Title: TBA

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Abstract:

Duality between the Ising model & 3d Quantum Gravity

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Renormalization '15 (PI)

Work with F. Costantino & V. Bonzom - arXiv:1504.02822 [math-ph]

Interface Quantum Gravity - Statistical Physics

Coarse-Graining & Renormalisation of LQG & Spinfoams

→ Main issue both at fundamental & effective level

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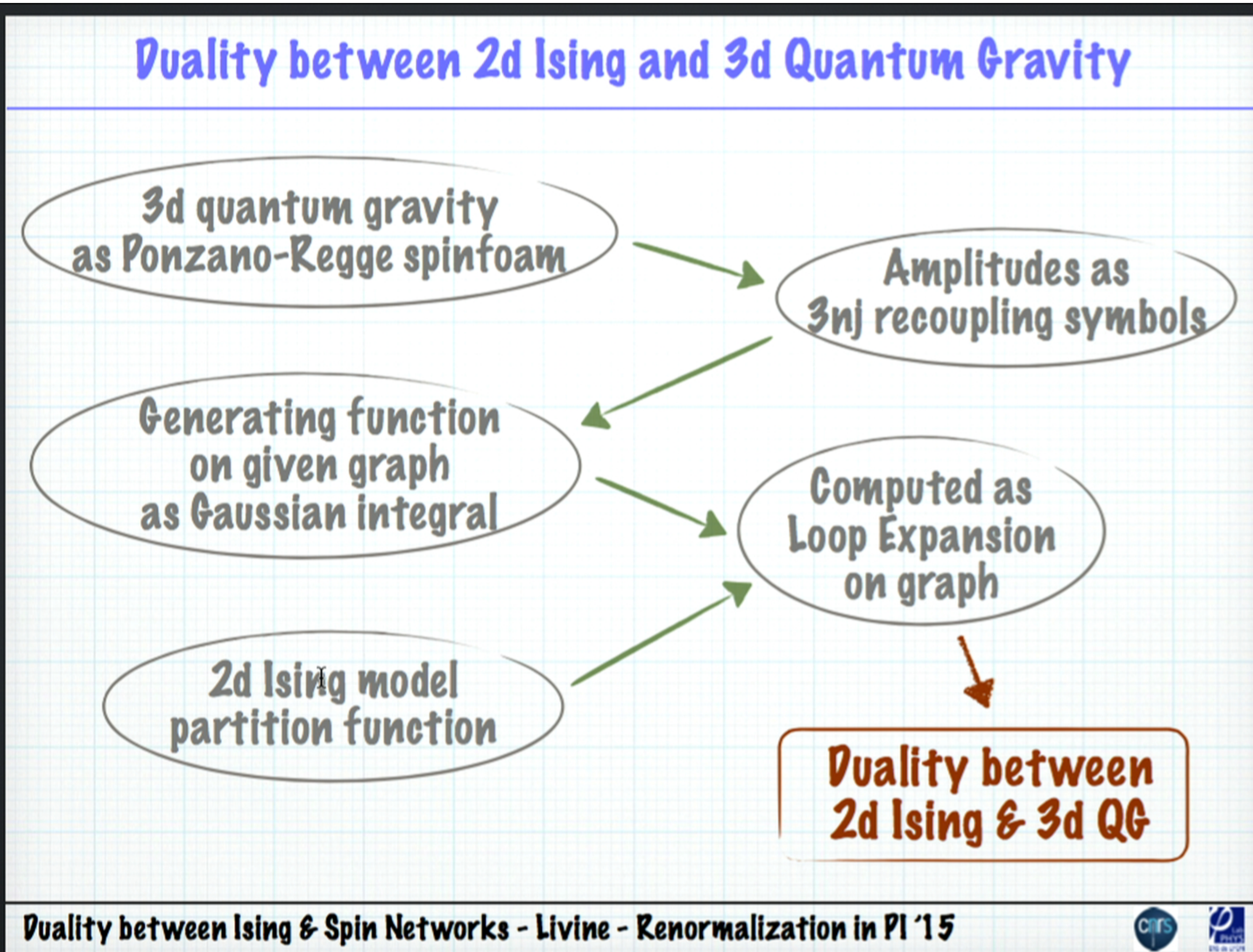
To study the dynamics and path integral for LQG:

1. Dynamics of QG degrees of freedom on fixed network or space-time triangulation (lattice gauge theory)
2. Dynamics of fluctuating graphs or triangulations (through matrix models, tensor models or GFTs)


