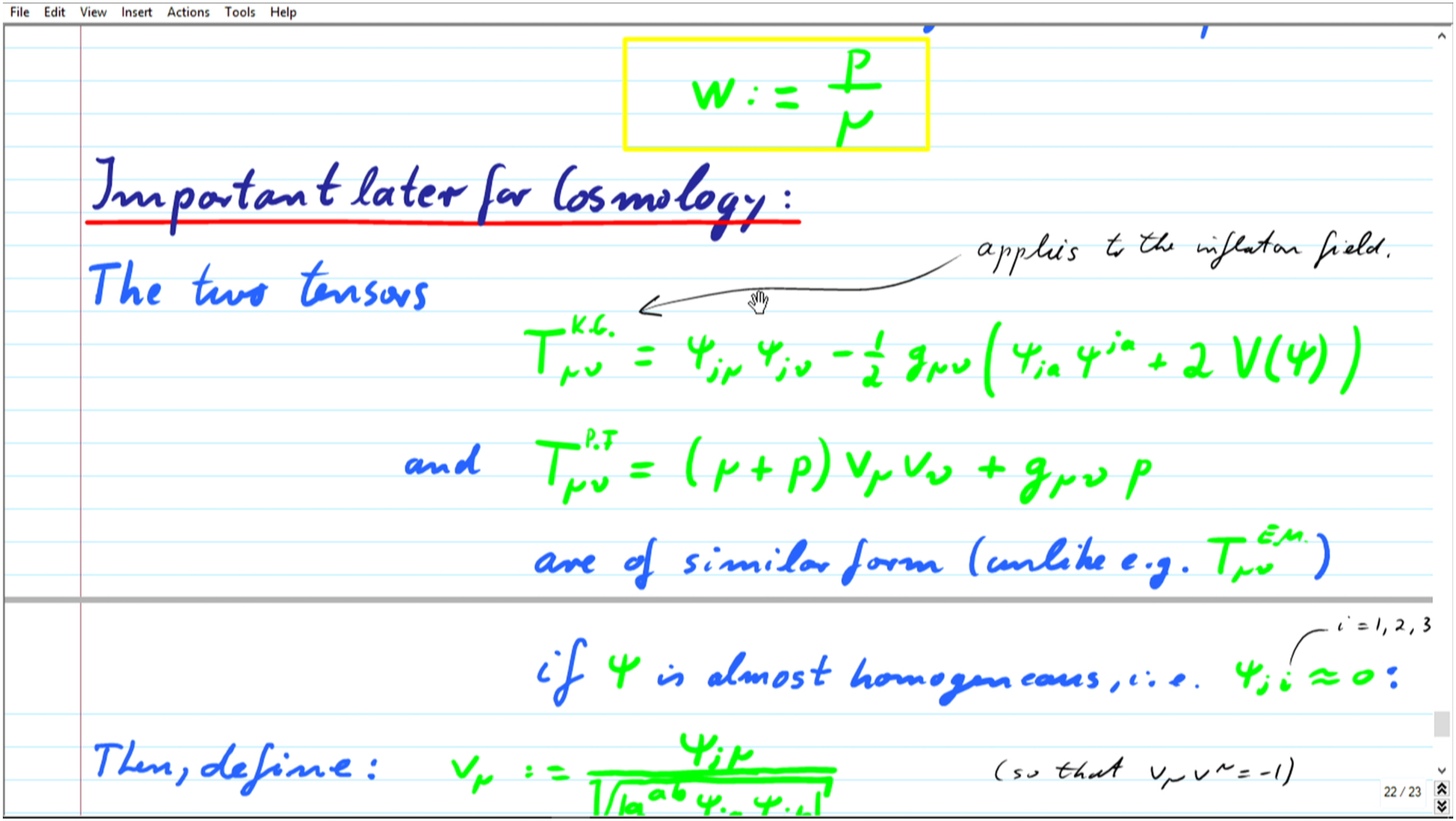
Important later for Cosmology:

The two tensors

applies inflaton field.

