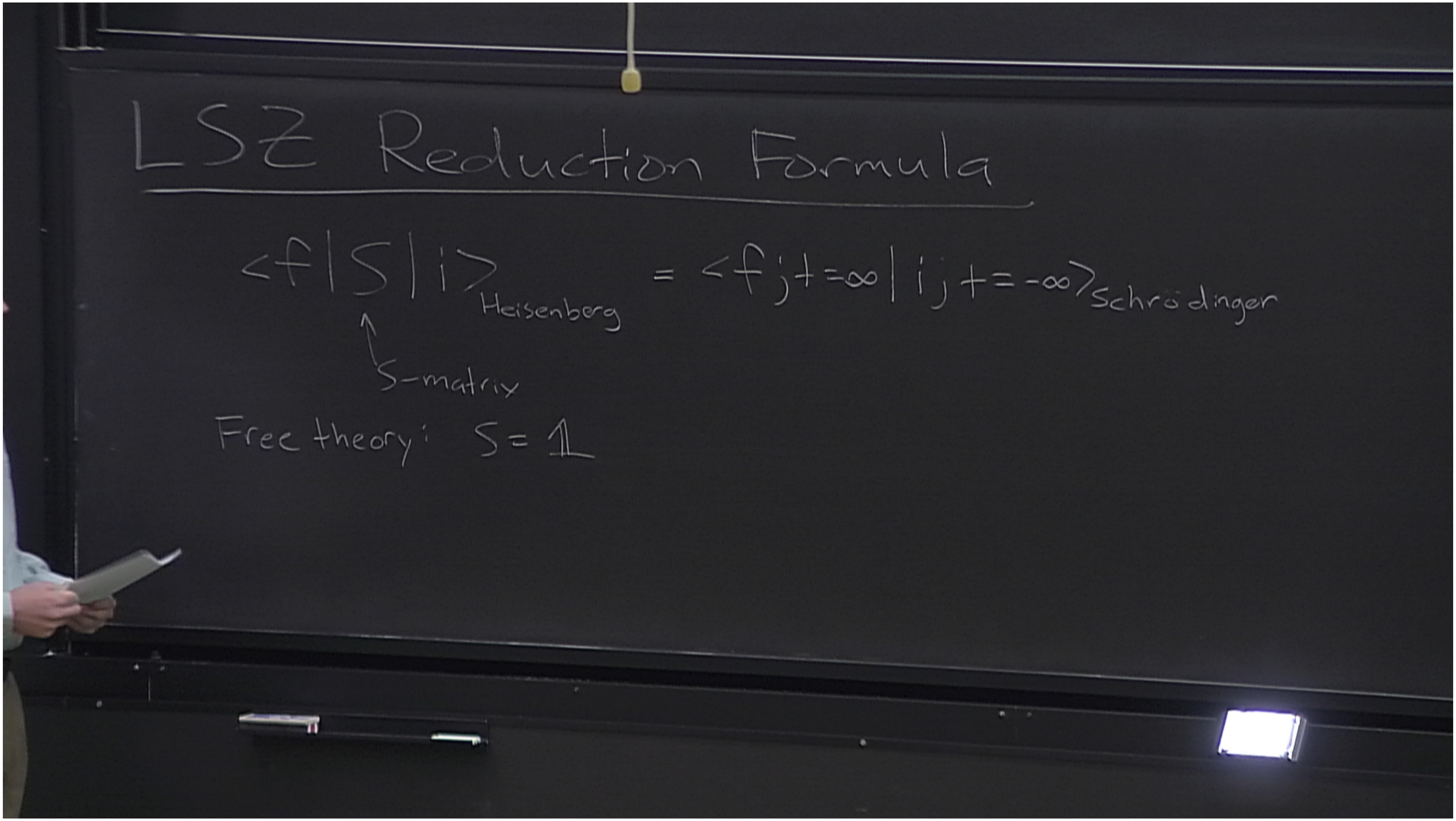
Title: PSI 2015/2016 Quantum Field Theory I - Daniel Wohns and Tibra Ali - Lecture 3

Date: Oct 15, 2015 09:00 AM

URL: http://pirsa.org/15100038

Abstract:


LSZ Reduction Formula
$\langle f | S | i \rangle_{\text{Heisenberg}} = \langle f; t = \infty | i; t = -\infty \rangle_{\text{Schrödinger}}$
S-matrix
Free theory: $S = \mathbb{1}$

LSZ Reduction Formula

$$\langle f|S|i\rangle_{\text{Heisenberg}} = \langle f; t=\infty \,|\, i; t=-\infty\rangle_{\text{Schrödinger}}$$

S-matrix

Free theory: $S = \mathbb{1}$

interacting theory $S = \mathbb{1} + iT$

transfer matrix

ödinger

$$T = (2\pi)^4 \delta^4(\Sigma p)\mathcal{M}$$

$$\langle f|S-\mathbb{1}|i\rangle = (2\pi)^4 \delta^4(\Sigma p)\,\underbrace{i\langle f|\mathcal{M}|i\rangle}_{=\,i\mathcal{M}(i\to f)}$$