Duality between 2d Ising and 3d Quantum Gravity
3d quantum gravity as Ponzano-Regge spinfoam
Amplitudes as 3nj recoupling symbols
Generating function on given graph as Gaussian integral
Computed as Loop Expansion on graph
2d Ising model partition function
Duality between 2d Ising & 3d QG
Duality between Ising & Spin Networks - Livine - Renormalization in PI '15

Duality between 2d Ising and 3d Quantum Gravity

Result:

- Generating function for spin network evaluations as Gaussian integral (using spinors)
- Ising partition function as odd-Grassmann Gaussian integral
- Equality between the two functions, realized through supersymmetry

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Applications:

- Import statistical physics tools to QG: criticality, phase diagrams, continuum limit
- Geometrical interpretation of Ising critical couplings (Fisher zeroes)
- Generalizable to 4d? to other stat phys models?

Duality between 2d Ising and 3d Quantum Gravity

Outline:

1. 3d QG Ponzano-Regge amplitudes as spin network evaluations
2. Generating function for spin networks: integral and result
3. Ising partition function: fermionic integral & loop expansion
4. Westbury theorem & Supersymmetry

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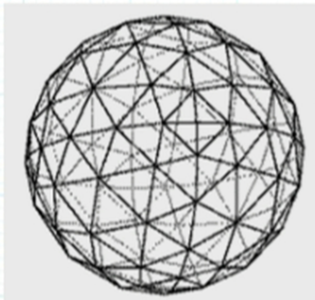
6. Link between Ising criticality and spin network saddle points
7. Application to tetrahedron graph, Fisher zeroes and 6j duality
8. Coarse-graining Ising and Pachner moves
9. Speculations on continuum limit & boundary CFT for 3d QG

3d Quantum Gravity: Spinfoams & Spin Networks

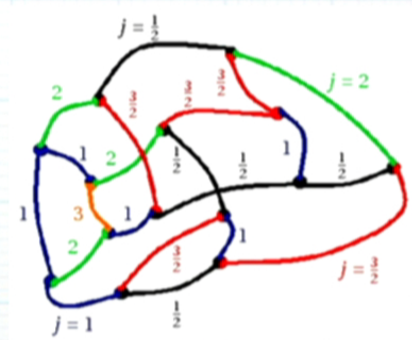
3d gravity as a TQFT can be exactly spinfoam quantized:



- 3d bulk triangulations or dual 2-complex
- Spins on edges j_e
- Amplitude as product of 6j-symbols

$$\mathcal{A}_\Delta = \sum_{\{j_e\}} \prod_e (2j_e + 1) \prod_T \{6j\}$$

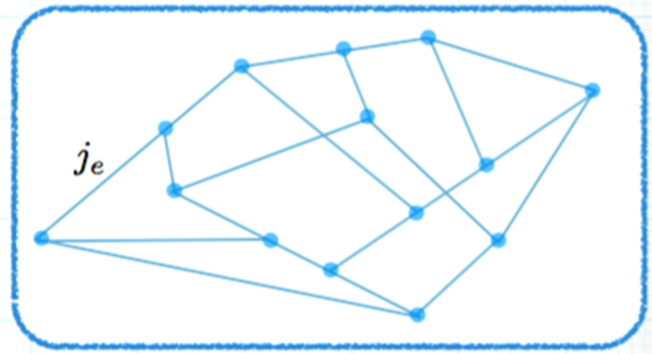
- Boundary 2d triangulated surface or dual 3-valent graph
- Spins on boundary edges or dual links: **boundary spin network**



Generating Function for Spin Network Evaluations

Consider 3-valent planar connected oriented boundary graph

Spin network evaluation is a 3nj symbol, obtained by gluing Clebsh-Gordan coefficients:



$$s^\Gamma(\{j_e\}) = \psi^\Gamma_{\{j_e\}}(\mathbb{1}) = \sum_{\{m_e\}} \prod_e (-1)^{j_e - m_e} \prod_v \begin{pmatrix} j_{e_1^v} & j_{e_2^v} & j_{e_3^v} \\ \epsilon^v_{e_1} m_{e_1^v} & \epsilon^v_{e_2} m_{e_2^v} & \epsilon^v_{e_3} m_{e_3^v} \end{pmatrix}$$

