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Abstract:

Spin Foams Without Spins

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Motivation

Towards simpler spin foam models:

- We like the EPRL-FK models
- Besides some heroic efforts, these models are hard to compute much with
- There are 'equivalent' simplicity constraints for Euclidean QG which are nicer [Dupuis, Livine]

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \qquad |z_L\rangle = \rho |z_R\rangle$$

Plan:

- Introduction to coherent state formulation of spin foams
- Computing with holomorphic simplicity constraints
- Formulating spin foams with generating functions

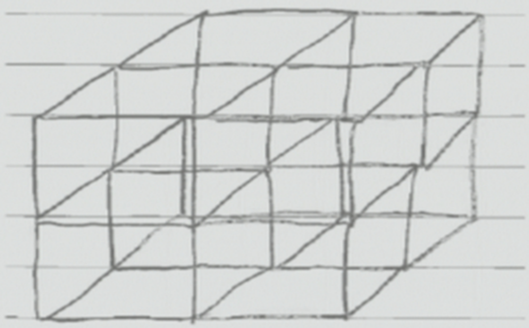
Lattice Gauge theory

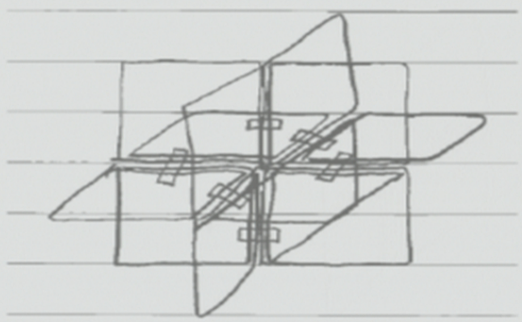
Wilson action

$$S(g_e) = \sum_f \mathrm{tr}(G_f) \qquad G_f = g_{e_1} g_{e_2} \cdots g_{e_k}$$

The Lattice Gauge theory partition function

$$Z_{\mathrm{YM}}(\beta) = \int \mathrm{d}g_e \, e^{\frac{\beta}{2} S(g_e)}$$





The usual Spin Foam Expansion

SU(2) character expansion

$$e^{\frac{\beta}{2}\mathrm{tr}(G)} = \frac{1}{(\beta/2)} \sum_{j\in\mathbb{N}/2} (2j+1) I_{2j+1}(\beta)\chi_j(G)$$

The modified Bessel function acts as an effective heat kernel because for large j

$$\frac{I_{2j+1}(\beta)}{I_1(\beta)} \approx e^{-\frac{2j(j+1)}{\beta}}$$

and for large β we get the delta function

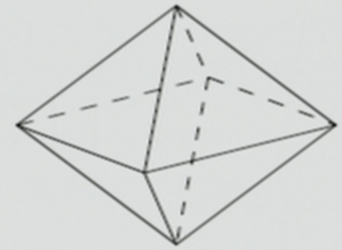
$$e^{\frac{\beta}{2}\mathrm{tr}(G)} \approx \frac{I_1(\beta)}{(\beta/2)}\delta(g)$$

At this point, integrating over products of $\chi(g)$ leads to recoupling coefficients and the usual spin foam sum. We want to avoid this.

The Dual of Lattice Gauge theory

SU(2) lattice Yang-Mills on a simple cubic lattice

$$Z_{\mathrm{YM}}(\beta) = \sum_{\{j_f\}} \sum_{\{\iota_e\}} (-1)^{\chi} \prod_f (2j_f+1) I_{2j_f+1}(\beta) \prod_v$$



Pros:

- This is a strong coupling expansion
- Geometric picture, spins as areas, 6j as tetrahedra,...
- Semiclassical limit: Regge action

Cons:

- Recoupling theory
- Numerically intractable
- Continuum limit?