matrix element

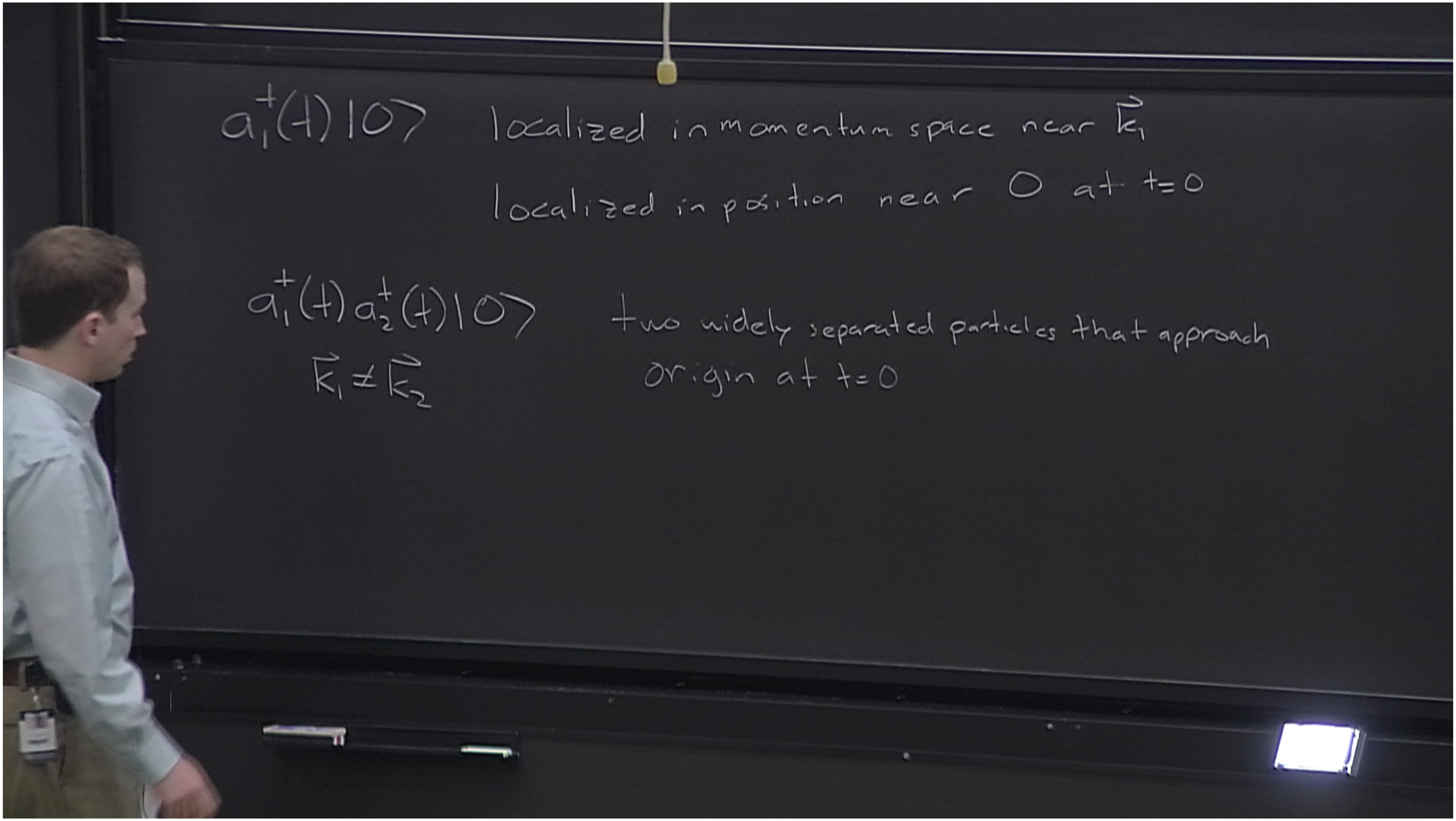
$$T = (2\pi)^4 \delta^4(\Sigma p)\, \mathcal{M}$$