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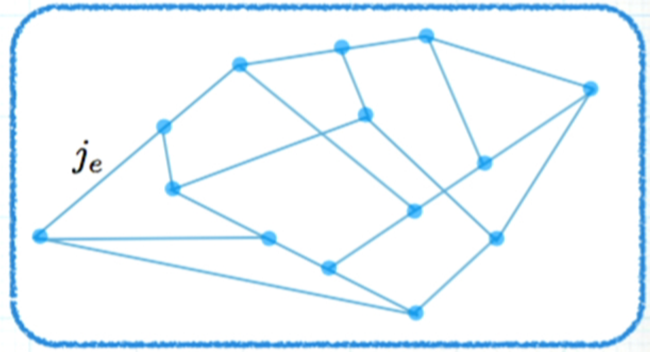

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A « technicality » :

choose Kasteleyn orientation on planar graph to fix signs, show evaluation is independent of choice of orientation & matches standard normalizations

Generating Function for Spin Network Evaluations

Consider **3-valent planar connected oriented boundary graph**

Define generating function for 3nj's using specific combinatorial weights:

$$Z_\Gamma^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v-2j_e)!}}\, s^\Gamma(\{j_e\}) \prod_e Y_e^{2j_e}$$

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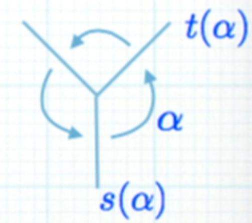
Get it from gluing the 3j-symbol generating functions using Gaussian weights:



$$\sum_{j_e,m_e} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \sqrt{(J+1)!} \prod_e \frac{Y_e^{j_e} z_e^{j_e+m_e} w_e^{j_e-m_e}}{\sqrt{(J-2j_e)!(j_e-m_e)!(j_e+m_e)!}}$$

$$= \exp \sum_\alpha X_\alpha (z_{s(\alpha)} w_{t(\alpha)} - w_{s(\alpha)} z_{t(\alpha)})$$

$$X_\alpha = \sqrt{Y_{s(\alpha)} Y_{t(\alpha)}}$$

Choose cyclic orientation (anti-clockwise) around each vertex

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Get it from gluing the 3j-symbol generating functions using Gaussian weights:

$$Z_\Gamma^{Spin}(\{Y_e\}) = \int \prod_{ev} \frac{d^2 z_{ev} d^2 w_{ev}}{\pi^2}\, e^{-\sum_{ev}(|z_{ev}|^2+|w_{ev}|^2)}$$
$$e^{-\sum_e (\bar{z}_{s(e)}\bar{w}_{t(e)} - \bar{w}_{s(e)}\bar{z}_{t(e)}) + \sum_\alpha X_\alpha (z_{s(\alpha)}w_{t(\alpha)} - w_{s(\alpha)}z_{t(\alpha)})}$$

Generating function as Vertex Integral

Generating function as half-edge integral :

$$Z^{Spin}(\{X_\alpha\}) = \int_{\mathbb{C}^{4E}} \prod_{e,v} \frac{e^{-\langle z_e^v | z_e^v \rangle}\, d^4 z_e^v}{\pi^2}\, e^{\sum_e \langle z_e^{s(e)} | z_e^{t(e)}]}\, e^{\sum_\alpha X_\alpha [z_{s(\alpha)} | z_{t(\alpha)} \rangle}$$

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Use a little spinorial trick :

$$\int_{\mathbb{C}^2} \frac{e^{-\langle z|z\rangle}\, d^4 z}{\pi^2}\, e^{\sum_{i\in I} [\zeta_i | z\rangle \langle \Omega | z_i \rangle + [\zeta_i | z] [\Omega | z_i\rangle} = e^{\sum_{i<j} [\zeta_i|\zeta_j\rangle [z_i|z_j\rangle}$$

Spinorial version of coherent intertwiner scalar product formula:

$$\int_{\mathrm{SU}(2)} dg\, e^{\sum_i [\zeta_i | g | z_i\rangle} = \sum_{J\in\mathbb{N}} \frac{1}{J!(J+1)!} \left(\sum_{i<j} [\zeta_i|\zeta_j\rangle [z_i|z_j\rangle \right)^J$$

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Generating function as vertex integral, as in spinfoams :

$$Z_\Gamma^{Spin}(\{X_\alpha = [\zeta_{s(\alpha)} | \zeta_{t(\alpha)} \rangle\}) = \int_{\mathbb{C}^{2V}} \prod_v \frac{e^{-\langle \xi_v | \xi_v \rangle} \, d^4 \xi_v}{\pi^2} \, e^{-\sum_e [\zeta_e^{s(e)} | \left(|\xi_{s(e)}\rangle \langle \xi_{t(e)}| + |\xi_{z(e)}] [\xi_{t(e)}| \right) | \zeta_e^{t(e)} \rangle}$$