$$T^{K.G.}_{\mu\nu} = \varphi_{;\mu} \varphi_{;\nu} - \frac{1}{2} g_{\mu\nu} \left(\varphi_{;\alpha} \varphi \quad V(\varphi) \right.$$

and $T^{P.F.}_{\mu\nu} = (\rho + p) v_\mu v_\nu + g_{\mu\nu}$



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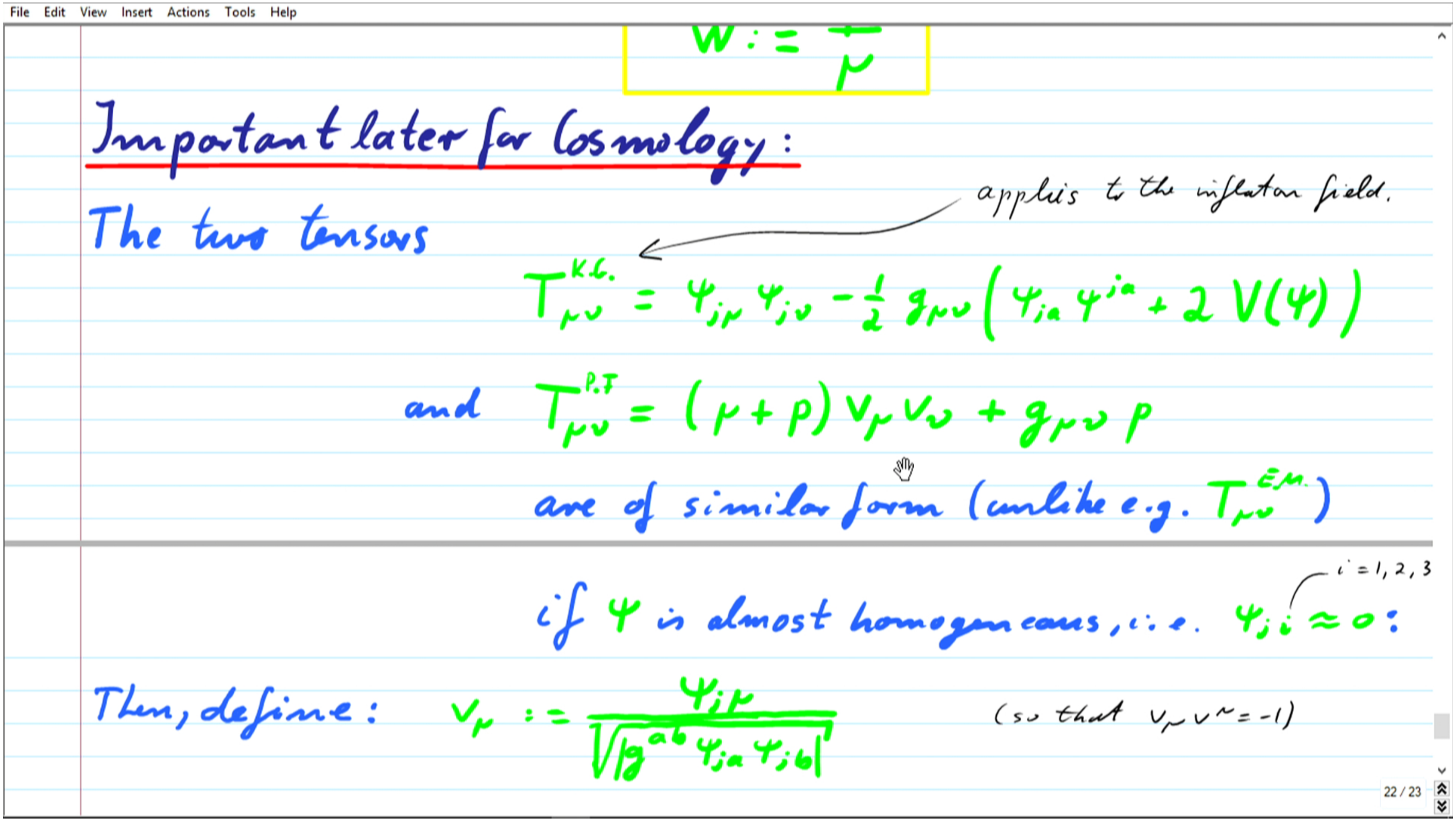
are of similar form (unlike e.g. $T^{E.M.}_{\mu\nu}$)

if ψ is almost homogeneous, i.e. $\psi_{;i} \approx 0$: $i = 1, 2, 3$

Then, define: $v_\mu := \frac{\psi_{;\mu}}{\sqrt{|g^{ab}\psi_{;a}\psi_{;b}|}}$ (so that $v_\mu v^\mu = -1$)