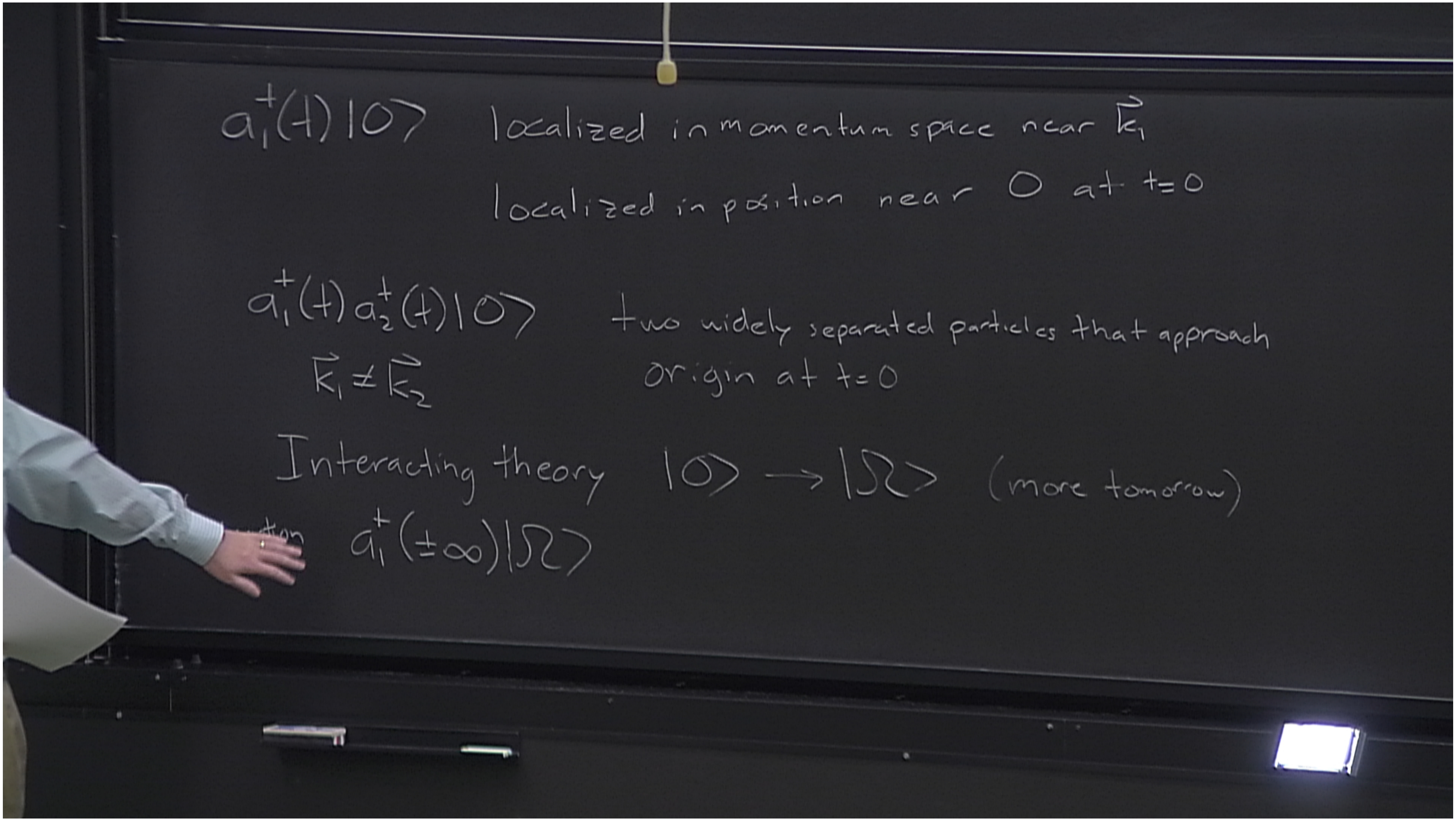
$$\langle f | S - \mathbb{1} | i \rangle = (2\pi)^4 \delta^4(\Sigma p)\, \underbrace{i \langle f | \mathcal{M} | i \rangle}_{= i\mathcal{M}(i \to f)}$$

matrix element

Free theory: $a_1^\dagger(t) = \int d^3k\, f_1(\vec{k})\, a_k^\dagger(t)$

$$f_1(\vec{k}) \propto \exp\left[-(\vec{k} - \vec{k}_1)^2 / 4\sigma^2\right]$$


$a_1^\dagger(t)|0\rangle$ localized in momentum space near $\vec{k}_1$
localized in position near 0 at $t=0$
$a_1^\dagger(t)\,a_2^\dagger(t)|0\rangle$ two widely separated particles that approach origin at $t=0$
$\vec{k}_1 \neq \vec{k}_2$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$\vec{k}_1 \neq \vec{k}_2$
Interacting theory $|0\rangle \longrightarrow |\Omega\rangle$ (more tomorrow)
$a_1^\dagger(\pm\infty)|\Omega\rangle$

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Interacting theory $|0\rangle \rightarrow |\Omega\rangle$ (more tomorrow)

Assumption $a_1^\dagger(\pm\infty)|\Omega\rangle$ are one particle states

$2 \to 2$ scattering

$$|i\rangle \propto a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)|\Omega\rangle$$

$$|f\rangle \propto a_3^\dagger(+\infty)\, a_4^\dagger(+\infty)|\Omega\rangle$$

Want: compute $\langle f|S|i\rangle$

Useful identity:

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i\int d^3k\, f_1(\vec{k})\, d^4x\, e^{-ik\cdot x}\underbrace{(\partial^2+m^2)\varphi(x)}_{\text{non-zero in interacting theory}}$$

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Assumption: $a_k^{\dagger}(t) = -i\int d^3x\, e^{-ikx}\left[\partial_0\varphi(x) + iE_{\vec{k}}\varphi(x)\right]$

Interacting theory $|0\rangle \to |\Omega\rangle$ (more tomorrow)