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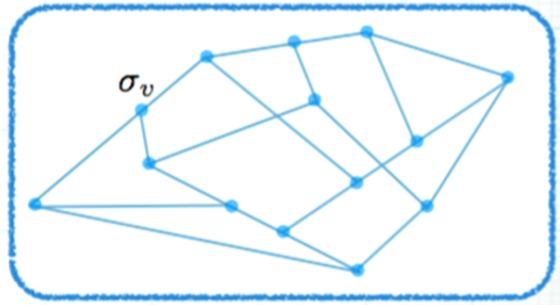
That's the usual spinfoam way !

Should be useful to generalize to arbitrary valence of nodes

The Ising Model Partition Function

On same graph, put « spins » on vertices: $\sigma_v = \pm 1 \in \mathbb{Z}_2$

$$Z_\Gamma^{Ising}(\{y_e\}) = \sum_\sigma \exp\left(\sum_e y_e \sigma_{s(e)} \sigma_{t(e)}\right)$$



Can define high temperature expansion...

$$Z_\Gamma^{Ising}(\{y_e\}) = (\prod_e \cosh(y_e)) \sum_\sigma \prod_e (1 + \tanh(y_e) \sigma_{s(e)} \sigma_{t(e)})$$

... as sum over loops:

$$Z_\Gamma^{Ising}(\{y_e\}) = 2^V (\prod_e \cosh(y_e)) \sum_{\gamma \in \mathcal{G}} \prod_{e \in \gamma} Y_e \quad \text{with } Y_e = \tanh y_e$$

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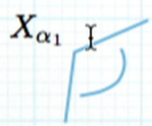
The Ising Model as a Fermion Path Integral

Two-level system naturally represented in terms of fermions.

Here explicitly: $Z_\Gamma^{Ising}(\{y_e\}) = 2^V \prod_e \cosh(y_e)\, Z_f(\{X_\alpha\})$

$$Z_f(\{X_\alpha\}) = \int \prod_{ev} d\psi_{ev} \, \exp\left(\sum_e \psi_{s(e)}\psi_{t(e)} + \sum_\alpha X_\alpha \, \psi_{s(\alpha)}\psi_{t(\alpha)} \right)$$

→ **We glue angles along edges to form loops, or vice-versa**



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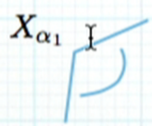

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And for our purpose:

$$(Z_f)^2 = \int \prod_{ev} [d\psi d\eta d\bar{\psi} d\bar{\eta}]_e^v\, e^{\sum_{e,v} \psi_e^v \bar{\eta}_e^v + \bar{\psi}_e^v \eta_e^v}$$

$$e^{-\sum_e \bar{\psi}_{s(e)}\bar{\psi}_{t(e)} + \bar{\eta}_{s(e)}\bar{\eta}_{t(e)}}\, e^{\sum_\alpha X_\alpha (\psi_{s(\alpha)}\psi_{t(\alpha)} + \eta_{s(\alpha)}\eta_{t(\alpha)})}$$

Matching Loop Expansions

All these Gaussian integrals can be computed explicitly !

$$(Z_f)^2\, Z_\Gamma^{Spin} = 1 \qquad Z_f = \sum_{\gamma \in \mathcal{G}} \prod_{\alpha \in \gamma} X_\alpha = \sum_{\gamma \in \mathcal{G}} \prod_{e \in \gamma} Y_e$$

$$(Z^{Ising})^2\, Z^{Spin} = 2^{2V} \prod_e \cosh(y_e)^2$$

⟶ **Duality between Ising model & Spin Evaluations**

- Westbury theorem
- Square lattice by Dittrich & Hnybida - arXiv:1312.4656

Duality through Supersymmetry

We can introduce a meta-theory combining

- Ising model ⟷ Fermions
- Spin networks ⟷ Bosons

$$\mathcal{Z}_\Gamma = (Z_f)^2\, Z^{Spin} = \int dz\, dw\, d\psi\, d\eta\, e^{S[\{z,w,\psi,\eta\}_{ev}]}$$

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$$\mathcal{Z}_\Gamma = (Z_f)^2\, Z^{Spin} = \int dz\, dw\, d\psi\, d\eta\, e^{S[\{z,w,\psi,\eta\}_{ev}]}$$