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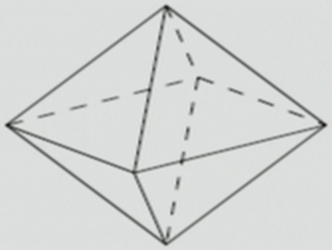

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Advantages of Coherent States

We want to avoid recoupling theory and sums over spins. This makes it easier to perform analytic calculations.
Plan:

- Introduce Bargmann-Fock representation on holomorphic functions on spinor space $\mathcal{L}^2_{\text{hol}}(\mathbb{C}^2, \mathrm{d}\mu)$
- Derive overcomplete set of states from a Gaussian integral
- Factorize the group elements in $\chi_j(g_1 g_2 \cdots g_k)$
- Obtain coherent projector onto SU(2) invariant subspace

What this achieves:

- Sum over spins are eliminated
- Sum over intertwiners replaced by integrals over spinors
- Vertex recoupling coefficients replaced by generating functions
- Retain geometric interpretation

The Holomorphic Representation

The space $L^2_{\text{hol}}(\mathbb{C}^2, \mathrm{d}\mu)$ of holomorphic functions is a representation space for SU(2) known as the Bargmann-Fock space

$$\rho(g) \cdot F(z) = F(g^{-1}z) \qquad \langle F|F'\rangle = \int \mathrm{d}\mu(z)\overline{F(z)}F'(z)$$

with Gaussian measure

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}^4 z}{\pi^2} e^{-\langle z|z\rangle}$$

where we use a use a bra-ket notation for spinors

$$\langle w| = (\bar{w}_0, \bar{w}_1), \qquad |z\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \qquad \langle w|z\rangle = \delta^{AB}\bar{w}_A z_B$$

Analytic expansion of $F(z)$, terms of different homogeneity (spin) are orthogonal (countable number of unitary irreducible representations).

The Reproducing Kernel

Standard one complex Gaussian integral

$$\int_{\mathbb{C}} \mathrm{d}\mu(\alpha)\, e^{a\alpha + b\bar{\alpha}} = e^{ab} \qquad \mathrm{d}\mu(\alpha) = \frac{\mathrm{d}^2\alpha}{\pi} e^{-|\alpha|^2}$$

Now in two dimensions $|x\rangle = (\alpha_0, \alpha_1)^t$, $\langle x| = (\bar{\alpha}_0, \bar{\alpha}_1)$

$$\int_{\mathbb{C}^2} \mathrm{d}\mu(x)\, e^{\langle z|x\rangle + \langle x|w\rangle} = e^{\langle z|w\rangle} \qquad \mathrm{d}\mu(x) = \frac{\mathrm{d}^4 x}{\pi^2} e^{-\langle x|x\rangle}$$

Hence we have the overcomplete set of states on spin $k/2$

$$\int_{\mathbb{C}^2} \mathrm{d}\mu(x)\, \frac{\langle z|x\rangle^k \langle x|w\rangle^{k'}}{k!k'!} = \frac{\langle z|w\rangle^k}{k!} \delta_{kk'}$$

Isolate the Group Elements

Trace in coherent states

$$\chi_j(G) = \int_{\mathbb{C}^2} \mathrm{d}\mu(z) \frac{\langle z|g_1 g_2 g_3 g_4|z\rangle^{2j}}{(2j)!}$$

Factorize all the group elements

$$\chi_j(G) = \int_{\mathbb{C}^{2\times 4}} \mathrm{d}\mu(z_i) \frac{\langle z_4|g_{e_1}|z_1\rangle^{2j} \langle z_1|g_{e_2}|z_2\rangle^{2j} \langle z_2|g_3|z_3\rangle^{2j} \langle z_3|g_{e_4}|z_4\rangle^{2j}}{(2j)!(2j)!(2j)!(2j)!}$$

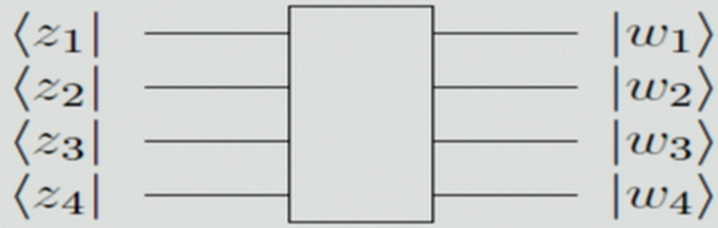
Collecting all group elements belonging to an edge we now have a coherent intertwiner on each edge of n wires carrying n spins j_i

$$P_{j_i}(z_i, w_i) = \int_{\mathrm{SU}(2)} \mathrm{d}g \prod_{i=1}^{n} \frac{\langle z_i|g|w_i\rangle^{2j_i}}{(2j_i)!}$$