Assumption $a_1^+(\pm\infty)|\Omega\rangle$ are one particle states

$$\boxed{a_1^+(+\infty) - a_1^+(-\infty) = -i\int d^3k\, f_1(k) \ldots}$$

Assumption: $a_k^+(t) = -i\int d^3x\, e^{-ikx}[\partial \ldots$

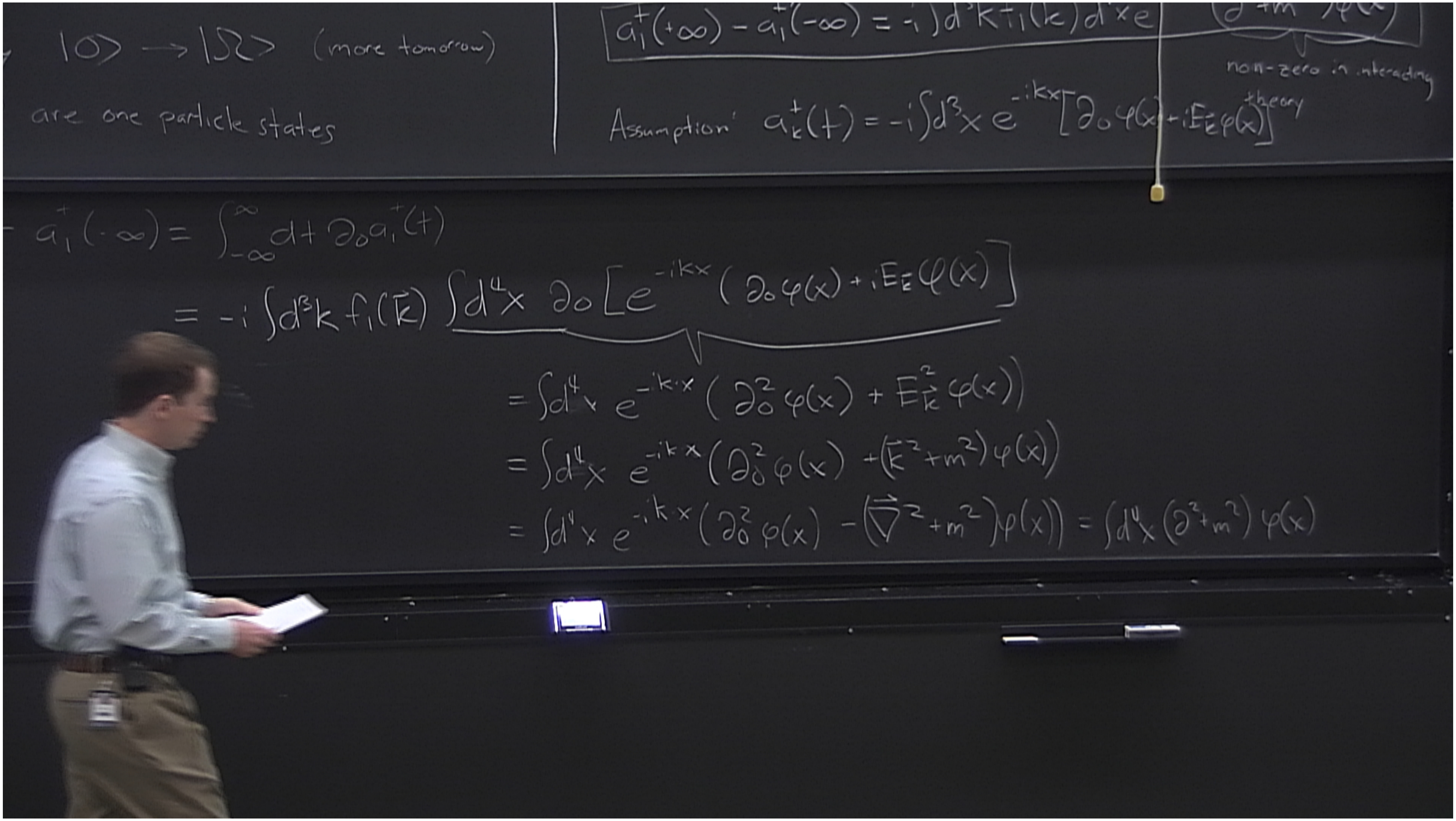
Proof: $a_1^+(+\infty) - a_1^+(-\infty) = \int_{-\infty}^{\infty} dt\, \partial_0 a_1^+(t)$

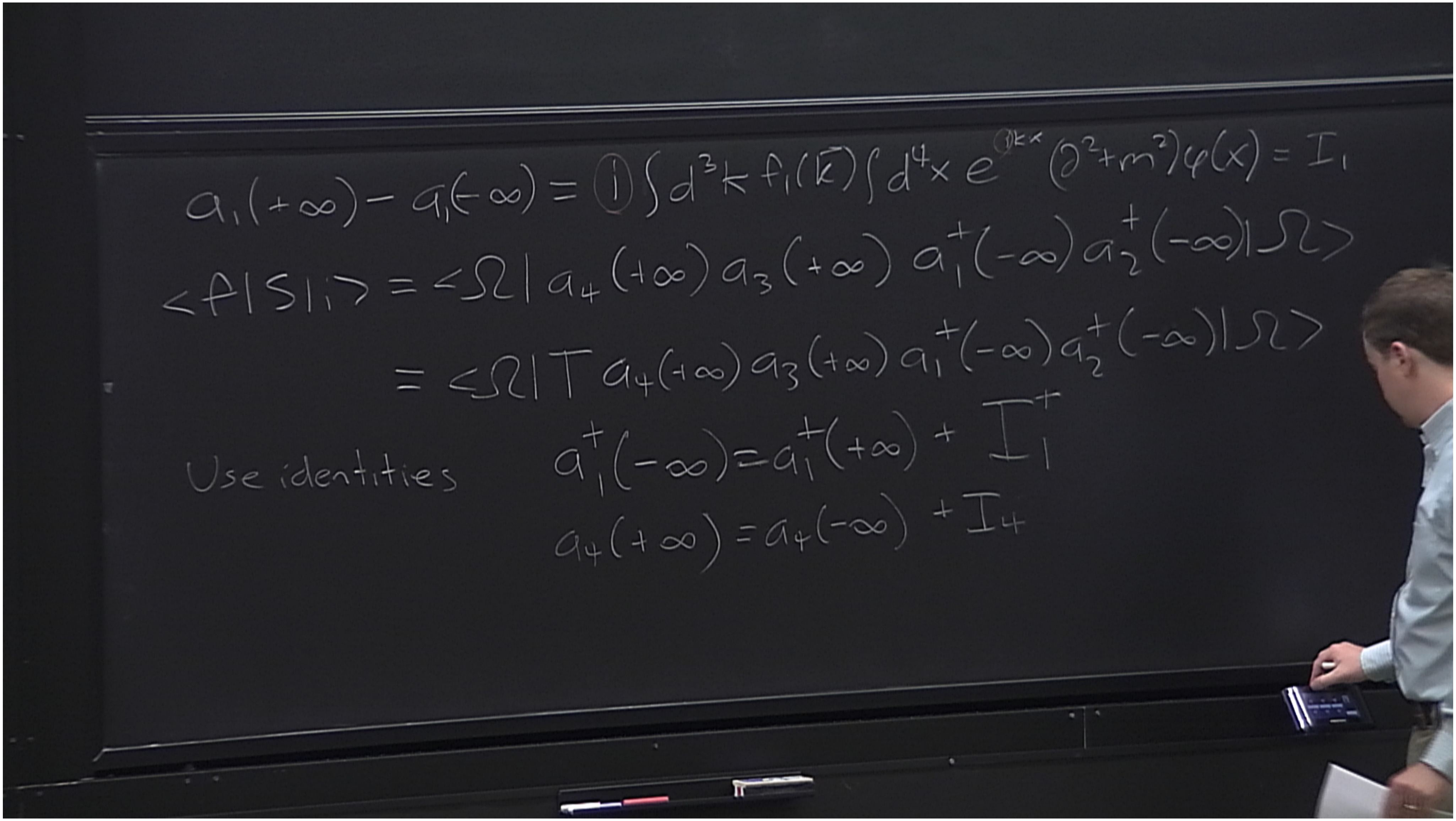
$$= -i\int d^3k\, f_1(\vec{k}) \underbrace{\int d^4x\, \partial_0\left[e^{-ikx}\left(\partial_0\varphi(x) + iE_{\vec{k}}\varphi(x)\right)\right]}$$