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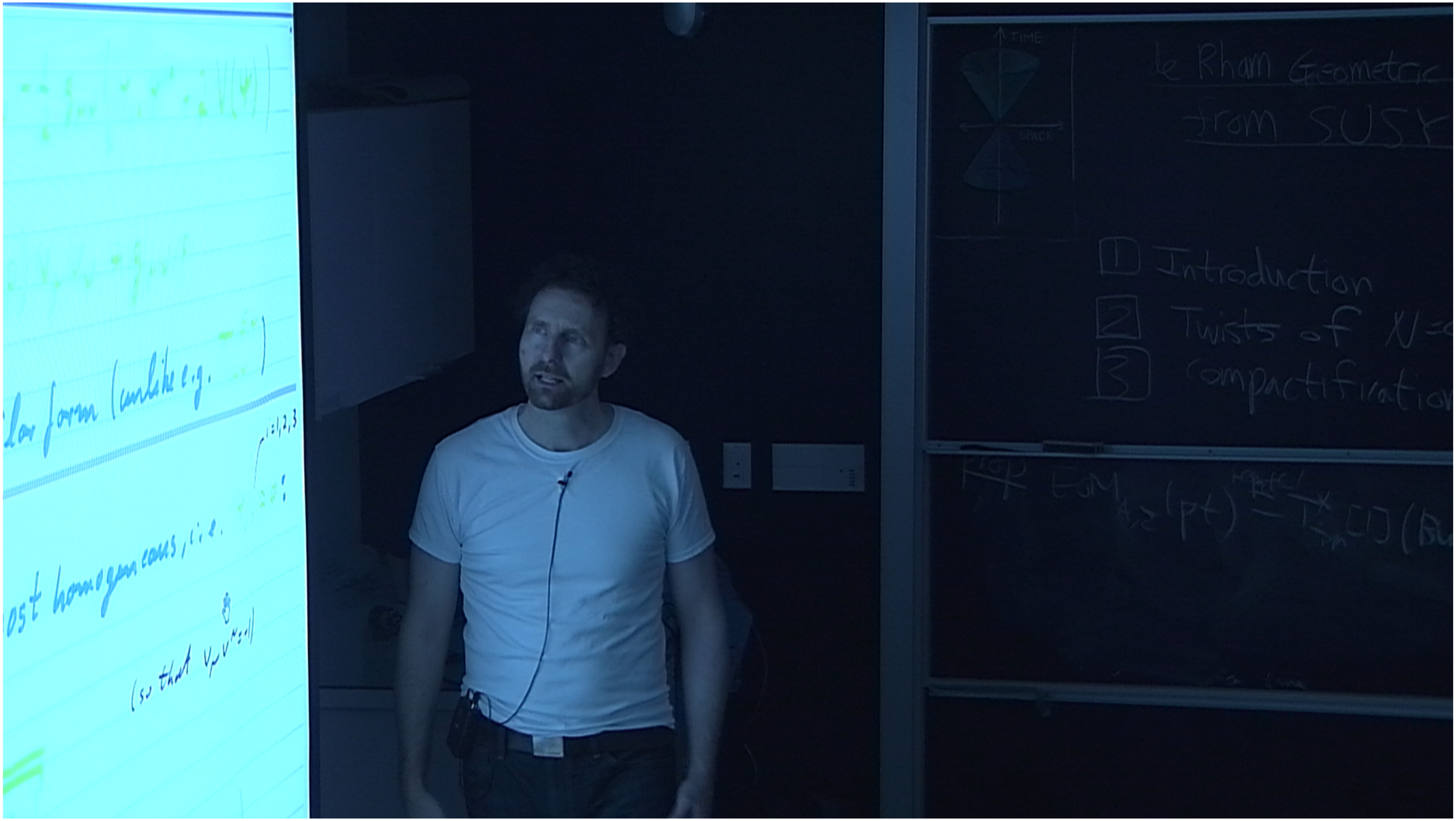
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