

Title: PSI 2015/2016 Quantum Field Theory I - Daniel Wohns and Tibra Ali - Lecture 11

Date: Oct 27, 2015 09:00 AM

URL: <http://pirsa.org/15100046>

Abstract:

Causality & Feynman propagator for fermions

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0, \quad (x-y)^2 < 0.$$

$$[\psi(x), \psi(y)] = 0, \quad (x-y)^2 < 0$$

$$[\psi_a(x), \bar{\psi}_b(y)] \neq 0, \quad (x-y)^2 < 0$$

Fermionic observables: $\mathcal{O}(x) = \bar{\psi}(x) \Gamma \psi(x)$
 $\hookrightarrow$ combination of γ -matrices.

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↳ Combination of γ -matrices.

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$$\{\psi_a(x), \bar{\psi}_b(y)\} = (i \not{\partial}_x + m)_{ab} [D(x-y) - D(y-x)]$$

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3 2E_p} e^{-ip(x-y)}$$

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$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3 2E_p} e^{-ip(x-y)} \quad x^0 = y^0$$

$$\gamma^\mu \psi(x)$$

↳ Combination of γ -matrices.

$$\text{for } (x-y)^2 < 0.$$

$$+m)_{ab} [D(x-y) - D(y-x)]$$

$$e^{-i\tau(x-y)}$$

$$x^0 = y^0$$

$$Q = \int d^3x \bar{\psi} \gamma^0 \psi$$

$$[AB, CD] = ABCD - CDAB$$

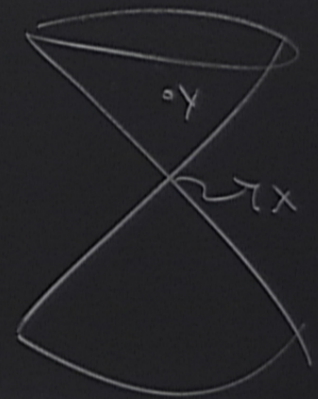
$$= ABCD + ACBD - ACBD - CDAB$$

$$= A\{B, C\}D - \dots$$

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Feynman propagator:

$$S_F(x-y)_{ab} = [\langle 0 | T(\psi_a(x) \bar{\psi}_b(y)) | 0 \rangle]$$

$$= \begin{cases} \langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle & \text{if } x^0 > y^0 \\ -\langle 0 | \bar{\psi}_a(y) \psi_b(x) | 0 \rangle & \text{if } y^0 > x^0 \end{cases}$$

$$S_F = \langle 0 | T(\psi \bar{\psi}) | 0 \rangle$$

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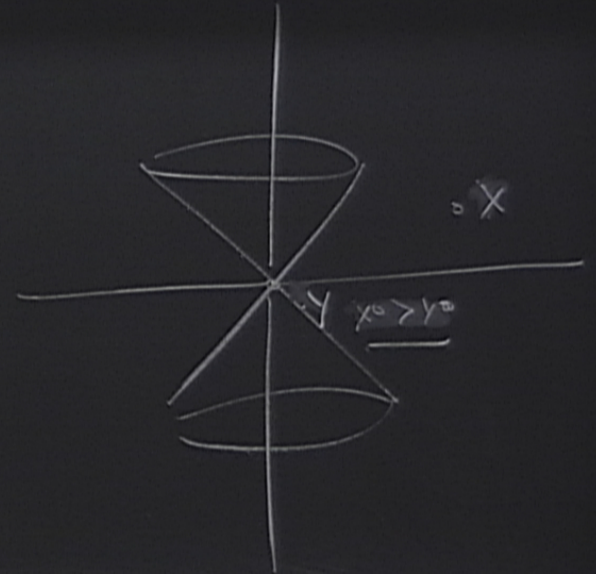
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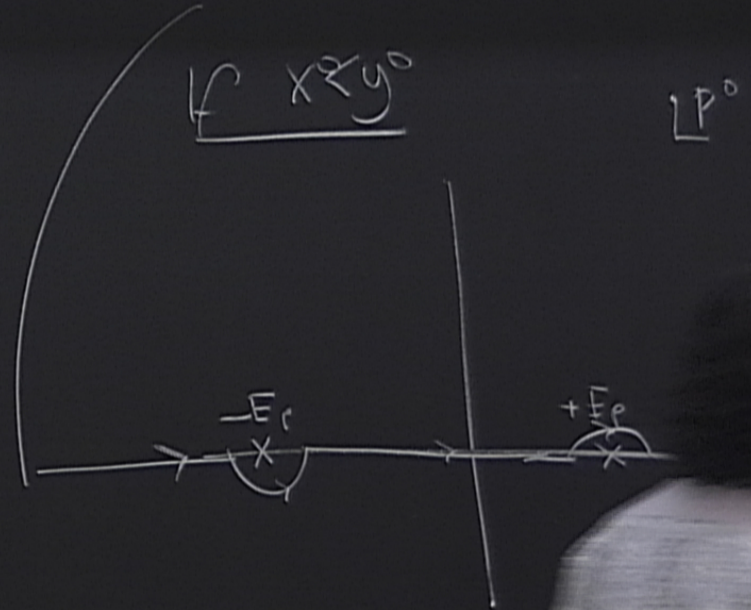
$$S_F = \langle 0 | T(\psi \bar{\psi}) | 0 \rangle$$