The Coherent Projector

By orthogonality we can sum over spins

$$P(z_i, w_i) = \int_{\mathrm{SU}(2)} \mathrm{d}g \, e^{\sum_i \langle z_i | g | w_i \rangle} =$$



This is a projector

$$P : L^2_{\mathrm{hol}}(\mathbb{C}^{2n}, \mathrm{d}\mu) \to \mathrm{Inv}_{\mathrm{SU}(2)} \left(L^2_{\mathrm{hol}}(\mathbb{C}^{2n}, \mathrm{d}\mu) \right)$$

onto the subspace of SU(2) invariant functions since

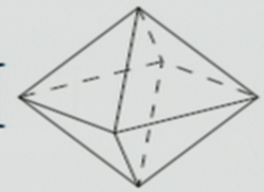
$$P \circ f = \int_{\mathbb{C}^{2n}} \mathrm{d}\mu(z_i) P(z_i, w_i) f(z_i) = \int_{\mathrm{SU}(2)} \mathrm{d}g \, f(g w_i)$$

and $P \circ P = P$. We now focus our study on P.

End of Introduction

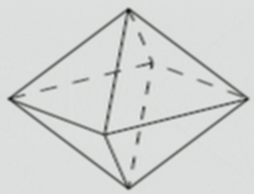
Nothing new has been done so far. A spin foam is defined by three quantities: (Δ, A_f, P) the two complex, the face wieghts and the projector.

$$Z = \sum_{\{j_f\}} \sum_{\{\iota_e\}} (-1)^{\chi} \prod_f A_f \prod_v$$



We have used a coherent basis of intertwiners which has a closed form.

$$\sum_{\{\iota_e\}} \prod_v$$



$$= \int \mathrm{d}\mu(z_f^e, w_f^e) \;\lrcorner_v P(z_f^e, w_f^e)$$

Let us now see the advantages of this.

Towards Simpler Models

A Tour of Results:

1. Coarse graining [Banburski, Chen, JH, Freidel]
2. Holomorphic simplicity constraints [Dupuis, Livine]
3. Generating functions [Freidel, JH, Bonzom, Livine]
4. Closure can be imposed strongly [Conrady, Freidel, Livine]

Final result: Spin foams without spins

$$Z_{\mathrm{YM}}(\beta) = \oint_{\gamma_0^F} \mathrm{d}A_f \oint_{\gamma_0^E} \mathrm{d}A_e \int_{\mathbb{C}^{2EF}} \mathrm{d}\mu(x_f^e)\delta\left(\sum_f |x_f^e\rangle\langle x_f^e| - 1\right) \prod_v A_v$$

with generating functions for vertex amplitudes

$$A_v = (1 + \sum_{\text{cycles}} A_C)^{-2}$$

Handling Face Weights

Recall the projector

$$P(z_i, w_i) = \int_{\mathrm{SU}(2)} \mathrm{d}g \, e^{\sum_i \langle z_i | g | w_i \rangle}$$

Tracing over one strand with different kernels gives

$$\int \mathrm{d}\mu(z,w)\delta_{BF}(z,w)e^{\langle z|g|w\rangle} = \delta(g)$$
$$\int \mathrm{d}\mu(z,w)\delta_{YM}(z,w,\beta)e^{\langle z|g|w\rangle} = e^{\frac{\beta}{2}\mathrm{tr}(g)}$$

where the kernels are

$$\delta_{BF}(z,w) = (\langle z_n|w_n\rangle - 1)e^{\langle w_n|z_n\rangle}$$
$$\delta_{\mathrm{YM}}(z_n, w_n, \beta) = \frac{1}{2\pi i}\oint_{\gamma_0} \mathrm{d}\tau \, \frac{\beta}{2}(\tau^{-1} - \tau)\, e^{\frac{\beta}{2}(\tau+\tau^{-1})+\tau\langle z|w\rangle}$$

