Title: Bose condensation in category theory

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Abstract: <p>The condensation of bosons can induce transitions between topological quantum field theories (TQFTs). This as been previously investigated through the formalism of Frobenius algebras and with the use of Vertex lifting coefficients. I will discuss an alternative, algebraic approach to boson condensation in TQFTs that is physically motivated and computationally efficient. With a minimal set of assumptions, such as commutativity of the condensation with the fusion of anyons, we can prove a number of theorems linking boson condensation in TQFTs with algebra extensions in conformal field theories and with the problem of factorization of completely positive matrices over the positive integers. I will present an algorithm for obtaining a condensed theory fusion algebra and its modular matrices. In addition, I will discuss how this formalism can be used to build multi-layer TQFTs which could be a starting point to build three-dimensional topologically ordered phases. Using this formalism, I will also give examples of bosons that cannot undergo a condensation transition due to topological obstructions.</p>

Bose condensation in category theory

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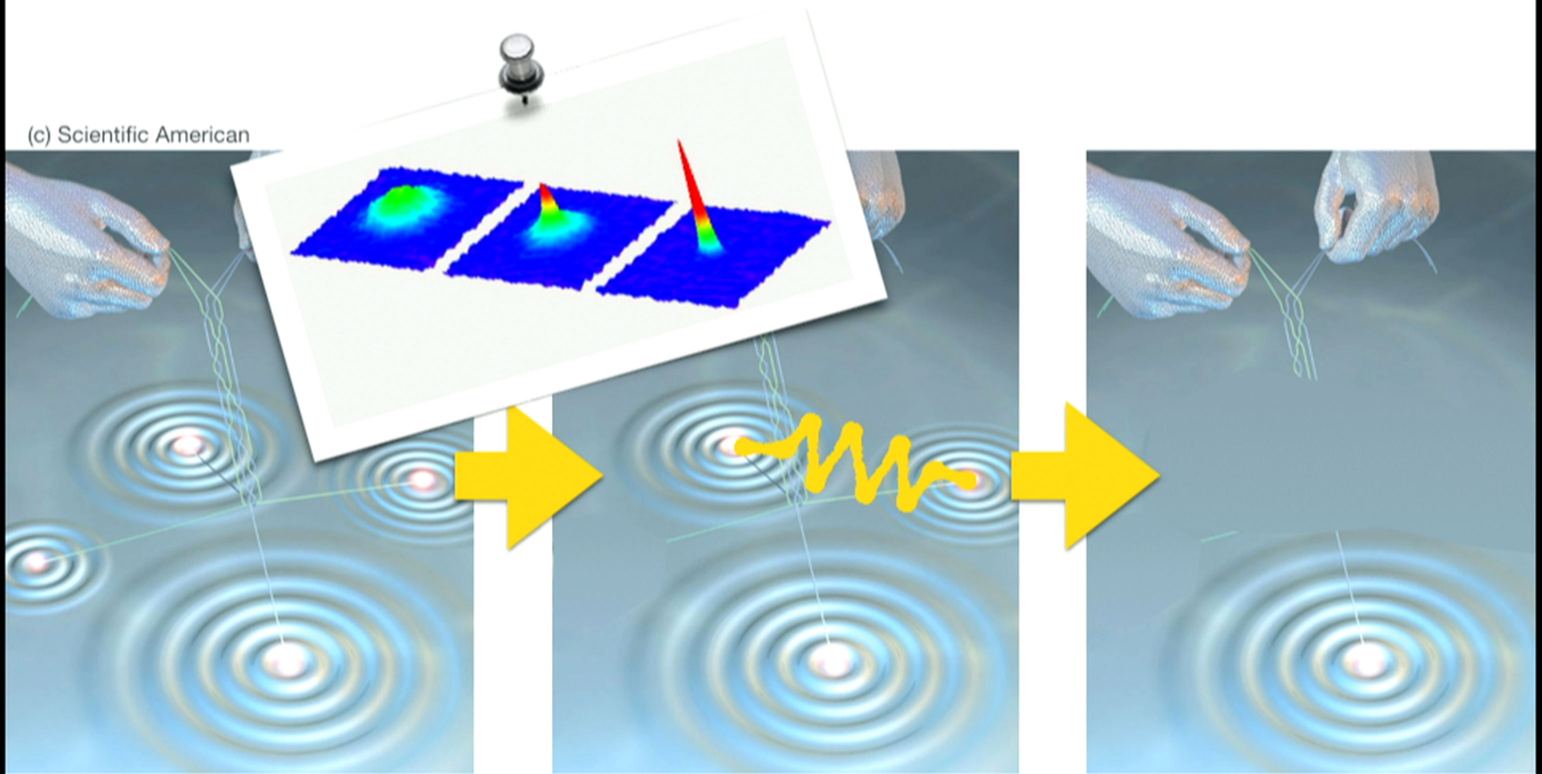
Condensation in topological fluids