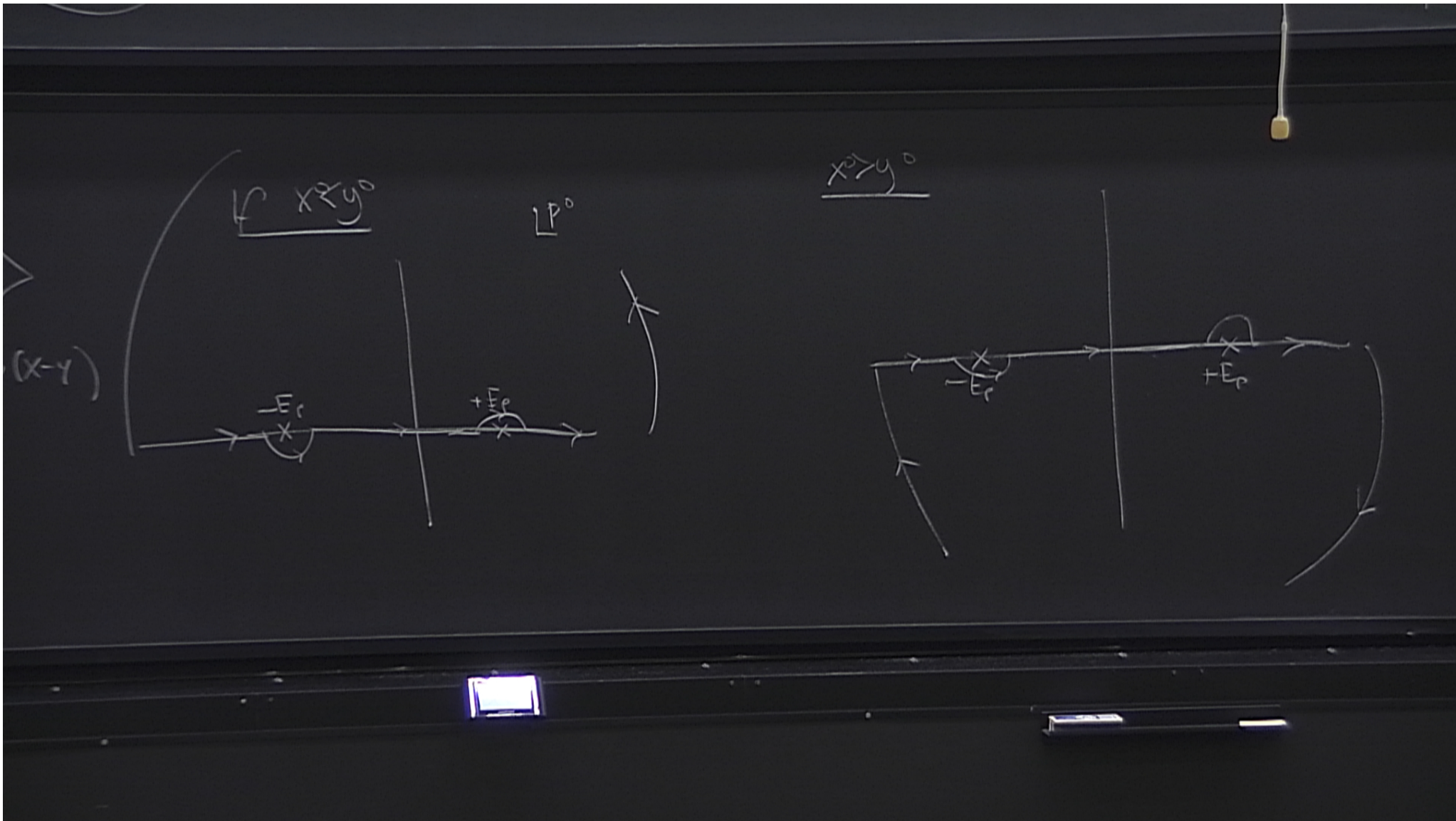


Explicitly:

$$S_F = \langle 0 | T(\psi(x) \bar{\psi}(y)) | 0 \rangle$$

$$= i \int \frac{d^4 p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

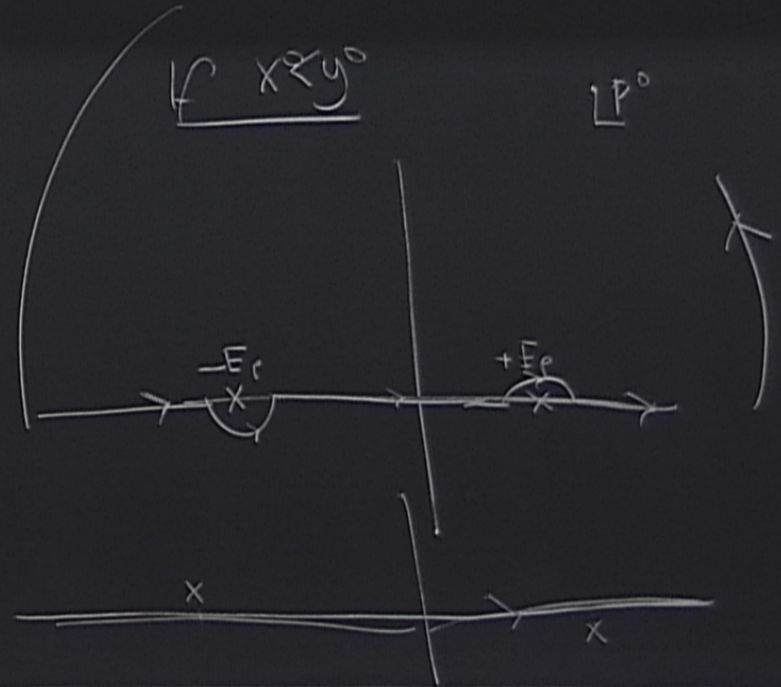




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Yukawa Theory

(1934) ~ Nucleons interact by exchanging
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A generalization of Yukawa interaction plays the role renormalizable coupling between chiral fermions

and

and Higgs field in the SM which
breaks the gauge symmetries of
the theory.

gauge
field theory

fields
fermions

$$L_{\text{incl.}} = L_f + L_\varphi + L_{\text{Yuk}}$$

$$L_f = \bar{\psi} (i \not{\partial} - m) \psi$$

$$L_\varphi = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} M^2 \varphi^2, \quad m > M$$

$$L_{\text{Yuk}} = -\lambda \varphi \bar{\psi} \psi$$

$$\mathcal{L}_{\text{incl.}} = \mathcal{L}_f + \mathcal{L}_\varphi + \mathcal{L}_{\text{Yuk}}$$

$$\mathcal{L}_f = \bar{\psi} (i \not{\partial} - m) \psi$$

$$\mathcal{L}_\varphi = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} M^2 \varphi^2, \quad m > M, \lambda > 0$$

$$\mathcal{L}_{\text{Yuk}} = -\lambda \varphi \bar{\psi} \psi \quad [\lambda] = 0$$