$$= \int d^4x\, e^{-ik\cdot x}\left(\partial_0^2\varphi(x) + E_{\vec{k}}^2\varphi(x)\right)$$

$$= \int d^4x\, e^{-ik\cdot x}\left(\partial_0^2\varphi(x) + (\vec{k}^2+m^2)\varphi(x)\right)$$

$$= \int d^4x\, e^{-ik\cdot x}\left(\partial_0^2\varphi(x) - (\vec{\nabla}^2+m^2)\varphi(x)\right) =$$


$|0\rangle \to |\Omega\rangle$ (more tomorrow)
are one particle states
$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i\int d^3k\, f_1(k)\, d^4x\, e \ldots (\partial^2+m^2)\varphi(x)$
non-zero in interacting theory
Assumption: $a_k^\dagger(t) = -i\int d^3x\, e^{-ikx}[\partial_0\varphi(x) + iE_k\varphi(x)]$
$a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt\, \partial_0 a_1^\dagger(t)$
$= -i\int d^3k\, f_1(\vec{k}) \int d^4x\, \partial_0[e^{-ikx}(\partial_0\varphi(x) + iE_k\varphi(x))]$
$= \int d^4x\, e^{-ik\cdot x}(\partial_0^2\varphi(x) + E_k^2\varphi(x))$
$= \int d^4x\, e^{-ik\cdot x}(\partial_0^2\varphi(x) + (\vec{k}^2+m^2)\varphi(x))$
$= \int d^4x\, e^{-ik\cdot x}(\partial_0^2\varphi(x) - (\vec{\nabla}^2+m^2)\varphi(x)) = \int d^4x\, (\partial^2+m^2)\varphi(x)$


$a_1(+\infty) - a_1(-\infty) = i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ikx}(\partial^2+m^2)\varphi(x) = I_1$
$\langle f|S|i\rangle = \langle\Omega| a_4(+\infty)\, a_3(+\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)|\Omega\rangle$
$= \langle\Omega|T\, a_4(+\infty)\, a_3(+\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)|\Omega\rangle$
Use identities
$a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + I_1^\dagger$
$a_4(+\infty) = a_4(-\infty) + I_4$

$$\langle f|S|i\rangle = \langle \Omega | T\, a_4(-\infty)\, a_3(-\infty)\, a_1^{+}(+\infty)\, a_2^{+}(+\infty) |\Omega\rangle$$

0 time ordering

$$\cancelto{0}{+\cdots} + \langle \Omega | T\, I_4 I_3 I_1^{+} I_2^{+} |\Omega\rangle$$

$$f_1(\vec{k}) \rightarrow \delta(\vec{k} - \vec{k}_1)$$

$$I_1^{+}$$

$$\langle f|S|i\rangle = \langle \Omega | T\, a_4(-\infty)\, a_3(-\infty)\, a_1(+\infty)\, a_2(+\infty) | \Omega \rangle$$