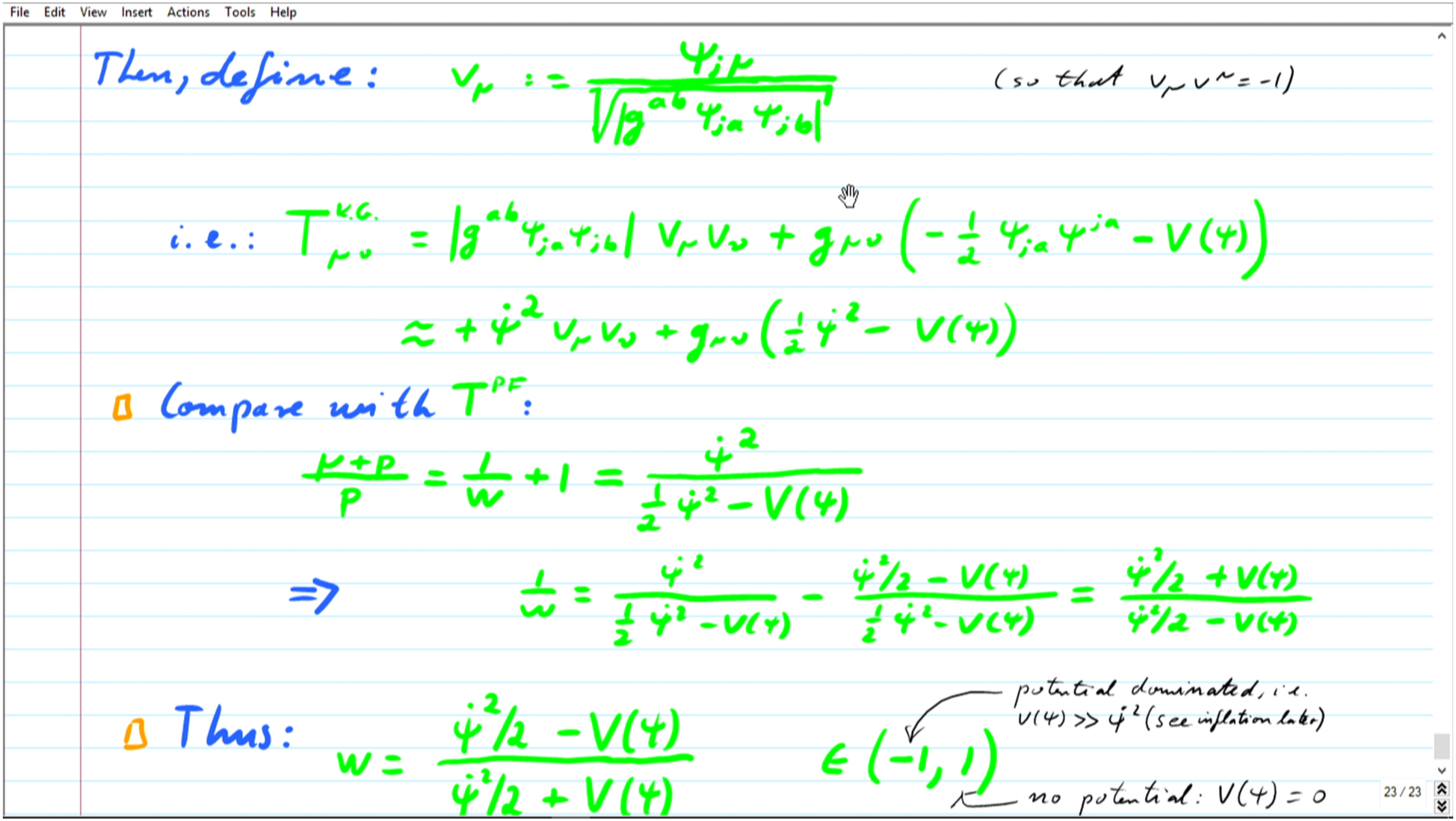
TIME
SPACE
de Rham Geometric
from SUSY
1 Introduction
2 Twists of N=
3 Compactification