$U(1)$ symmetry

$$\psi \rightarrow e^{i\alpha} \psi$$

$$\bar{\psi} \rightarrow \bar{\psi} e^{-i\alpha}$$

LSZ:

$$\langle f | T(\exp[-i \int d^4x \mathcal{H}_{int}^I(x)]) | i \rangle$$

LSZ:

$$\langle f | T(\exp[-i \int d^4x \mathcal{H}_{int}^I(x)]) | i \rangle \quad \square$$

$$T[\psi_a(x) \bar{\psi}_b(y)] = : \psi_a(x) \bar{\psi}_b(y) : + \psi_a(x) \bar{\psi}_b(y)$$

$$\overline{\psi_a(x) \psi_b(y)} = \begin{cases} \{ \psi_a^{(+)}(x), \bar{\psi}_b^{(-)}(y) \} & \text{if } x^0 > y^0 \\ - \{ \bar{\psi}_a^{(+)}(y), \psi_b^{(-)}(x) \} & \text{if } x^0 < y^0 \end{cases}$$

$\bar{\psi} \psi$

$$\overline{\psi}_a(x) \psi_b(y)$$

$$x^0 > y^0$$

$$x^0 < y^0$$

$$\overline{\psi} \psi = \overline{\psi} \psi = 0$$

$$T[\psi_1 \overline{\psi}_2 \overline{\psi}_3 \dots] =: \psi_1 \overline{\psi}_2 \overline{\psi}_3 \dots : + \text{all possible contractions.}$$

$$\begin{aligned} & : \overline{\psi}_1 \psi_2 \overline{\psi}_3 \dots : \\ &= - : \overline{\psi}_1 \overline{\psi}_2 \psi_3 \dots : \\ &= - \overline{\psi}_1 \overline{\psi}_2 : \psi_3 \dots : \end{aligned}$$

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$$\underline{\psi\psi \rightarrow \psi\psi}$$

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x)$$

$\nearrow b$ $\nearrow et$

$$\overline{\psi}(x) = \overline{\psi}^{(-)}(x) + \overline{\psi}^{(+)}(x)$$

$\nwarrow b^+$ $\nwarrow c$

$$\underline{\psi\psi \rightarrow \psi\psi}$$

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x)$$

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$$\overline{\psi}(x) = \overline{\psi}^{(-)}(x) + \overline{\psi}^{(+)}(x)$$

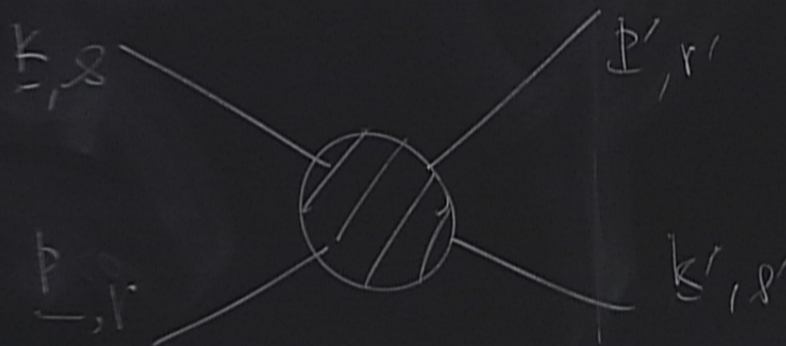
$$\varphi(x) = \underset{a}{\varphi^{(+)}(x)} + \underset{at}{\varphi^{(-)}(x)}$$