0 time ordering

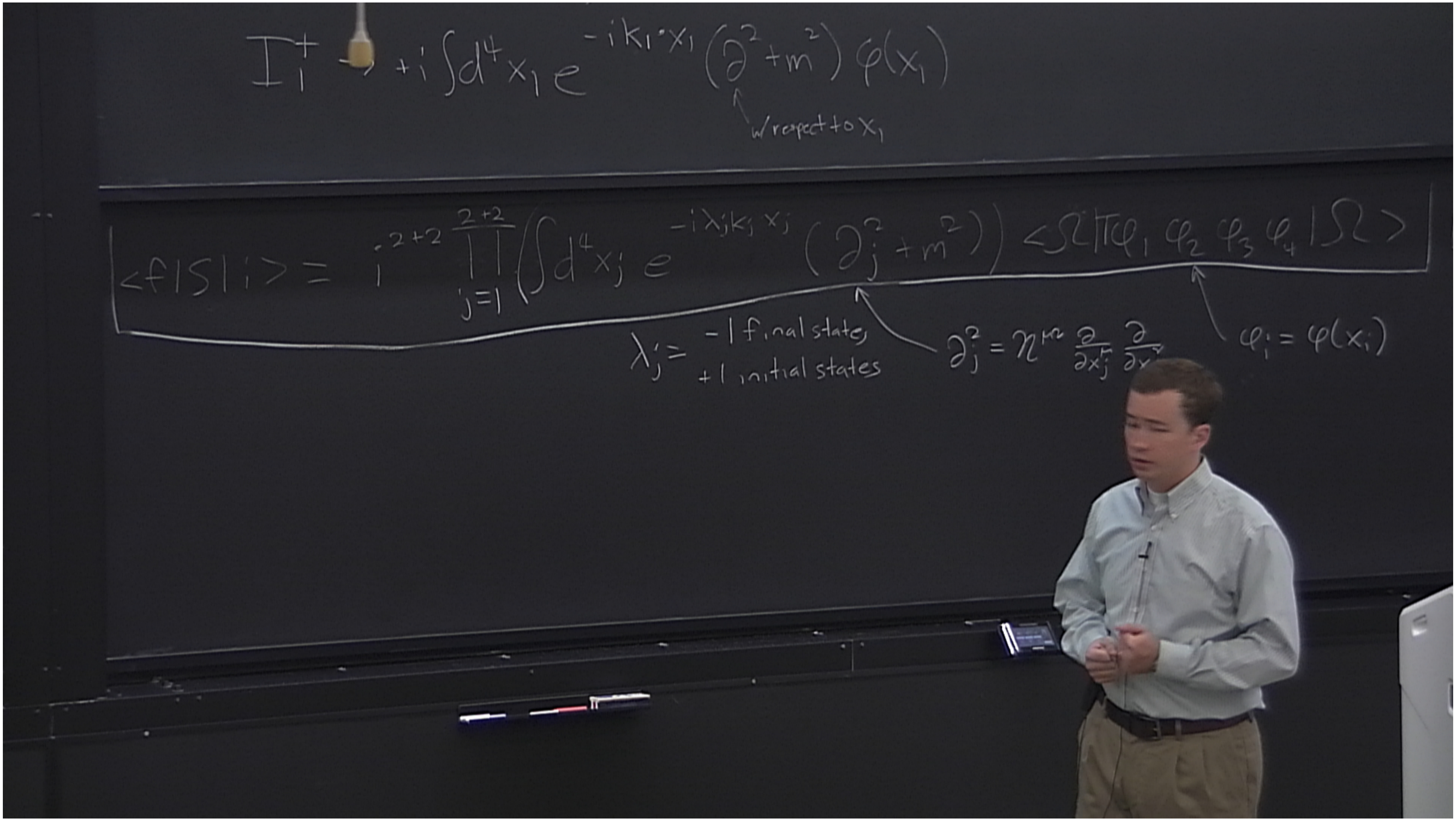
$$+ \cdots + \langle \Omega | T\, I_4 I_3 I_1^{\dagger} I_2^{\dagger} | \Omega \rangle$$

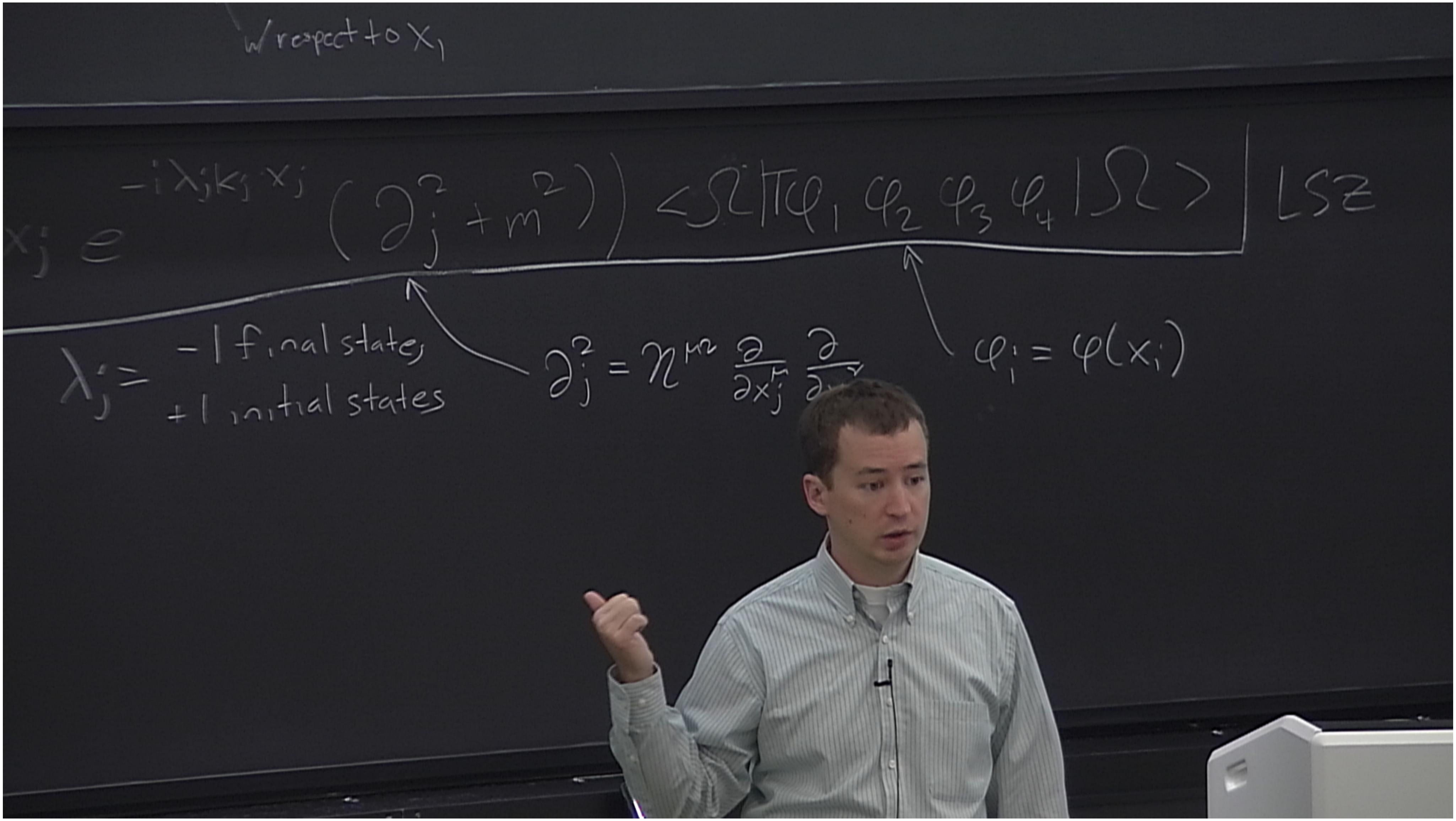
$$f_1(\vec{k}) \to \delta(\vec{k} - \vec{k}_1)$$