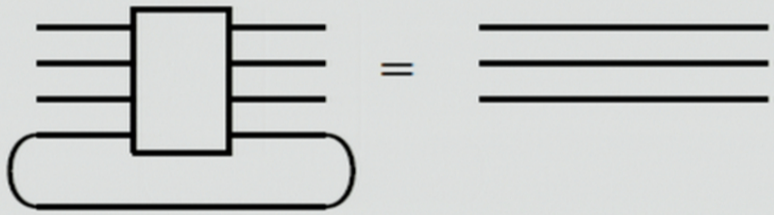
The point is that face weights are not a problem.

The Loop Identity

Applying these kernels to the BF projector

$$\int \mathrm{d}\mu(z_n, w_n)\delta_{BF}(z_n, w_n)P(z_i, w_i) = \int_{\mathrm{SU(2)}} \mathrm{d}g\, e^{\sum_{i=1}^{n-1}\langle z_i|g|w_i\rangle}\delta(g)$$
$$= e^{\sum_{i=1}^{n-1}\langle z_i|w_i\rangle}$$

which is the loop identity (responsible for triangulation invariance)



For Yang-Mills we get

$$\int \mathrm{d}\mu(z_n, w_n)\, \delta_{\mathrm{YM}}(z_n, w_n, \beta)P(z_i, w_i) = \frac{\beta}{2r} I_1(2r)$$

where $r^2 = \det(\beta/2 + \sum_i |w_i\rangle\langle z_i|)$. What about with simplicity constraints?

Holomorphic Simplicity Constraints

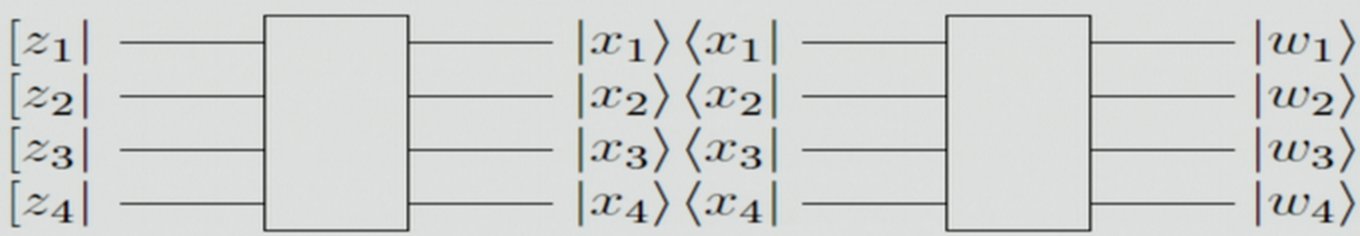
For Euclidean spin foam models with gauge group $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ the projector is

$$P(z_i^L, w_i^L)P(z_i^R, w_i^R)$$

and the simplicity constraints of Dupuis, Livine are something like $|z_i^L\rangle = \rho |z_i^R\rangle$. However in order to apply this on the boundary of a vertex we must use

$$P \circ P = \int \mathrm{d}g \mathrm{d}h \int \prod_i \mathrm{d}\mu(x_i) e^{\sum_i [z_i|g|x_i\rangle + \sum_i \langle x_i|h|w_i\rangle}$$

$= \int \mathrm{d}\mu(x_i)$



and we should impose $|x_i^L\rangle = \rho |x_i^R\rangle$

Internal Simplicity Constraints

Naively setting $|z_i^L\rangle = \rho|z_i^R\rangle$ on the spinors contracted at at a vertex we get [Banburski,Chen,JH,Freidel]

$$P(\rho z_i, \rho w_i)P(z_i, w_i)$$

and the contracted spinors also combine in a twisted-Regge action in the large spin limit, hence these simplicity constraints have a similar interpretation. Integrating out the group elements

$$P(\rho z_i, \rho w_i)P(z_i, w_i) = \sum_J {}_2F_1(-J-1, -J; 2; \rho^4)\frac{(z_i|w_i)^J}{J!(J+1)!}$$

- the loop identity is relatively easy to compute
- compute degrees of divergence of Pachner moves
- get nonlocality: propose truncation
- nontrivial competition of degree of face weight and ρ
- more information: Linqing's talk on Friday

Generating Functions and Gauge Fixing

That ends the discussion on coarse graining in holomorphic spin foam models.

