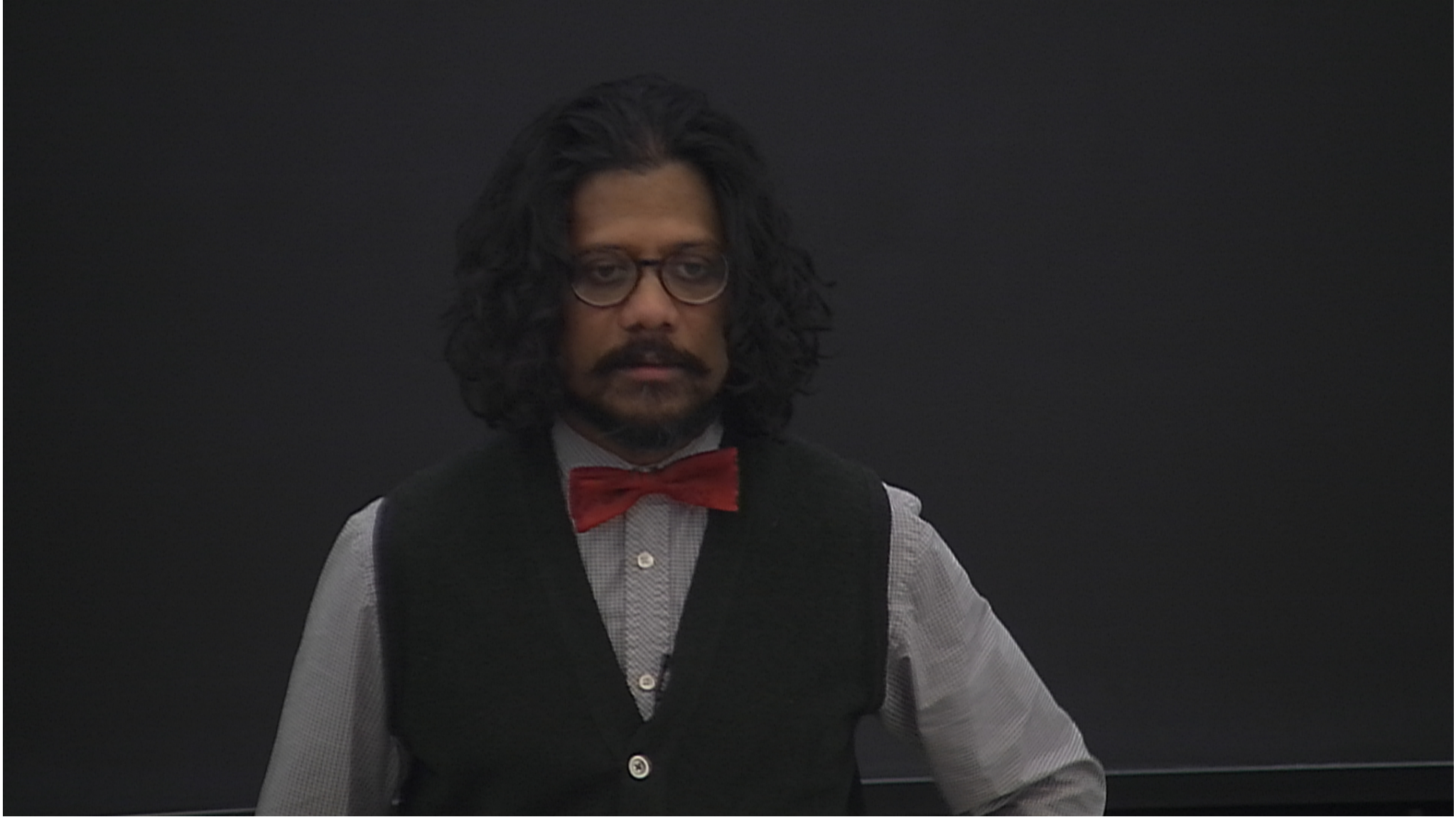


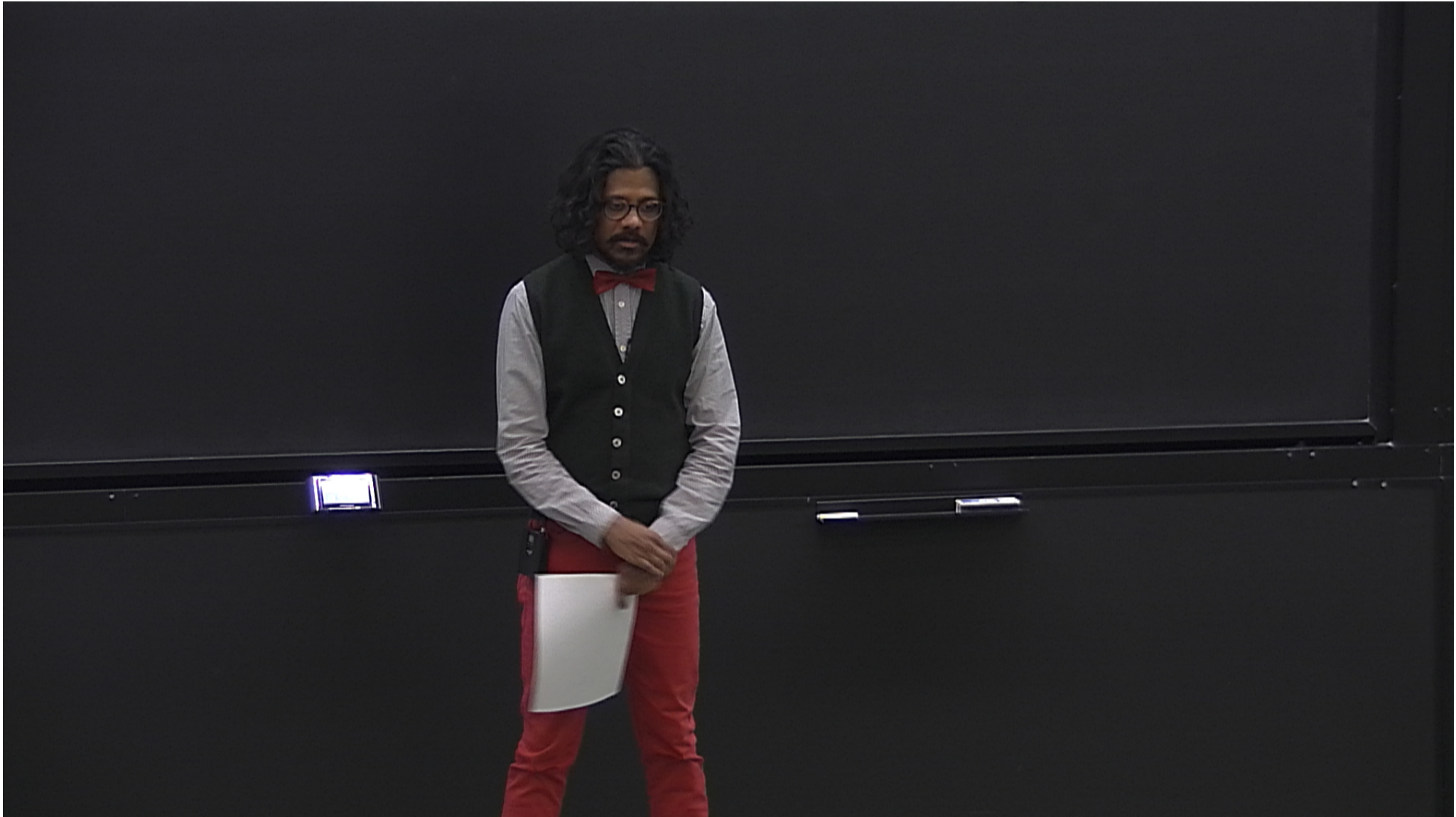
Title: PSI 2015/2016 Quantum Field Theory I - Daniel Wohns and Tibra Ali - Lecture 10

Date: Oct 26, 2015 09:00 AM

URL: <http://pirsa.org/15100045>

Abstract:





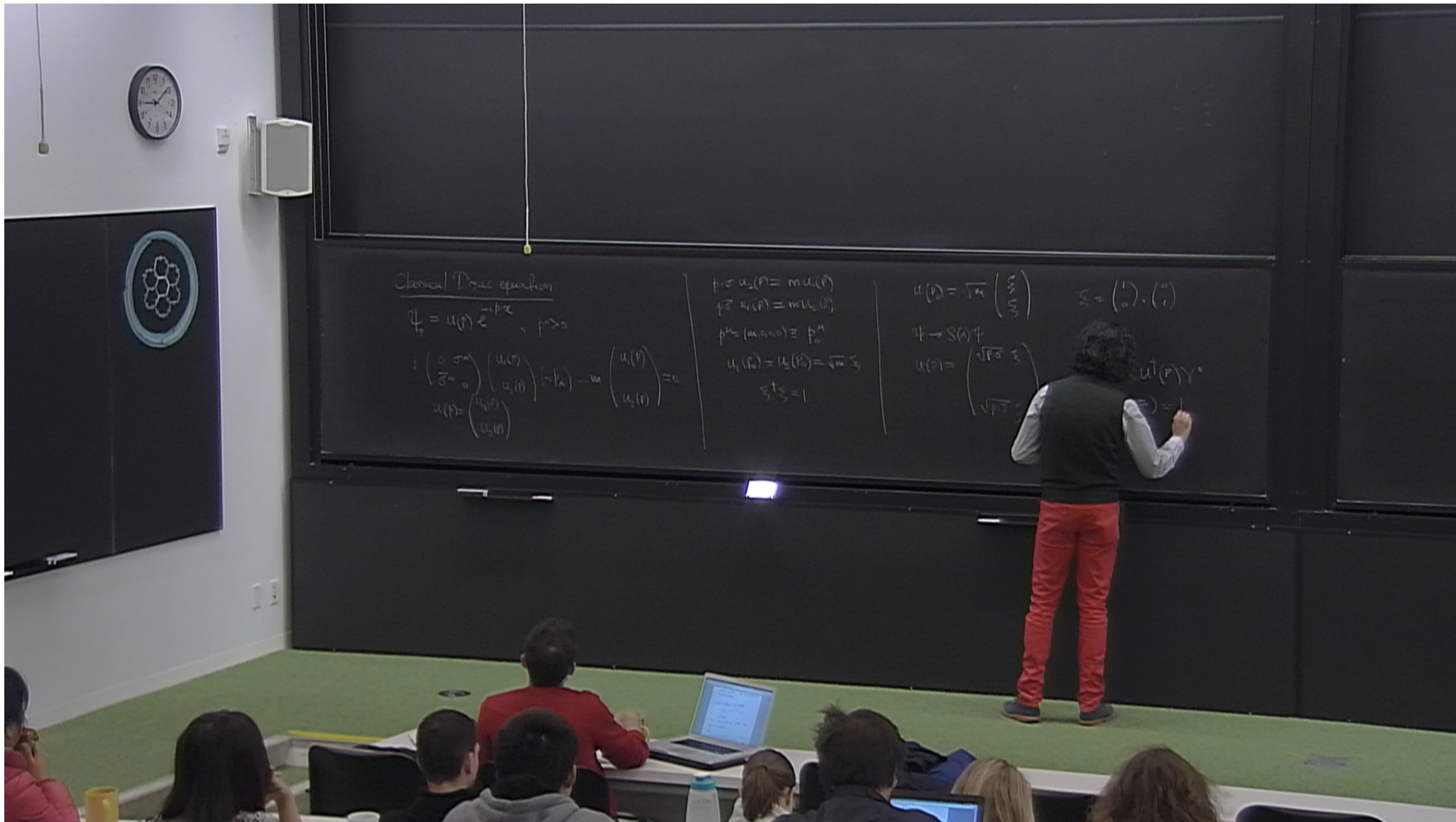
Classical Dirac equation:

$$\psi_p = u(p) e^{-ip \cdot x}, \quad p^0 > 0$$

$$i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix} (-ip_\mu) - m \begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix} = 0$$

$$u(p) = \begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix}$$

$$p \cdot \sigma u_2(p)$$



$$p \cdot \sigma u_2(p) = m u_1(p)$$

$$p \cdot \bar{\sigma} u_1(p) = m u_2(p)$$

$$p^\mu = (m, 0, 0, 0) \equiv p_0^\mu$$

$$u_1(p_0) = u_2(p_0) = \sqrt{m} \xi$$

$$\xi^\dagger \xi = 1$$

$$u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$\psi \rightarrow S(\lambda) \psi$$

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix}$$

$$u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}$$

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\psi \rightarrow S(\lambda) \psi$$