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$$\approx +\dot{\varphi}^2 v_\mu v_\nu + g_{\mu\nu}\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)$$

- Compare with T^{PF}:

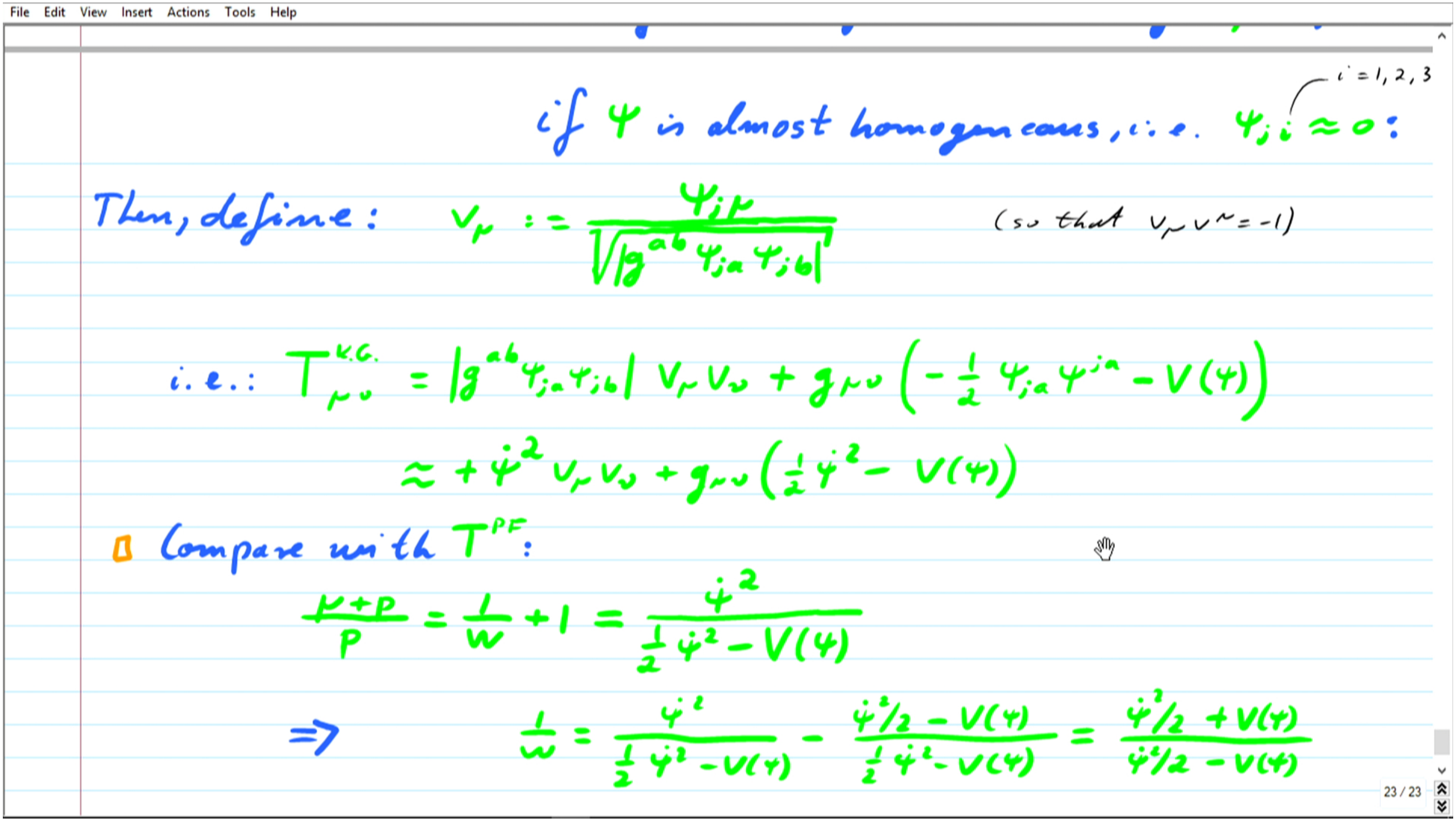
$$\frac{\rho+p}{p} = \frac{1}{w} + 1 = \frac{\dot{\varphi}^2}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)}$$

$$\Rightarrow \qquad \frac{1}{w} = \frac{\dot{\varphi}^2}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)} - \frac{\dot{\varphi}^2/2 - V(\varphi)}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)} = \frac{\dot{\varphi}^2/2 + V(\varphi)}{\dot{\varphi}^2/2 - V(\varphi)}$$