I will now show:

1. How to express the spin foam representation in terms of generating functions
2. How to gauge fix $SL(2,\mathbb{C})$ invariance by imposing closure constraints strongly [Conrady, Freidel, Livine]

Generating Functionals

Recall the U(n) invariant scalar product

$$(z_i|w_i) = \sum_{i,j} \epsilon_{AB}\epsilon_{CD}\bar{z}_i^A \bar{z}_j^B w_i^C w_j^D$$

The vertex amplitude is the contraction in the pattern of a graph Γ. Assign a spinor x_j^i to vertex i directed to vertex j then

$$\int \mathrm{d}\mu(w_j^i) e^{\sum_{i\in\Gamma}(w_j^i|x_j^i)}\Big|_{w_i^j=\epsilon_{ij}\check{w}_j^i} = \frac{1}{\det\left(1+X^\Gamma\right)}$$

where the 2x2 block elements are

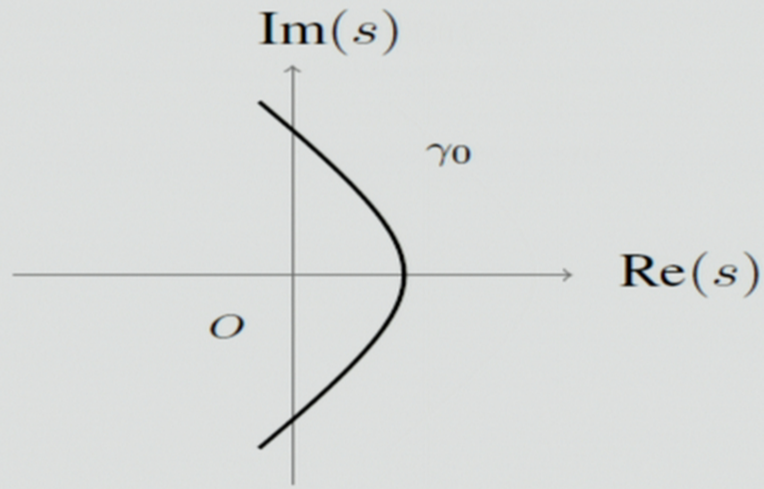
$$X_{ij}^\Gamma = \epsilon_{ij}|x_j^i\rangle[x_i^j|$$

We need the U(n) invariants to be exponentiated!

An Integral representation of the Projector

An integral representation of the reciprocal Gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\oint_{\gamma_0} \mathrm{d}s\, s^{-z} e^{s},$$

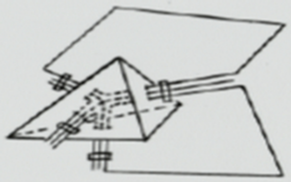


Therefore replacing the $(J+1)!$ in the projector

$$P(z_i, w_i) = \sum_{J=0}^{\infty} \frac{(z_i|w_i)^J}{J!(J+1)!} = \frac{1}{2\pi i}\oint_{\gamma_0} \frac{\mathrm{d}s}{s^2}\, e^{s+\frac{1}{s}(z_i|w_i)}$$