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix}$$

$$\bar{u}(p) = u^\dagger(p) \gamma^0$$

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 = m^2$$

$$\bar{u}(p) u(p) = 2m \xi^\dagger \xi = 2m$$

$$\xi^{(1)}, \xi^{(2)}, \quad \xi^{(s)\dagger} \xi^{(r)} = \delta^{sr}$$

$$\bar{u}^{(s)}(p) u^{(r)}(p) = 2m \delta^{sr}$$

$$u^{(s)\dagger}(p) u^{(r)}(p) = 2E_p \delta^{sr}$$

Negative frequency solutions

$$\psi_p(x) = v(p) e^{+ip \cdot x}, \quad p^0 > 0$$

$$v^{(s)}(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \sigma} \eta^s \end{pmatrix}$$

$$\text{where } \eta^{s\dagger} \eta^r = \delta^{sr}$$

$$\bar{v}^s(p) v^r(p) = -2m \delta^{sr}$$

$$v^{s\dagger}(p) v^r(p) = +2E_p \delta^{sr}$$

$$v^{(s)}(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \sigma} \eta^s \end{pmatrix}$$

where $\eta^s \eta^r = \delta^{sr}$

$$v^r(p) = -2m \delta^{sr}$$

$$v^s(p) = +2E_p \delta^{sr}$$

$$\bar{u}^s(p) v^r(p) = \bar{v}^s(p) u^r(p) = 0$$

$$u^{+s}(p) v^r(p) \neq 0, \quad v^{+s}(p) u^r(p) \neq 0$$

$$\left. \begin{aligned} u^{+s}(-p) v^r(p) &= 0 \\ v^{+s}(-p) u^r(p) &= 0 \end{aligned} \right\}$$

Spin Sums

$$\sum_{s=1}^2 u^s(p) \bar{u}^s(p) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = (\not{p} + m \mathbb{1}), \quad \not{p} = p_\mu \gamma^\mu$$

$$\sum_{s=1}^2 v^s(p) \bar{v}^s(p) = (\not{p} - m \mathbb{1})$$

$$|p(x) - v(p)| \leq \dots, p > 0$$

Spin Sums

$$\sum_{s=1}^2 u^s(p) \bar{u}^s(p) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = (\not{p} + m \mathbb{1}), \quad \not{p} = p_\mu \gamma^\mu$$

$$\sum_{s=1}^2 v^s(p) \bar{v}^s(p) = (\not{p} - m \mathbb{1})$$

$$p = p_\mu \gamma^\mu$$

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{s=1}^2 \left\{ u^s(\underline{k}) b^s(\underline{k}) e^{-ik \cdot x} + v^s(\underline{k}) c^{s*}(\underline{k}) e^{+ik \cdot x} \right\}$$

$$E_k^2 = m^2 + |\underline{k}|^2$$

$$\not{p} = p_\mu \gamma^\mu$$

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{s=1}^2 \left\{ u^s(k) b^s(k) e^{-ik \cdot x} + v^s(k) c^{s*}(k) e^{+ik \cdot x} \right\}$$

$$E_k^2 = m^2 + |\mathbf{k}|^2$$

$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{s=1}^2 \left\{ \bar{u}^s(k) b^s(k) e^{+ik \cdot x} + \bar{v}^s(k) c^s(k) e^{-ik \cdot x} \right\}$$

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{s=1}^2 \left\{ u^s(k) b^s(k) e^{-ik \cdot x} + v^s(k) c^{s*}(k) e^{+ik \cdot x} \right\}$$

$$E_k^2 = m^2 + |\mathbf{k}|^2$$

$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{s=1}^2 \left\{ \bar{u}^s(k) b^s(k) e^{+ik \cdot x} + \bar{v}^s(k) c^s(k) e^{-ik \cdot x} \right\}$$

$$U^S = \begin{pmatrix} p \cdot \sigma & \xi^S \\ p \cdot \bar{\sigma} & \xi^S \end{pmatrix}$$

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Symmetries of Dirac theory (Weyl Theory)

$$\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$$

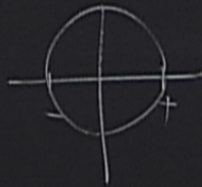
1. Global $U(1)$ (or phase rotation) symmetry.

Internal symmetry.