- Thus: $w = \frac{\dot{\varphi}^2/2 - V(\varphi)}{\dot{\varphi}^2/2 + V(\varphi)} \qquad \in (-1, 1)$

potential dominated, i.e. $V(\varphi) >> \dot{\varphi}^2$ (see inflation later)

no potential: $V(\varphi) = 0$



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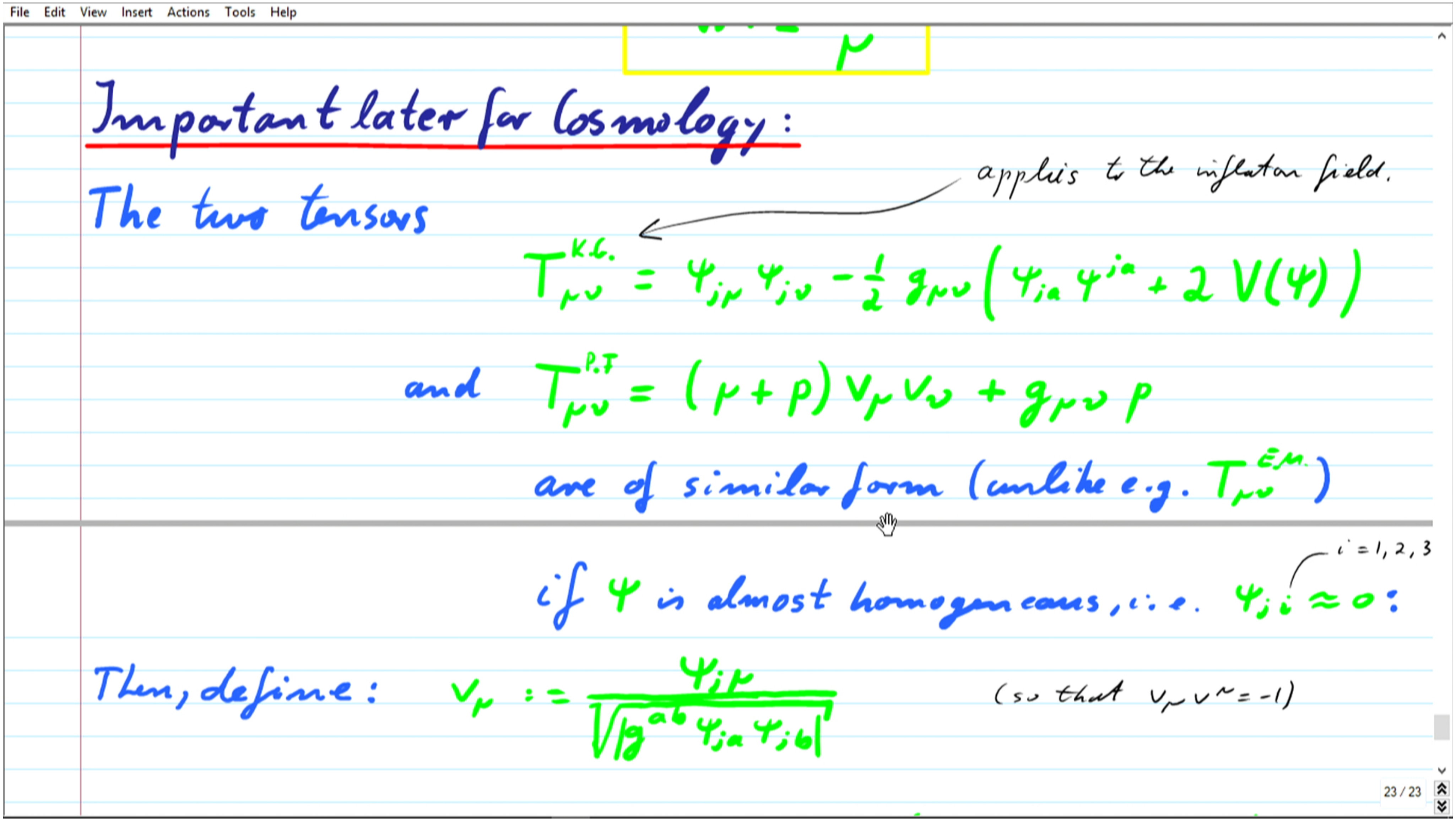