$\nwarrow b^+$ $\nwarrow$

$$|ia\rangle = b_s^\dagger(k) b_r^\dagger(p) |0\rangle$$

$$|final\rangle = b_{s'}^\dagger(k') b_{r'}^\dagger(p') |0\rangle$$

$$\langle final| = \langle 0| b_{r'}(p') b_{s'}(k')$$



$$H_{int} = \lambda \int d^3x \psi(x) \bar{\psi}(x) \psi(x)$$

$$|in\rangle = b_s^\dagger(k) b_r^\dagger(p) |0\rangle$$

$$|final\rangle = b_{s'}^\dagger(k') b_{r'}^\dagger(p') |0\rangle$$

$$\langle final| = \langle 0| b_{r'}(p') b_{s'}(k')$$



$$H_{int} = \lambda \int d^3x \phi(x) \psi(x) \bar{\psi}(x)$$

$$\frac{O(X)}{O(X)}$$

$$\phi^{(+)} |in\rangle = 0$$

$$\langle final| \phi^{(-)} = 0$$

$\mathcal{O}(\lambda^2)$:

$$i\mathcal{T} = \frac{(-i\lambda)^2}{2!} \langle f | T \int d^4x_1 d^4x_2 (\bar{\psi}\psi)_1 (\bar{\psi}\psi)_2 | i \rangle$$

$$= \frac{(-i\lambda)^2}{2!} \langle f | \int d^4x_1 d^4x_2 : (\bar{\psi}\psi)_1 (\bar{\psi}\psi)_2 : | i \rangle \bar{\psi}_1 \psi_2$$

$$= -\frac{(i\lambda)^2}{2!} \int d^4x_1 d^4x_2 \int \frac{d^3k_1 d^3k_2}{(2\pi)^6 2E_{k_1} 2E_{k_2}} \langle f | \sum_{t,t'} \bar{\psi}_1^{(-)} \cdot u_t(k_1) \bar{\psi}_2^{(-)} \cdot u_{t'}(k_2) e^{-ik_1 x_1 - ik_2 x_2} | i \rangle$$

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$$\times b_L(k_1) b_{L'}(k_2) \left[b_s^\dagger(k) b_r^\dagger(p) |0\rangle \right] \varphi_1 \varphi_2$$

k_1, k_2

$$\times b_t(\underline{k}_1) b_{t'}(\underline{k}_2) \left[b_s^\dagger(\underline{k}) b_r^\dagger(\underline{p}) |0\rangle \right] \varphi_1 \varphi_2$$

||

$$(-2\pi)^6 4E_{\underline{k}} E_{\underline{p}} \left[\delta_{t,s} \delta_{t',r} \delta^3(\underline{k}_1 - \underline{k}) \delta^3(\underline{k}_2 - \underline{p}) |0\rangle \right]$$

$$- \delta_{t',s} \delta_{tr} \delta^3(\underline{k}_1 - \underline{p}) \delta^3(\underline{k}_2 - \underline{k}) |0\rangle$$