(c) Scientific American



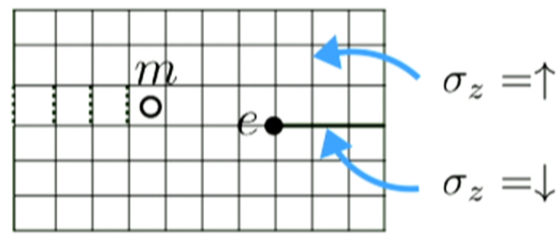
$\mathcal{A}$ $\mathcal{T}$ $\mathcal{U}$

Condensation Remove confined excitations

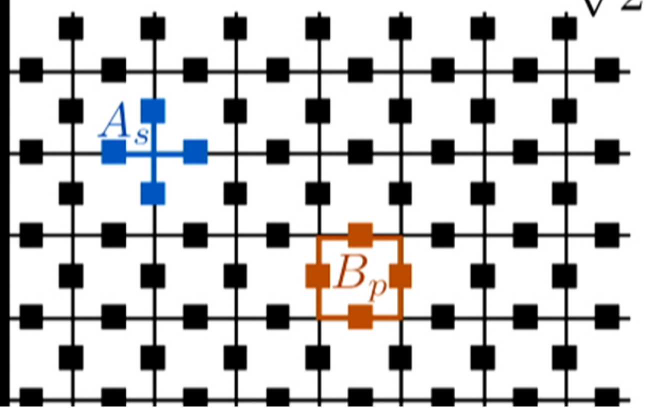
Example 1: Toric Code

$$H = -J_{\mathrm{e}} \sum_s A_s - J_{\mathrm{m}} \sum_p B_p \qquad A_s := \prod_{j \in s} \sigma_j^z \qquad B_p := \prod_{j \in p} \sigma_j^x$$

boson *e*
boson *m*
fermion *f* = *e* x *m*



field $h\sigma_x$: $\quad |\rightarrow\rangle = \dfrac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$



everywhere open σ_z strings

condenses *e* particles

σ_x strings cost linear energy

confines *m* particles

toric code $\xrightarrow{\text{condense } e \text{ (or } m\text{)}}$ **trivial theory**