$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_\alpha X_\alpha S_\alpha$$

We define a supersymmetry generator, acting on each half-edge $i = (ev)$:

$$\begin{array}{rcl} Qz_i &=& \psi_i \\ Qw_i &=& \eta_i \\ Q\psi_i &=& w_i \\ Q\eta_i &=& -z_i \end{array}$$

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$$S = \sum_{e,v} \lambda_{e,v} K_{e,v} + \sum_e \mu_e S_e - \sum_\alpha X_\alpha S_\alpha$$

All terms are both Q-closed & Q-exact: $QK_{e,v} = QS_e = QS_\alpha = 0$

$$\begin{array}{lcl} K_{e,v} & = & Q\ (\psi\bar{w} - \eta\bar{z}) \\ S_e & = & Q\ (\overline{z\psi} + \overline{w\eta}) \\ S_\alpha & = & Q\ (z\psi + w\eta) \end{array} \longrightarrow \frac{\partial \mathcal{Z}_\Gamma}{\partial \lambda_{e,v}} = \frac{\partial \mathcal{Z}_\Gamma}{\partial \mu_e} = \frac{\partial \mathcal{Z}_\Gamma}{\partial X_\alpha} = 0$$

What to do with this Ising - Spin Network duality ?

Applications:

- Map spin averages to Ising correlations
- Higher order supersymmetric actions
- Phase diagram and critical Ising couplings
- Continuum Limit of QG Amplitudes

Mapping Spin Averages to Ising correlations

Compare spin insertions in both partition functions :

$$\langle \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_n} \rangle = \frac{1}{Z^{Ising}} \sum_{\sigma} \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_n} \, e^{\sum_e y_e \sigma_s(e) \sigma_t(e)}$$

$$\langle j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} \rangle = \frac{1}{Z^{Spin}} \sum_{\{j_e\}} j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} \, s(\Gamma, \{j_e\}) \mathcal{W}(\{j_e\}) \prod_e (\tanh y_e)^{2 j_e}$$

Can get general relation :

$$\langle j_e \rangle = \sinh y_e \big(\sinh y_e - \cosh y_e \, \langle \sigma_{s(e)} \sigma_{t(e)} \rangle \big)$$

$$\langle \sigma_v \sigma_w \rangle_c^{(\mathcal{P})} = \frac{-2^{n-1}}{\prod_{e \in \mathcal{P}} \sinh(2 j_e)} \langle \prod_{e \in \mathcal{P}} (2 j_e) \rangle_c^{(\mathcal{P})}$$

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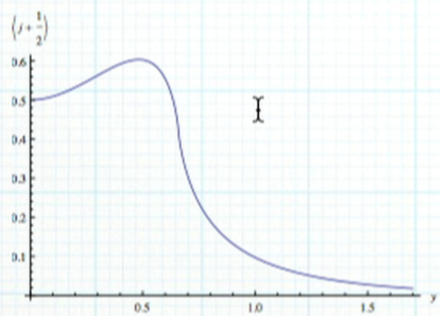
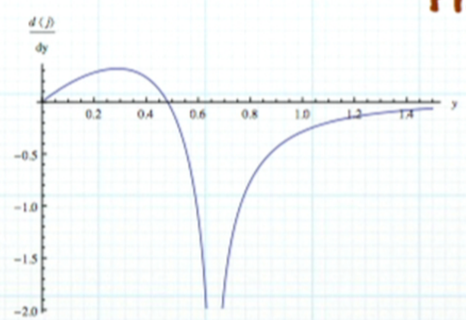
$$\langle j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} \rangle = \frac{1}{Z^{Spin}} \sum_{\{j_e\}} j_{e_1}^{n_1} j_{e_2}^{n_2} \cdots j_{e_k}^{n_k} \; s(\Gamma, \{j_e\}) \mathcal{W}(\{j_e\}) \prod_e (\tanh y_e)^{2j_e}$$

Get exact formula for spin average :

Phase Transition !!





Critical Ising & Spin Network Saddle Points

Let's come back to the combinatorial definition of the generating function of spin network evaluations:

$$Z_\Gamma^{Spin}(\{Y_e\}) = \sum_{\{j_e\}} \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v-2j_e)!}}\, s^\Gamma(\{j_e\}) \prod_e Y_e^{2j_e}$$

Spin distribution defined by statistical weight ?

$$\rho(\{j_e\}) = \sqrt{\frac{\prod_v (J_v+1)!}{\prod_{ev}(J_v-2j_e)!}} \prod_e Y_e^{2j_e}$$

Saddle point? Geometrical Interpretation?