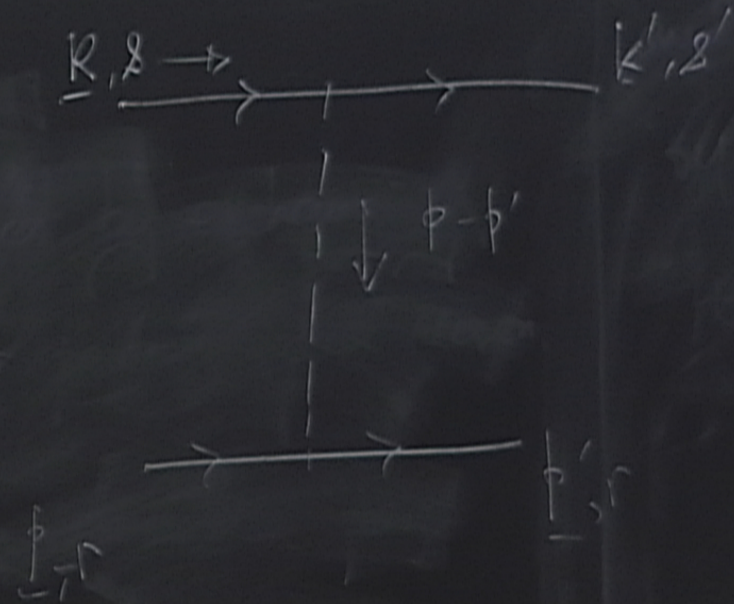
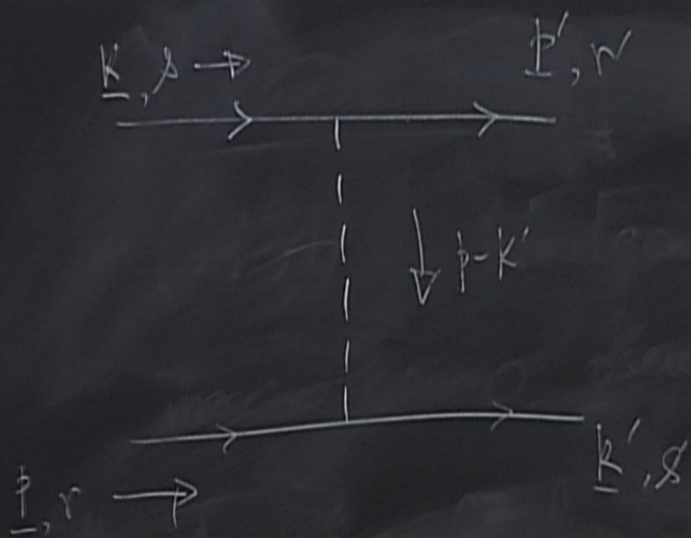
$$iT = -\frac{(-i\lambda)^2}{2!} \int d^4x_1 d^4x_2 \langle 0 | b_{r'}(p') b_{r''}(k') \left\{ \bar{\psi}_1^{(+)} u_2(k) \bar{\psi}_2 u_{r'}(p) \right. \\ \left. e^{-i k \cdot x_1 - i p \cdot x_2} + \bar{\psi}_1^{(-)} u_{r'}(p) \bar{\psi}_2^{(-)} u_2(k) e^{-i p \cdot x_1 - i k \cdot x_2} \right\} | 0 \rangle$$

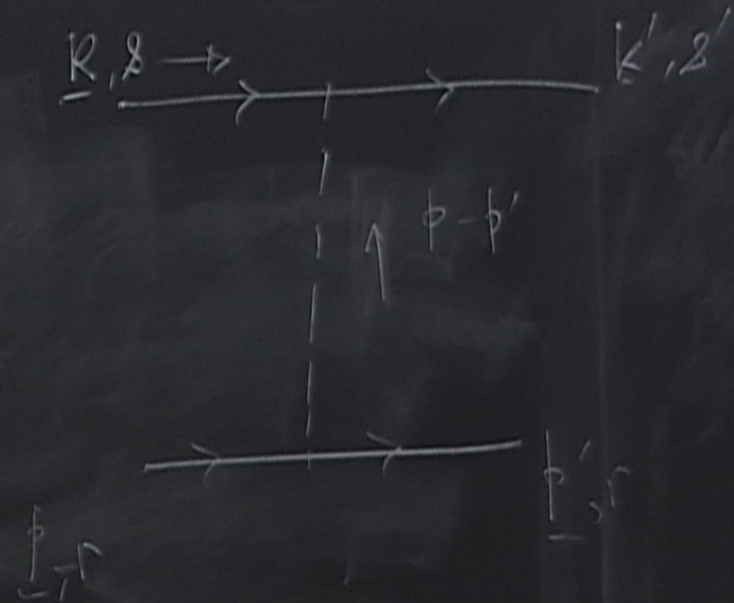
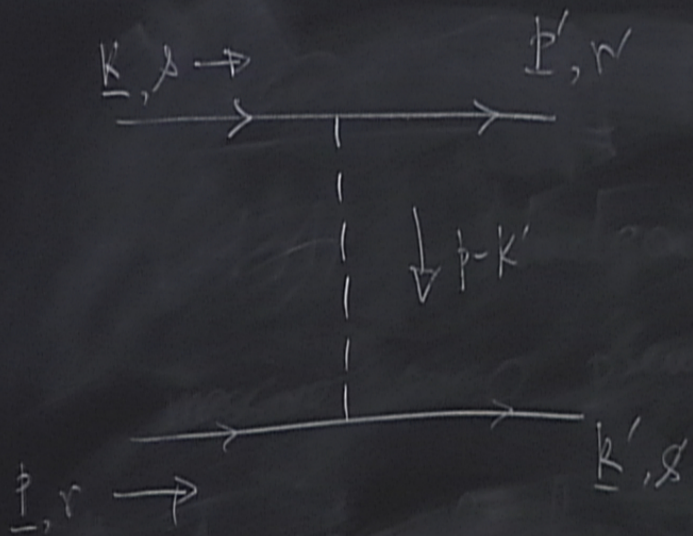
$$|0\rangle \prod \varphi_1 \varphi_2$$