The Vertex Amplitude

Contracting projectors at a vertex



$$= \oint_{\gamma_0^4} \frac{\mathrm{d}s_i}{(2\pi i)^4} \frac{e^{\sum_i s_i}}{\det\left(D(s_i) + X^\Gamma\right)}$$

where $D(s_i) = \mathrm{diag}(s_1, s_2, s_3, s_4)$. For a tetrahedron

$$\oint_{\gamma_0^4} \frac{\mathrm{d}s_i}{(2\pi i)^4} \frac{e^{s_1+s_2+s_3+s_4}}{(s_1 s_2 s_3 s_4 - s_1 A_{234} - s_2 A_{134} - s_3 A_{124} - s_4 A_{123} + A_{1234} - A_{1342} - A_{1423})^2}$$

where for each cycle of the graph

$$A_{12\ldots p} = [x_p^1 | x_2^1\rangle [x_1^2 | x_3^2\rangle \cdots [x_{p-1}^p | x_1^p\rangle$$

Gauge Fixing

We want to reduce the spinors x_i

$$P \circ P = \frac{1}{(2\pi i)^2} \oint_{\gamma_0} \oint_{\gamma_0} \frac{\mathrm{d}s\mathrm{d}t}{(st)^2} \int_{\mathbb{C}^{2n}} \mathrm{d}\mu(x_i) e^{s+t+\frac{1}{s}(z_i|x_i)+\frac{1}{t}(x_i|w_i)}$$

by imposing the closure constraints

$$\sum_i |x_i\rangle\langle x_i| = 1$$

in the path integral by inserting the following identity

$$\int_{\mathrm{GL}(2,\mathbb{C})} \frac{\mathrm{d}^8 g}{|\det(g)|^4} \delta^{(4)}\left(\sum_i g^{-1}|x_i\rangle\langle x_i|(g^{-1})^\dagger - 1\right) = \frac{\mathrm{Vol}(\mathrm{U}(2))}{2}$$

The projector over the Grassmannian

The result is:

$$P(z_i, w_i) = \oint_{\gamma_0} \oint_{\gamma_0} \frac{\mathrm{d}s\,\mathrm{d}t}{(2\pi i)^2} K_n(st) \int_{\mathbb{C}^{2n}} \mathrm{d}\hat{\Omega}(x_i) e^{\frac{1}{s}(z_i|x_i) + \frac{1}{t}(x_i|w_i)}$$

with the normalized measure over $\mathrm{Gr}(2, n)$

$$\mathrm{d}\hat{\Omega}(x_i) = 2 \frac{\mathrm{Vol}(\mathrm{Gr}(2,n))}{\mathrm{Vol}(\mathrm{U}(2))} \delta^{(4)} \left(\sum_i |x_i\rangle\langle x_i| - 1 \right) \prod_i \mathrm{d}^4 x_i$$

with the kernel

$$K_n(st) \equiv \frac{{}_2F_1(n-1, n-2; 1; st)}{(n-1)(n-2)(st)^2}$$

Spin Foams without Spins

$$Z_{\mathrm{YM}}(\beta)=\oint_{\gamma_0^F}\frac{\mathrm{d}\tau_f}{2\pi i}\delta_{\mathrm{YM}}(\tau_f,\beta)\oint_{\gamma_0^{2E}}\frac{\mathrm{d}s_e\mathrm{d}t_e}{(2\pi i)^2}K_{n_e}(s_et_e)\int_{\mathrm{Gr}(2,n_e)^E}\mathrm{d}\hat{\Omega}(x_f^e)\prod_v A_{\Gamma_v}(x_f^e,s_e,t_e,\tau_f)$$

- γ_0 is a contour which encircles the origin in a counter-clockwise manner
- $\delta_{\mathrm{YM}}(\tau_f,\beta)$ is responsible for the Yang-Mills regularization which approaches BF theory in the $\beta\to\infty$ limit
- $\mathrm{d}\hat{\Omega}(x_i)$ imposes closure $\delta^{(4)}\left(\sum_i |x_i\rangle\langle x_i| - 1\right)\prod_i \mathrm{d}^4x_i$
- The vertex amplitudes $A_{\Gamma_v}(x_f^e, s_e, t_e, \tau_f)$ are given by the spin network generating functional of the boundary graph Γ_v dual to v

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Conclusions

- What is the physical meaning of the contour variables?
- What does the canonical action of the conformal group represent?
- Can we evaluate the generating functions as in the 2d Ising model?
- Can we also impose matching constraints?