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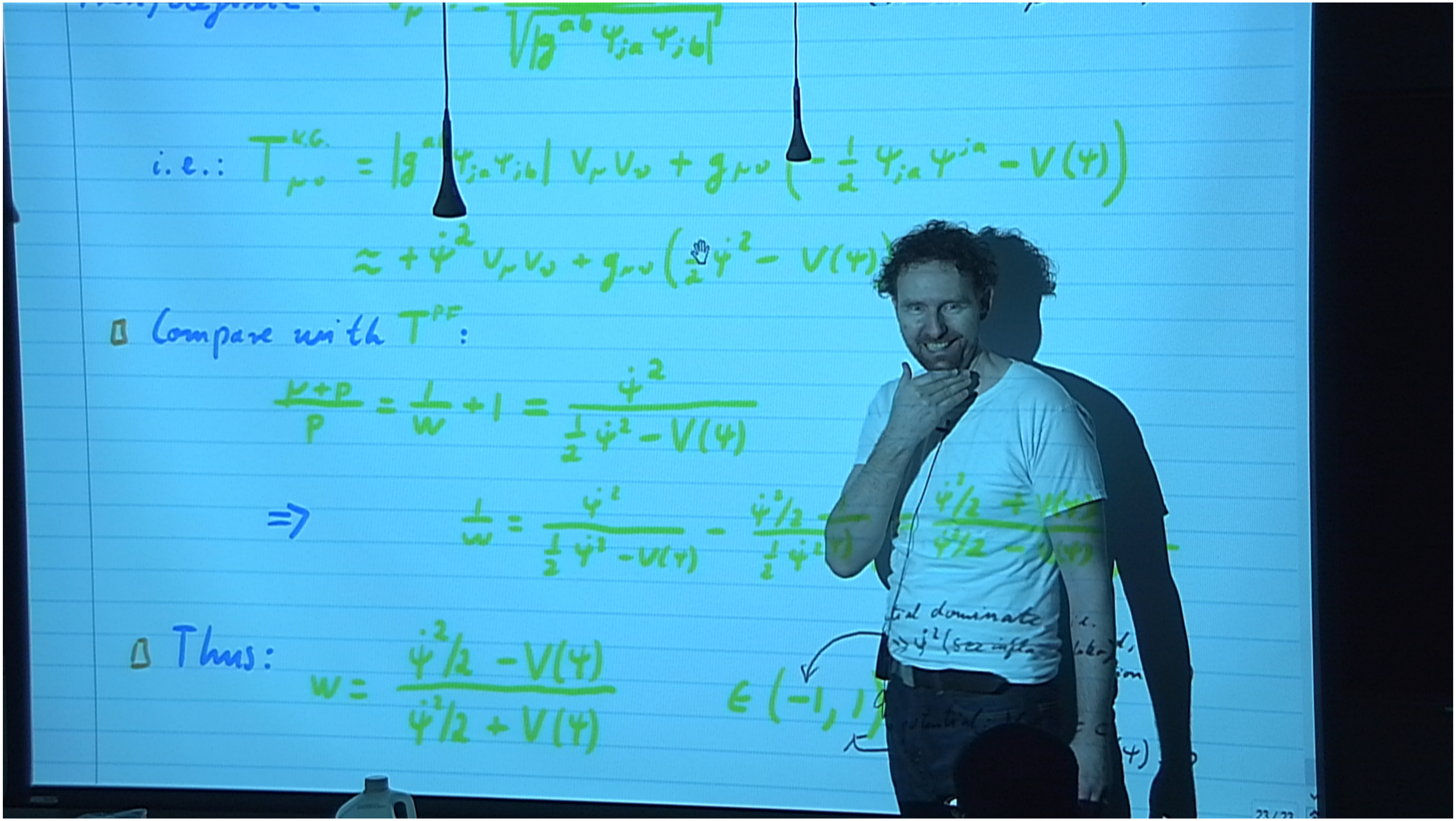

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potential:

(ψ)

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