$$\prod \varphi_1 \varphi_2 = i \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq \cdot (x-y)}}{q^2 - M^2 + i\epsilon}$$

$$\begin{aligned}
 i\mathcal{T} &= (-i\lambda)^2 \int \frac{d^4 q}{q^2 - M^2 + i\epsilon} \frac{(2\pi)^4}{(2\pi)^4} \left[\underbrace{(\bar{u}_{r'}(p') u_s(k)) (\bar{u}_{s'}(k') u_r(p))}_{\text{Term 1}} \delta^4(p' - k + q) \delta^4(k' - p - q) \right. \\
 &\quad \left. - \underbrace{(\bar{u}_{s'}(k') u_s(k)) (\bar{u}_r(p') u_r(p))}_{\text{Term 2}} \delta^4(k' - k + q) \delta^4(p' - p + q) \right] \\
 &= (2\pi)^4 \delta^4(p + k - p' + k') \left\{ (-i\lambda)^2 \left[\frac{(\bar{u}_{r'}(p') u_s(k)) (\bar{u}_{s'}(k') u_r(p))}{(k' - p)^2 - M^2 + i\epsilon} \right. \right. \\
 &\quad \left. \left. - \frac{(\bar{u}_{s'}(k') u_s(k)) (\bar{u}_r(p') u_r(p))}{(p' - p)^2 - M^2 + i\epsilon} \right] \right\} \rightarrow i\mathcal{M}
 \end{aligned}$$

$$\begin{aligned}
 &= (-i\lambda)^2 \int \frac{d^4 q}{q^2 - M^2 + i\varepsilon} \left[\frac{(\bar{u}_{r'}(p') u_s(k)) (\bar{u}_{s'}(k') u_r(p))}{(k' - p')^2 - M^2 + i\varepsilon} \delta^4(p' - k + q) \delta^4(k' - p - q) \right. \\
 &\quad \left. - \frac{(\bar{u}_{s'}(k') u_s(k)) (\bar{u}_{r'}(p') u_r(p))}{(p' - p)^2 - M^2 + i\varepsilon} \delta^4(k' - k + q) \delta^4(p' - p - q) \right] \\
 &= (2\pi)^4 \delta^4(p + k - p' - k') \left\{ (-i\lambda)^2 \left[\frac{(\bar{u}_{r'}(p') u_s(k)) (\bar{u}_{s'}(k') u_r(p))}{(k' - p')^2 - M^2 + i\varepsilon} \right. \right. \\
 &\quad \left. \left. - \frac{(\bar{u}_{s'}(k') u_s(k)) (\bar{u}_{r'}(p') u_r(p))}{(p' - p)^2 - M^2 + i\varepsilon} \right] \right\} \rightarrow i\mathcal{M}
 \end{aligned}$$