Example 2: Chiral *p*-wave superconductor

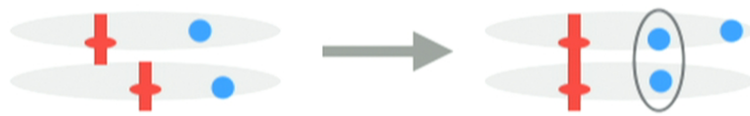
Vortex: Majorana anyon σ

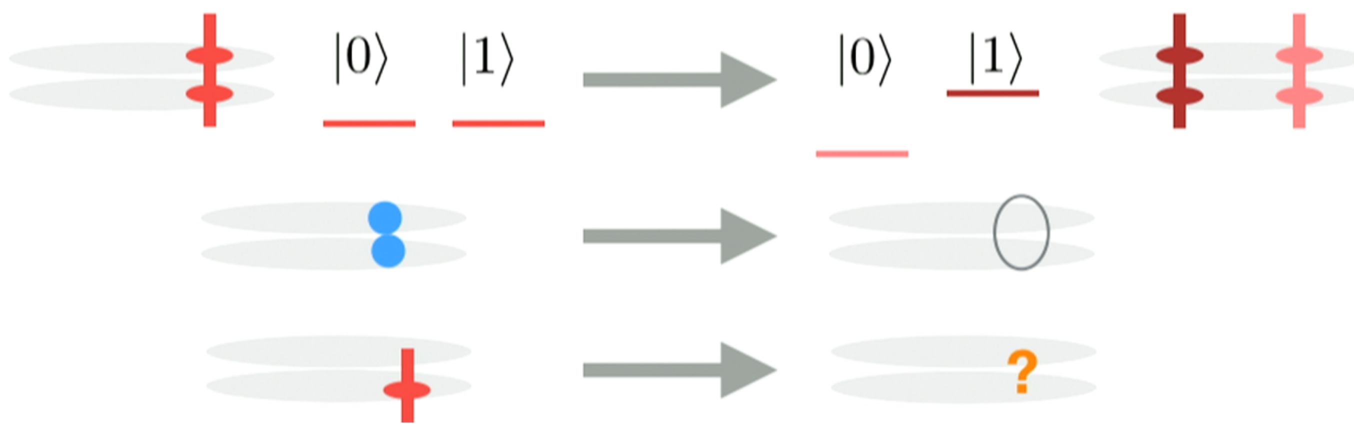
Bogoliubov quasiparticle: Fermion ψ

Majorana: indefinite parity

Condensate: Cooper pairs

Double layer
interlayer coupling

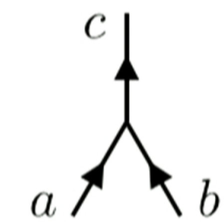




Outline

1. Primer on category theory
2. Axioms for condensation transitions
3. Algorithm for condensation
4. CFT interpretation
5. No-go theorem for condensing bosons

Fusion category



Finite set of anyon types: $a, b, c \ldots$

Fusion rules: $a \times b = \sum_c N_{ab}^c c, \qquad N_{ab}^c \in \mathbb{Z}_{\geq 0}$

Unique vacuum 0: $0 \times a = a, \quad \forall a$

Unique antiparticle: $\bar{a} \times a = 0 + \ldots$

Commutativity $a \times b = b \times a$

Associativity $(a \times b) \times c = a \times (b \times c)$

Quantum dimension: largest eigenvalue of $(N_a)_{bc} \ = \ N_{ab}^c$

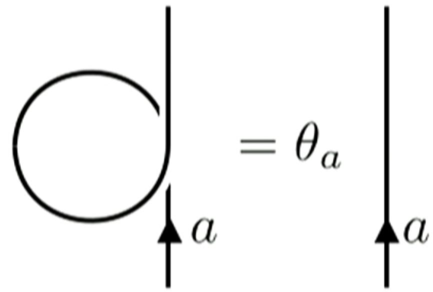
$$d_a d_b = \sum_c N_{ab}^c d_c$$

$d_a =$ 

Total quantum dimension $D = \sqrt{\sum_a d_a^2}$

Braided fusion category

Topological spin



Boson: $\theta_a = 1$ Fermion: $\theta_a = -1$

Modular matrices
(here: both unitary)

$$T_{ab} = \theta_a \delta_{ab}$$

$$S_{ab} = \frac{1}{\mathcal{D}} \sum_c \frac{\theta_a \theta_b}{\theta_c} N_{ab}^c d_c$$