$$I_1^{\dagger} \to +i \int d^4x_1\, e^{-ik_1\cdot x_1} \left(\partial^2 + m^2\right) \varphi(x_1)$$

w/ respect to x_1

$2+2$


$I_1^+ \rightarrow +i\int d^4x_1\, e^{-ik_1\cdot x_1}(\partial^2+m^2)\varphi(x_1)$
w/respect to x_1
$\langle f|S|i\rangle = i^{2+2}\prod_{j=1}^{2+2}\left(\int d^4x_j\, e^{-i\lambda_j k_j\cdot x_j}(\partial_j^2+m^2)\right)\langle\Omega|T\varphi_1\varphi_2\varphi_3\varphi_4|\Omega\rangle$
$\lambda_j =$ -1 final states
+1 initial states
$\partial_j^2 = \eta^{\mu\nu}\frac{\partial}{\partial x_j^\mu}\frac{\partial}{\partial x_j^\nu}$
$\varphi_i = \varphi(x_i)$


w/ respect to x_1
$x_j\, e^{-i\lambda_j k_j \cdot x_j} (\partial_j^2 + m^2)) \langle \Omega | T \varphi_1 \varphi_2 \varphi_3 \varphi_4 | \Omega \rangle$
LSZ
$\lambda_j =$ -1 final states, +1 initial states
$\partial_j^2 = \eta^{\mu\nu} \frac{\partial}{\partial x_j^\mu} \frac{\partial}{\partial x_j^\nu}$
$\varphi_i = \varphi(x_i)$

states

Assumptions: $\varphi(x)|\Omega\rangle$ $\quad a_p^\dagger(\pm\infty)|\Omega\rangle$

$$\varphi(x) = e^{i\hat{p}\cdot\hat{x}}\,\varphi(0)\,e^{-i\hat{p}\cdot\hat{x}}$$

$$\langle\Omega|\varphi(x)|\Omega\rangle = \langle\Omega|\,e^{i\hat{p}\cdot\hat{x}}\,\varphi(0)\,\underbrace{e^{-i\hat{p}\cdot\hat{x}}|\Omega\rangle}_{|\Omega\rangle \text{ by translation inv.}}$$

$$= \langle\Omega|\varphi(0)|\Omega\rangle$$

$$\stackrel{?}{=} 0$$

anslation
V.

$$\langle \Omega | \varphi(0) | \Omega \rangle = V$$

Lorentz invariant

$$\tilde{\varphi}(x) = \varphi(x) - V \implies \langle \Omega | \tilde{\varphi}(x) | \Omega \rangle = 0$$