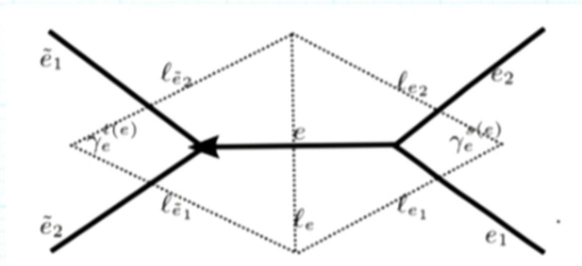
Critical Ising & Spin Network Saddle Points

We proceed as usual:

- Large spin approx, Stirling formula
- Look for stationary point(s)
- Interpret spins as lengths

We get a stationary point when spins j_e are length of a triangulation if the edge couplings Y_e are determined by the condition in terms of the triangulation angles:

$$Y_e^2 = \tan\frac{\gamma_e^{s(e)}}{2}\,\tan\frac{\gamma_e^{t(e)}}{2}$$



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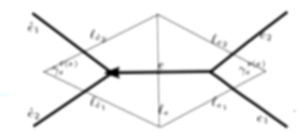
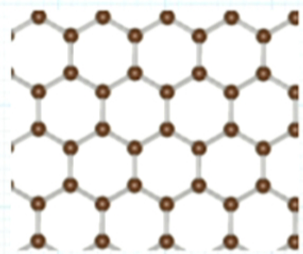


- Regular honeycomb network

$$Y = \frac{1}{\sqrt{3}} = Y^{critical}$$



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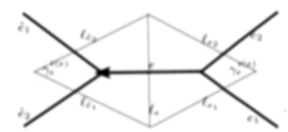

- Regular honeycomb network

$$Y = \frac{1}{\sqrt{3}} = Y^{critical}$$

- Also isoradial graphs !

$$Y_e^c = \tan\frac{\gamma_e}{2} = \tan\frac{\theta_e}{2}$$

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More general ?!?

Critical Ising & Spin Network Saddle Points

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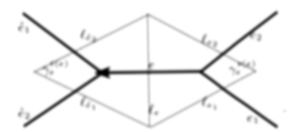

Admissible geometric couplings Y_e^{geom} → Scale invariant saddle points in spins j_e → Pole in the generating function $Z^{Spin} \to \infty$ → Zero of Ising partition function i.e. critical couplings Y_e^c

Test all this on the ...

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The Tetrahedron & the 6j Symbol

Test all this on the Tetrahedron !

- Look at generating function for 6j symbols
- Study saddle points of combining both weight & 6j symbol with Regge action at large spins
- Provide geometrical interpretation for Fisher zeroes on tetrahedron graph

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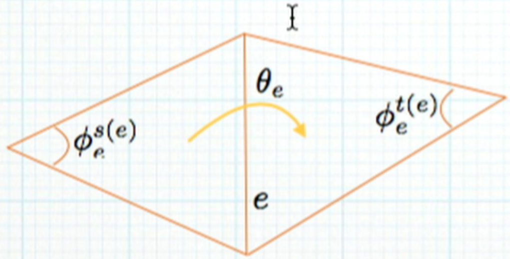
Critical couplings for Ising are complex, with phase given by dihedral angles

$\epsilon = \pm$ **global sign**



$$Y_e^c = e^{\epsilon \frac{i}{2} \theta_e} \sqrt{\tan \frac{\phi_e^{s(e)}}{2} \tan \frac{\phi_e^{t(e)}}{2}}$$

only depends on geometry up to global scale factor !

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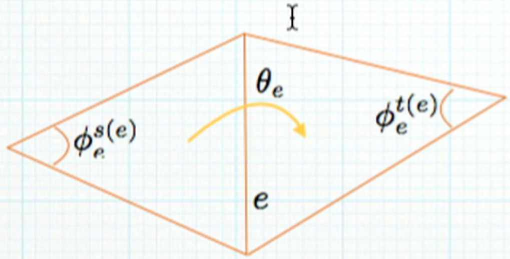

can see it on spherical tetrahedron ...!