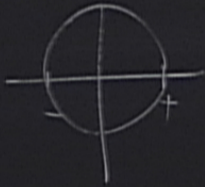
$$\left. \begin{aligned} \psi &\rightarrow e^{i\alpha} \psi \\ \bar{\psi} &\rightarrow e^{-i\alpha} \bar{\psi} \end{aligned} \right\} \alpha \rightarrow \text{independent of } x^\mu$$

$$\begin{aligned}
 \partial_\mu \mathcal{T}^\mu &= (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \bar{\psi} \gamma^\mu \partial_\mu \psi \\
 &= \frac{1}{i} [-m \bar{\psi} \gamma^\mu \psi + \bar{\psi} \gamma^\mu m \psi] \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \partial_\mu \mathcal{T}^\mu &= (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \bar{\psi} \gamma^\mu \partial_\mu \psi \\
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 &= (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \bar{\psi} \gamma^\mu \partial_\mu \psi \\
 &= \frac{1}{i} [-m \bar{\psi} \gamma^\mu \psi + \bar{\psi} \gamma^\mu m \psi] \\
 &= 0
 \end{aligned}$$

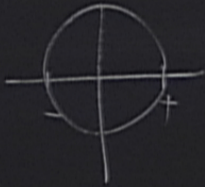


2. If $m=0$.

$$\psi(x) \rightarrow e^{i\alpha \gamma_5} \psi(x)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi} e^{i\alpha \gamma_5}$$

$$\begin{aligned}
 &= (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \bar{\psi} \gamma^\mu \partial_\mu \psi \\
 &= \frac{1}{i} [-m \bar{\psi} \gamma^\mu \psi + \bar{\psi} \gamma^\mu m \psi] \\
 &= 0
 \end{aligned}$$



2. If $m=0$.

$$\psi(x) \rightarrow e^{i\alpha \gamma_5} \psi(x)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi} e^{i\alpha \gamma_5}$$

$$\gamma^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi$$

2 Translational Symmetry:

$$x^\mu \rightarrow x^\mu - \varepsilon^\mu$$

$$\psi(x) \rightarrow \psi'(x) = \psi(x + \varepsilon) \\ \simeq \psi(x) + \varepsilon^\mu \partial_\mu \psi$$

$$\delta\psi = \varepsilon^\mu \partial_\mu \psi$$

$$\delta\mathcal{L} = \varepsilon^\mu \partial_\mu \mathcal{L}$$

$$\begin{aligned} J^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} \varepsilon^\nu \partial_\nu \psi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_a)} \varepsilon^\nu \partial_\nu \bar{\psi}_a - \varepsilon^\mu \mathcal{L} \\ &= \varepsilon^\nu i \bar{\psi} \gamma^\mu \partial_\nu \psi \end{aligned}$$

$$\delta \mathcal{L} = \epsilon^\mu \partial_\mu \mathcal{L}$$

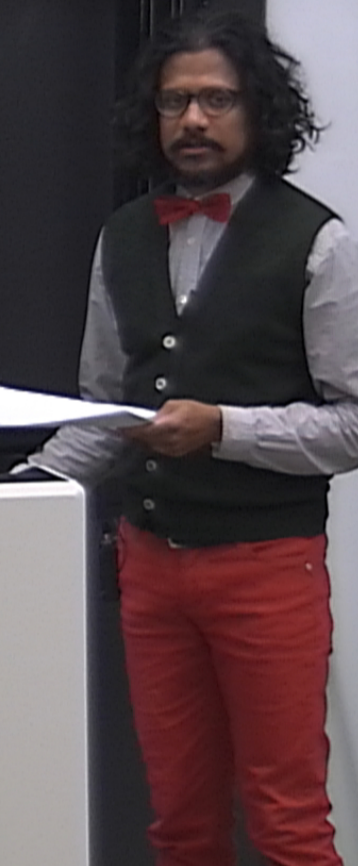
$$J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \epsilon^\nu \partial_\nu \psi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi}_a)} \epsilon^\nu \partial_\nu \bar{\psi}_a - \epsilon^\mu \mathcal{L}$$

$$= \epsilon^\nu i \bar{\psi} \gamma^\mu \partial_\nu \psi$$

$$\boxed{T^\mu_\nu = i \bar{\psi} \gamma^\mu \partial_\nu \psi}$$

Energy-momentum tensor.