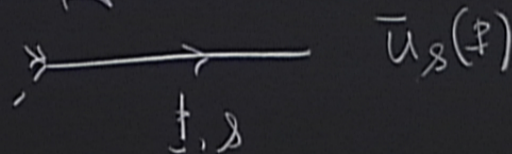




1. For each incoming fermion $(\not{p}, s)$ we write $u_s(\not{p})$



2. Outgoing



3. We label fermion lines by an arrow. Arrow follows direction of time. We write down the above factors from left to right

$\underline{k}, s \rightarrow$ $\underline{p}', r$

$\underline{k}, s \rightarrow$ $\underline{k}', s'$

$\downarrow p-k'$

$\uparrow p-p'$

$\underline{p}, r \rightarrow$ $\underline{k}', s$

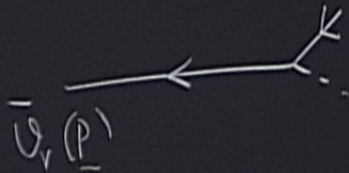
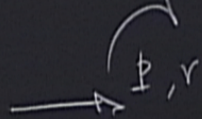
$\underline{p}, r$

$\underline{p}', r$

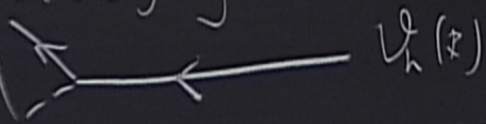
$\rightarrow$ time

while following the Fermion line
opposite to the direction of the arrow.

4. Incoming antifermion $\bar{\psi}_r(p)$



5. Outgoing anti-fermion $\bar{\psi}(p)$

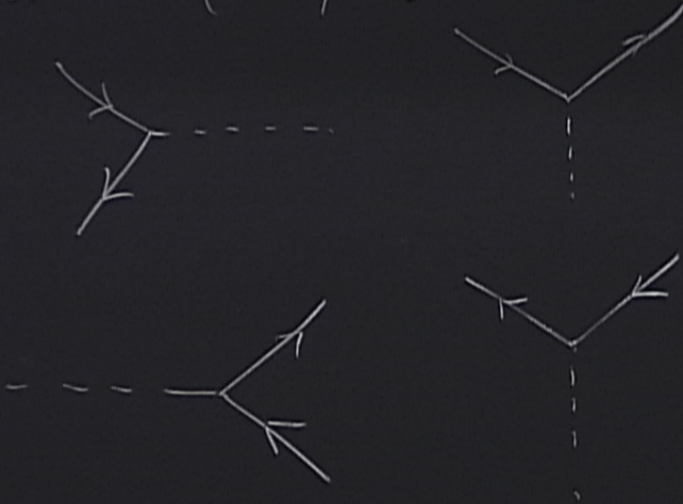


as direction
to right

6. Associate $(-ix)$ with

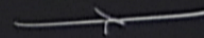
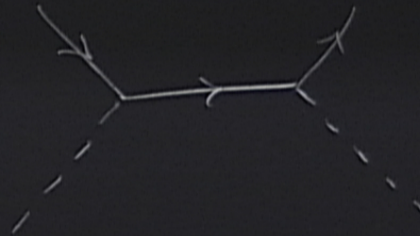


6. Associate $(-ix)$ with



7.

8.



$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

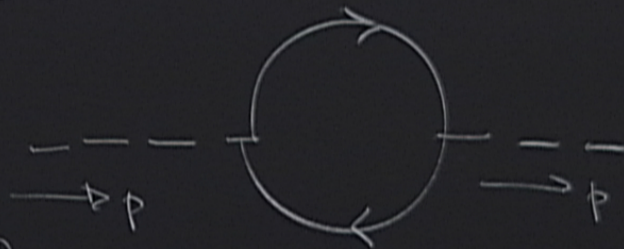
$$p^2 - m^2 + i\epsilon$$

9. Impose

9. Impose momentum conservation at each vertex.

10. minus sign due to Fermi-statistics.

11. Integrate over undetermined loop momentum



12. (-1) with fermion loops.