Duality between Ising & Spin Networks - Livine - Renormalization in PI '15

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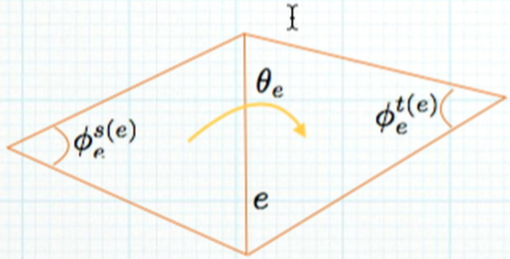

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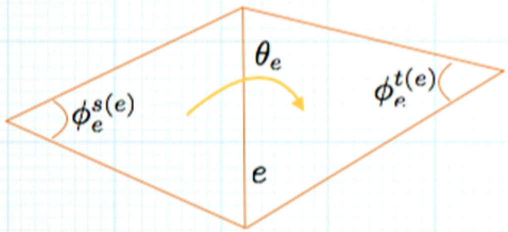
The Tetrahedron & the 6j Symbol

Critical couplings for Ising are complex, with phase given by 3d dihedral angles and modulus given by 2d triangle angles



$$Y_e^c = e^{\epsilon \frac{i}{2}\theta_e} \sqrt{\tan\frac{\phi_e^{s(e)}}{2} \tan\frac{\phi_e^{t(e)}}{2}}$$

These are roots of the tetrahedron loop polynomial :

$$P[Y_e] = 1 + Y_1Y_2Y_6 + Y_1Y_3Y_5 + Y_2Y_3Y_4 + Y_4Y_5Y_6 + Y_1Y_4Y_2Y_5 + Y_2Y_5Y_3Y_6 + Y_1Y_4Y_3Y_6$$

Direct proof is painful...

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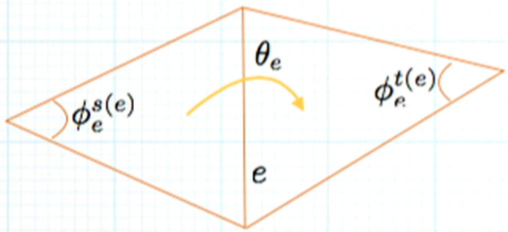

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and this only gives a 5d manifold within the 10d space of solutions

Have to go to complex tetrahedra ! Work in progress

The Tetrahedron & the 6j Symbol

Can have more fun on tetrahedron with high T/low T duality

Use loop expansion of 2d Ising to show duality identity on the partition function :

High T loop expansion:

$$Z_\Gamma(y_e) = \sum_{\{\sigma_{v=\pm 1}\}} e^{\sum_e y_e \sigma_{s(e)} \sigma_{t(e)}} = 2^V \prod_e \cosh y_e \sum_{C \subset \Gamma} \prod_{e \in C} \tanh y_e$$

Low T cluster expansion:

$$Z_\Gamma(y_e) = 2 \prod_e e^{y_e} \sum_{C^* \subset \Gamma^*} \prod_{e \in C^*} e^{-2y_e}$$

The Tetrahedron & the 6j Symbol

Can have more fun on tetrahedron with high T/low T duality

Use loop expansion of 2d Ising to show duality identity on the partition function :

$$Z_\Gamma(y_e) = \frac{2\prod_e e^{y_e}}{2^{V^*}\prod_e \cosh\tilde{y}_e} Z_{\Gamma^*}(\tilde{y}_e)$$

with dual couplings $Y_e = \tanh y_e = e^{-2\tilde{y}_e}, \quad \tilde{Y}_e = \tanh\tilde{y}_e = e^{-2y_e}$

$$Y = \mathcal{D}(\tilde{Y}) = \frac{(1-\tilde{Y})}{(1+\tilde{Y})}$$

Duality transform is involution, relating the graph and its dual

$$\tilde{Y} = \mathcal{D}(Y) = \frac{(1-Y)}{(1+Y)}$$

Its fixed point is critical Ising coupling for square lattice : $Y_c = -(1\pm\sqrt{2})$

The Tetrahedron & the 6j Symbol

Can have more fun on tetrahedron with high T/low T duality

Apply to 6j generating function :

$$4^3 \sum_{\{j_e\}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \prod_v \Delta_v(j_e) \prod_e (-1)^{2k_e} T(2j_e+1, 2k_e+1) = \begin{Bmatrix} k_4 & k_5 & k_6 \\ k_1 & k_2 & k_3 \end{Bmatrix} \prod_{v^*} \Delta_{v^*}(k_e)$$

with transform coefficients given by power series :

$$Y \frac{(1-Y)^{2j}}{(1+Y)^{2(j+1)}} = \sum_{k \in \mathbb{N}/2} (-1)^{2k} T(2j+1, 2k+1)\, Y^{2k+1}$$

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Not yet sure what to do with new relation or what it means, but ...