$$T_{\mu\nu} \propto \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$$



$$\psi_p(x) = \psi(p) e^{+ip \cdot x}, \quad p^0 > 0$$

$$\psi^*(p) \psi(p) = \delta^3(p) \delta(p^0)$$

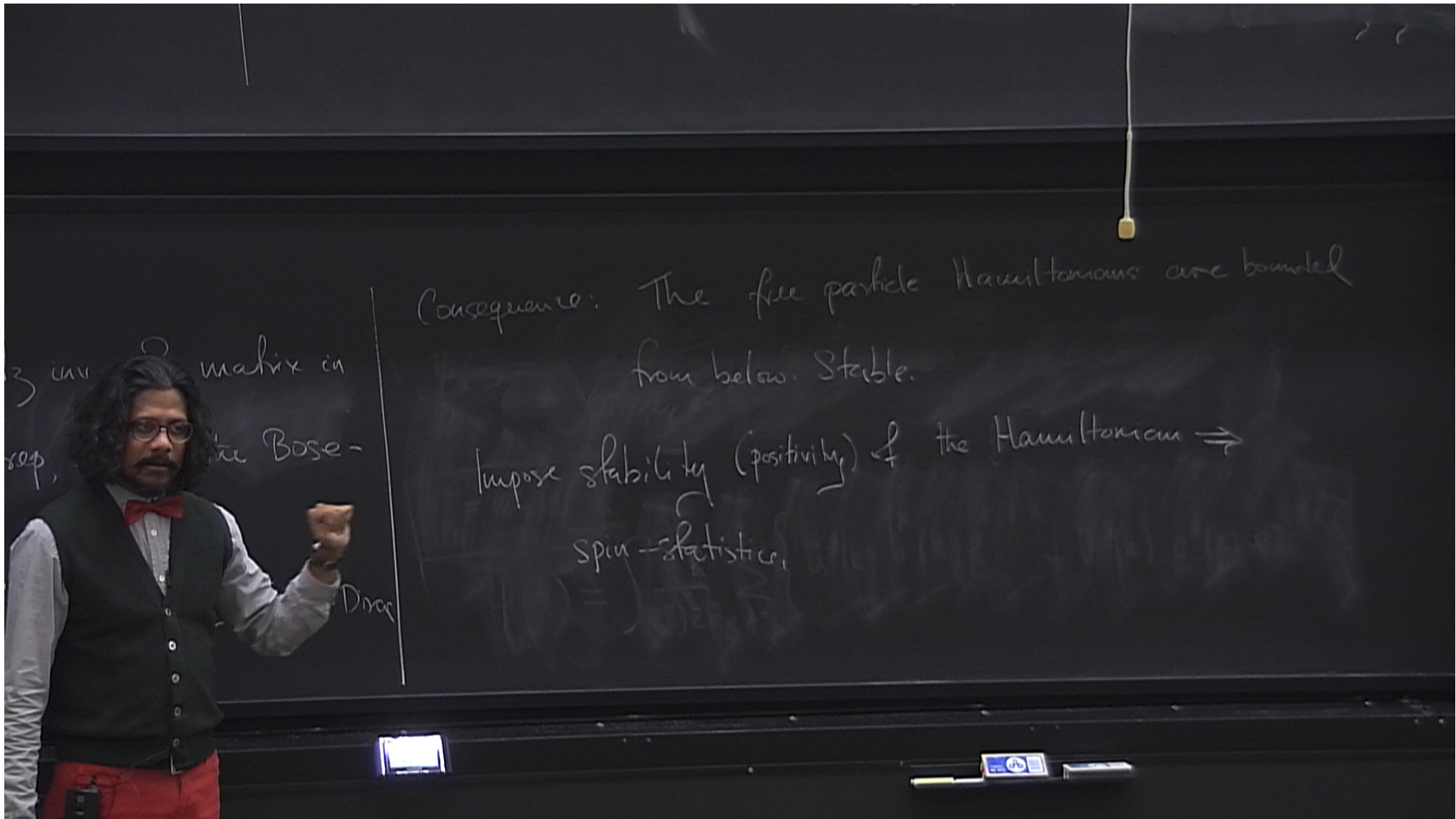
$$P_\nu = \int d^3x T_{\nu}^{\mu} \sim \text{Energy-Momentum 4-vector}$$

$$P_0 = H$$

Quantization of Spinors

Spin-statistics theorem: Lorentz inv. S matrix in $(3+1)$ dim $\Rightarrow$ integer spin rep. satisfy the Bose-Einstein statistics, half-integer spin rep. satisfy Fermi-Dirac statistics.