Verlinde formula

$$N_{ab}^c = \sum_x \frac{S_{ax} S_{bx} S_{\bar{c}x}}{S_{0x}}$$

Our Examples

1. Toric code

$$e \times m = f$$
$$e \times e = m \times m = f \times f = 0$$

$$d_e = d_m = d_f = 1$$

$$\theta_e = \theta_m = 1, \quad \theta_f = -1$$

2. Chiral p-wave superconductor (Ising TQFT)

$$\sigma \times \sigma = 0 + \psi$$ non-Abelian

$$\sigma \times \psi = \sigma$$

$$\psi \times \psi = 0$$

$$d_\sigma = \sqrt{2}, \quad d_\psi = 1$$

$$\theta_\sigma = e^{\mathrm{i}\pi/8}, \quad \theta_\psi = -1$$

Condensation

1. Identify some boson(s) with vacuum
2. Derive new consistent **fusion category** $\mathcal{T}$

Restriction map $\mathcal{A} \longrightarrow \mathcal{T}$ $\boxed{a \mapsto \sum_{t \in \mathcal{T}} n_a^t t,}$ $\forall a \in \mathcal{A}$

restriction coefficients: main players in this talk $n_a^t \in \mathbb{Z}_{\geq 0}$

Basic assumptions: Fusion

restriction $\quad a \mapsto \sum_{t\in\mathcal{T}} n_a^t t, \quad \forall a \in \mathcal{A}$

and

fusion $\quad N_{ab}^c$ in $\mathcal{A}$ $\qquad \tilde{N}_{rs}^t$ in $\mathcal{T}$

commute!

condensation

$$\boxed{\sum_{r,s\in\mathcal{T}} n_a^r n_b^s \tilde{N}_{rs}^t = \sum_{c\in\mathcal{A}} N_{ab}^c n_c^t}$$

implications: $\quad d_a = \sum_{r\in\mathcal{T}} n_a^r d_r \qquad d_t = \frac{1}{q}\sum_{a\in\mathcal{A}} n_a^t d_a$

$q := \sum_a n_a^\varphi d_a$

Basic assumptions: Braiding

Consistent spin assignment required for anyons in $\mathcal{U}$

 $t = \sum_a$  $a \;\; n_a^t$

$= \sum_a \theta_a$  $a \;\; n_a^t$

$$\boxed{\begin{aligned} \sum_a \theta_a n_a^t d_a &= q d_t \theta_t && \text{for anyons in } \mathcal{U} \\ &= 0 && \text{for confined anyons in } \mathcal{T}/\mathcal{U} \end{aligned}}$$

We can prove this under reasonable assumptions (U is modular tensor category, …) for

1. “Simple current condensate” (abelian Boson condenses)
2. theories with one confined particle
3. … soon more?

Towards a linear algebra problem

nonnegative, symmetric integer matrix $M_{ac} := \sum_{t \in \mathcal{U}} n_a^t n_c^t$

restrict to anyons in $\mathcal{U}$

Can show:

$$[M, S] = 0 \qquad [M, T] = 0$$

$$\tilde{S} n = n S$$

$$\tilde{T} n = n T$$

with $\tilde{S}$, $\tilde{T}$ modular matrices of $\mathcal{U}$

Algorithm to find condensations

Given modular tensor category $\mathcal{A}$

1) Find nonnegative symmetric integer M with

$$[M,S]=0 \qquad [M,T]=0 \qquad M_{1,1}=1$$

2) Find n that satisfy $\quad M = nn^{\mathsf{T}}$

!!! Integer matrix
!!! factorization is hard

3) Find unitary solutions $\tilde{S},\ \tilde{T}$ to

$$\tilde{S}n = nS \qquad \tilde{T}n = nT$$

Confirm that solutions describe modular tensor category

$$\tilde{S}^2 = \Theta(\tilde{S}\tilde{T})^3 = \tilde{C} \text{ is permutation matrix} \qquad \Theta = e^{-\mathrm{i}\pi c/4}$$

Verlinde-Formula: $\quad \tilde{N}^t_{rs} = \sum_x \frac{\tilde{S}_{rx}\tilde{S}_{sx}\tilde{S}_{\bar{t}x}}{\tilde{S}_{0x}} \in \mathbb{Z}_{\geq 0}$