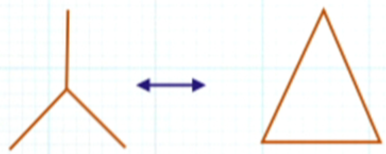
From Coarse-Graining Ising to boundary Pachner moves

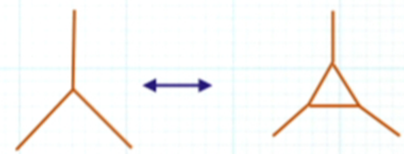
Natural application of duality between Ising models & spin networks :

COARSE-GRAINING

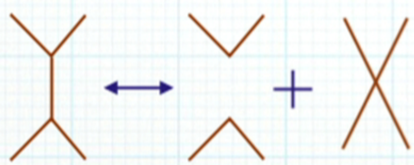
Star-Triangle relation → 3-1 Pachner move

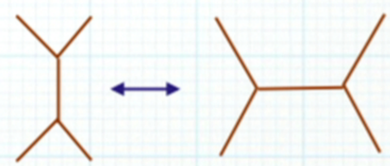
Related to Yang-Baxter equation





Tutte recursion relation → 2-2 Pachner move

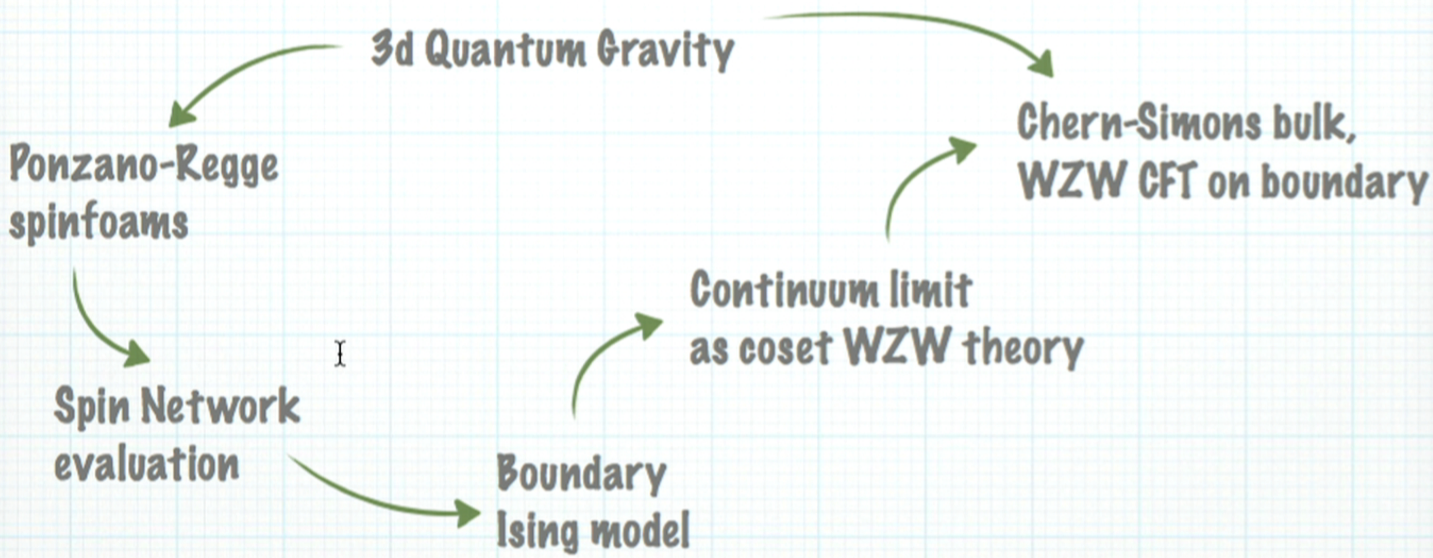




Continuum limit and boundary CFT

Use known continuum limit of Ising models to derive boundary CFT description of Ponzano-Regge spinfoam models at critical point

Let's try to close the loop :



Ising-QG Duality: Extensions & Prospects

- Technial improvements: arbitrary valence, non-planar graphs, magnetic field (Lee-Yang theorem), q-deformation, dual Potts model ?

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- Sum over all triangulations à la Kazakov
- Continuum limit of Ising model as WZW coset model, boundary CFT for spinfoams & models for conformal gravity

Duality between 2d Ising and 3d Quantum Gravity

Thank you for your attention !!