Conseq



$\psi(x)$, $\phi(x)$
Spinor Scalar

$\hat{\psi}(x)$, $\hat{\phi}(x)$

$$\hat{\phi}(x) \xrightarrow{\Lambda} \hat{U}_{\Lambda} \hat{\phi}(x) \hat{U}_{\Lambda}^{-1} = \hat{\phi}(\Lambda^{-1}x)$$

$$\hat{\psi}(x) \xrightarrow{\Lambda} \hat{U}_{\Lambda} \hat{\psi}_a(x) \hat{U}_{\Lambda}^{-1} = S_{ab}(\Lambda) \hat{\psi}_b(\Lambda^{-1}x)$$

$$\{b^s(p), b^{rt}(k)\} = (2\pi)^3 2E_p \delta^{sr} \delta^3(p-k)$$

$$\{c^s(p), c^{rt}(k)\} = (2\pi)^3 2E_p \delta^{sr} \delta^3(p-k)$$

$$\begin{aligned} \{b, b\} &= \{c, c\} = \{b^\dagger, b^\dagger\} = \{c^\dagger, c^\dagger\} \\ &= \{b, c^\dagger\} = \{b^\dagger, c\} = 0 \end{aligned}$$

$$b^\dagger(p_1, s_1) b^\dagger(p_2, s_2) |0\rangle = |(p_2, s_2), (p_1, s_1)\rangle$$

$$= - |(p_1, s_1), (p_2, s_2)\rangle$$

$$H = \int d^3x \{ \pi_a \dot{\varphi}_a - \mathcal{L} \}$$

$$= \int d^3x \{ -i \bar{\psi} \gamma^i \partial_i \psi + m \bar{\psi} \psi \}$$

$$= \{b, c^\dagger\} = \{b^\dagger, c\} = 0$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_r \left\{ \begin{array}{l} + \\ - \end{array} \right\}$$

$$(i\not{\partial} - m)u_s e^{-ipx} = 0$$

$$(i\not{\partial} - m)v_s e^{+ipx} = 0$$

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r=1}^2 \left\{ b_r^\dagger(p) b_r(p) - c_r(p) c_r^\dagger(p) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r=1}^2 \left\{ b_r^\dagger(p) b_r(p) + c_r^\dagger(p) c_r(p) \right\}$$

$$-(2\pi)^3 \delta_{rr} \delta(p-p')$$

$$2\pi \delta(p-p')$$

$$\{b, c^\dagger\} = \{b^\dagger, c\} = 0$$

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \left\{ \begin{array}{l} + \\ - \end{array} \right\}$$

$$(i\not{\partial} - m)u_s e^{-ipx} = 0$$

$$(i\not{\partial} - m)v_s e^{ipx} = 0$$

$$H = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left\{ b_r^\dagger(p) b_r(p) - c_r(p) c_r^\dagger(p) \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3 E_p} \frac{1}{2} \sum_{r=1}^2 \left\{ b_r^\dagger(p) b_r(p) + \underbrace{c_r^\dagger(p) c_r(p)}_{-(2\pi)^3 2E_p \delta_{rr} \delta(0)} \right\} E_p$$

Zero-point energy

Normal ordering:

$$:ab^{\dagger}: = - :b^{\dagger}a: = -b^{\dagger}a$$

important.

$$b_s^{\dagger}(\mathbf{k})|0\rangle = |\mathbf{k}, s\rangle$$

$$:H: |(\mathbf{k}, s)\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r E_p \left(b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) + c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}) \right) b_s^\dagger(\mathbf{k}) |0\rangle$$

$$= E_p |(\mathbf{k}, s)\rangle$$

$$:Q: = \int d^3x :J^0:$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r \left\{ b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) - c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r \left\{ b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) - c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}) \right\}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{r=1}^2 \left\{ b_r(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + c_r^\dagger(\mathbf{p}) v_r(\mathbf{p}) e^{+ipx} \right\}$$

$$= \psi^{(+)} + \psi^{(-)}$$

Causality $\leftrightarrow$ Feynman propagator:

$\mathcal{O}(x)$

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0 \quad \text{if } \underline{(x-y)^2} < 0$$

$$[\varphi(x), \varphi(y)] = 0 \quad \text{if } (x-y)^2 < 0$$

$$\frac{(x-y)^2 < 0}{(x-y)^2 < 0}$$

$$[\psi_a(x), \psi_b^\dagger(y)] \neq 0 \text{ for } (x-y)^2 < 0$$

$$\{\psi_a(x), \psi_b^\dagger(y)\} = 0 \quad " \quad " \quad "$$

$$0 = \bar{\psi}^\dagger \Gamma \psi$$

↪ some γ matrix stuff.