Back to examples! Toric code

$$S_{\mathrm{TC}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

0 e m f

$$T_{\mathrm{TC}} = \mathrm{diag}(1, 1, 1, -1)$$

1) $[M, S] = 0$

$[M, T] = 0$

2 solutions

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2) $M = nn^{\mathsf{T}}$

$n = (1, 1, 0, 0)$ $\qquad$ $n = (1, 0, 1, 0)$

e condensed,
m, *f* confined

m condensed,
e, *f* confined

3) $\tilde{S},\ \tilde{T}$ trivial

Back to examples! Double layer Ising

$$S_{\mathrm{I}} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad T_{\mathrm{I}} = \mathrm{diag}(\overset{0}{1}, \overset{\sigma}{e^{\mathrm{i}\pi/8}}, \overset{\psi}{-1})$$

$$S = S_{\mathrm{I}} \otimes S_{\mathrm{I}} \qquad T = T_{\mathrm{I}} \otimes T_{\mathrm{I}}$$

1) $[M, S] = 0$

$[M, T] = 0$

unique solution

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 00 \\ 0\sigma \\ 0\psi \\ \sigma 0 \\ \sigma\sigma \\ \sigma\psi \\ \psi 0 \\ \psi\sigma \\ \psi\psi \end{array} \end{array}$$

2) $M = nn^{\mathsf{T}}$

unique solution $\quad n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

- $\psi\psi$ condenses
- single Majorana anyons confined
- $\sigma\sigma$ splits

3) $\tilde{S}n = nS$ unique solution

$$a \times f = b$$
$$a \times a = b \times b = f$$
$$a \times b = 0, \quad f \times f = 0$$

C = 2 theory from Kitaev's 16-fold way

gauged chiral *d*-wave SC

Comments

Algorithm might find spurious solutions, but will find all possible condensations

Central charge is conserved under condensation (mod 8)

Total quantum dimensions are constrained by “size” of condensate q

$$q = D_{\mathcal{A}}/D_{\mathcal{U}}$$
$$q = D_{\mathcal{A}}^2/D_{\mathcal{T}}^2$$

$$D_{\mathcal{A}} > D_{\mathcal{U}}$$

D theorem

No-go theorem for condensing bosons

Can show that some bosons cannot condense

Known example: **Multiple layers of Fibonacci** $\tau \times \tau = 0 + \tau$

$$S_{\text{Fib}} \propto \begin{pmatrix} 1 & \phi \\ \phi & -1 \end{pmatrix} \qquad \begin{aligned} \theta_\tau &= e^{\mathrm{i}4\pi/5} \\ d_\tau &= \phi \end{aligned} \qquad \phi = \frac{1+\sqrt{5}}{2}$$

In 5 layers, $\tau\tau\tau\tau\tau$ is boson, for $\theta_\tau^5 = 1$

Consider the (00000), $(\tau 0000)$ element of

$$M S_{\text{Fib}^{(5)}} = S_{\text{Fib}^{(5)}} M$$

$$\sum_{a \in \mathcal{A}} n_a^\varphi S_{\text{Fib}^{(5)};a,(\tau 0000)} = \frac{1}{D} \sum_{a \in \mathcal{A}} d_a M_{a,(\tau 0000)}$$

$$\left(\phi - n^\varphi_{(\tau\tau\tau\tau\tau)} \phi^4\right) \qquad \geq 0$$

No-go theorem for condensing bosons

Proof for **any number** of layers of Fibonacci

Our formalism

One column

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Commutative algebras in Fibonacci categories

Thomas Booker[a,1], Alexei Davydov[b,*,2]

Fibonacci is $SO(3)_3$.

We generalized the proof: **No condensation is possible in any $SO(3)_k$ with k odd for any number of layers.**

Conclusions

Axiomatic, algebraic formulation of condensation in category theory

"numerically" tractable

relates to problem of factorizing integer matrices

analytically useful (e.g., proof of noncondensability)

Future directions

understand layer-graded condensation for topological order, e.g., on the surface of topological superconductors

non-modular theories and fermion condensation

bulk-boundary correspondence in topologically ordered states