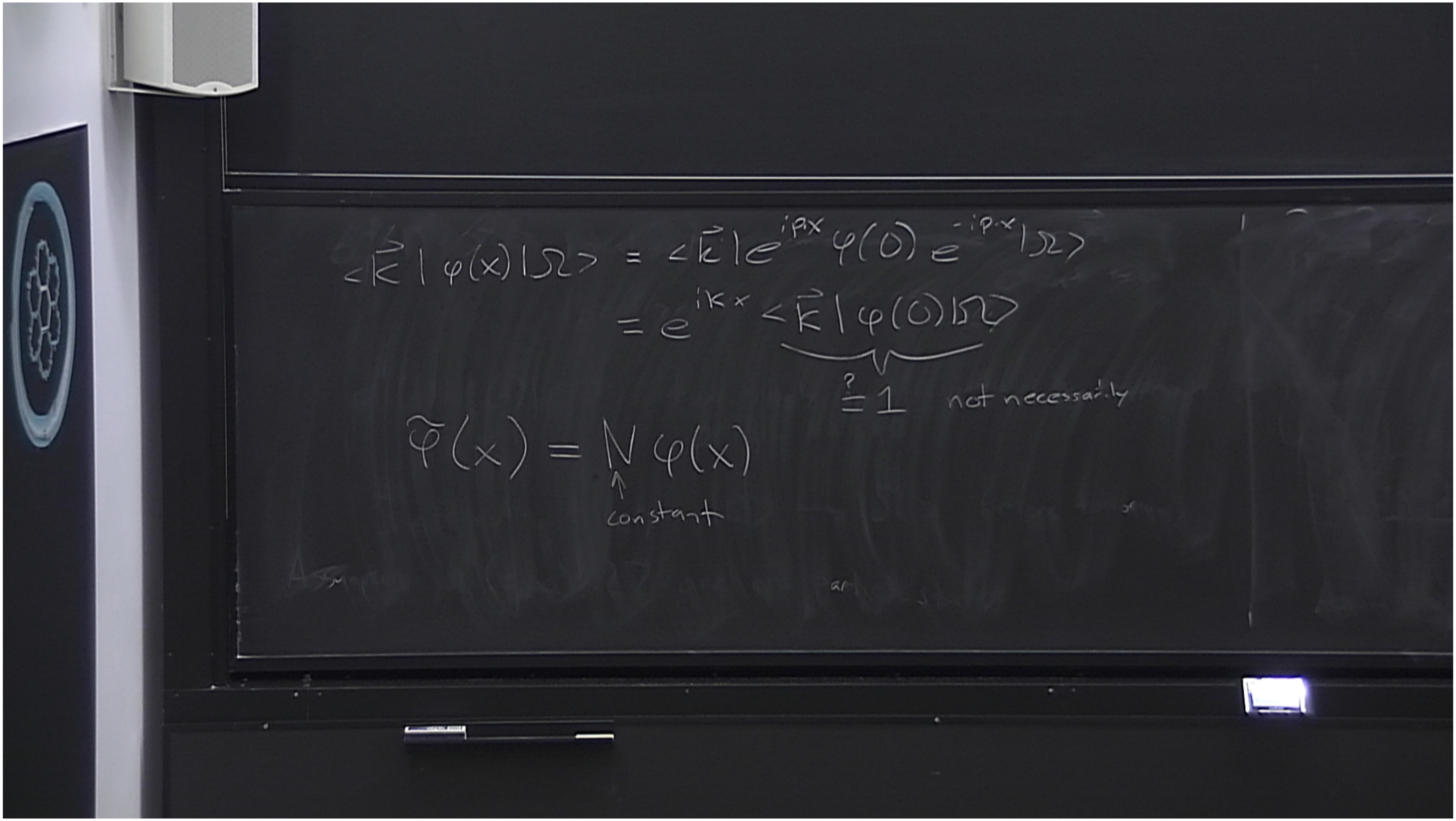
$$\langle \Omega | \varphi(0) | \Omega \rangle = v$$

Lorentz invariant

$$\tilde{\varphi}(x) = \varphi(x) - v \implies \langle \Omega | \tilde{\varphi}(x) | \Omega \rangle = 0$$

Drop ~ in what follow

lation


$\langle \vec{k} | \varphi(x) | \Omega \rangle = \langle \vec{k} | e^{ip\cdot x} \varphi(0) e^{-ip\cdot x} | \Omega \rangle$
$= e^{ik\cdot x} \underbrace{\langle \vec{k} | \varphi(0) | \Omega \rangle}_{\stackrel{?}{=} 1}$ not necessarily
$\tilde{\varphi}(x) = N \varphi(x)$
constant

nslation

$$\langle \vec{k} | \varphi(x) | \Omega \rangle = e^{i k \cdot x} \quad \text{in free theory}$$

$$\langle \vec{k} | = \langle k |$$

$$\langle \vec{p}, \sigma | \varphi(x) | \Omega \rangle = \langle \vec{p}, \sigma | e^{i P \cdot x} \varphi(0) e^{-i P \cdot x} | \Omega \rangle$$

$\vec{p}$: total 3-momentum

σ: all other parameters

$$= e^{i p \cdot x} \underbrace{\langle \vec{p}, \sigma | \varphi(0) | \Omega \rangle}_{A_n(\vec{p})}$$

$$\langle \vec{p}, \sigma | \, a_1^{+}(\pm\infty) | \Omega \rangle = 0$$

Need: $\langle \Psi | \, a_1^{+}(\pm\infty) | \Omega \rangle = 0$

multiparticle wave-packet

$$|\Psi\rangle = \sum_{\sigma} \int d^3p \, \Psi_{\sigma}(\vec{p}) \, |\vec{p}, \sigma\rangle$$

wave-packets

$$\langle \Psi | a_1^\dagger(t) | \Omega \rangle = -i \sum_\sigma \int d^3p\, \Psi_\sigma^*(\vec{p}) \int d^3k\, f_1(\vec{k}) \int d^3x \Big[e^{-ik\cdot x}\, \partial_0 \underbrace{\langle \vec{p},\sigma | \varphi(x) | \Omega \rangle}_{A_\sigma(\vec{p})\, e^{ip\cdot x}} - \langle p,\vec{\sigma} | \varphi(x) | \Omega \rangle\, \partial_0 e^{-ik\cdot x} \Big]$$

$$[\;] = i(p^0 + k^0)\, e^{i(p-k)\cdot x} A_\sigma(\vec{p})$$

$$\int d^3x\, e^{i(\vec{k}-\vec{p})\cdot\vec{x}} = (2\pi)^3 \delta(\vec{k}-\vec{p})$$

w/respect to x_1

$$\langle \Psi | a_1^+(t) | \Omega \rangle = \sum_\sigma \int d^3p \, (2\pi)^3 (p^0 + k^0) \, \psi_\sigma^*(\vec{p}) \, f_1(\vec{p}) \, A_\sigma(\vec{p}) \, e^{i(p^0-k^0)}$$

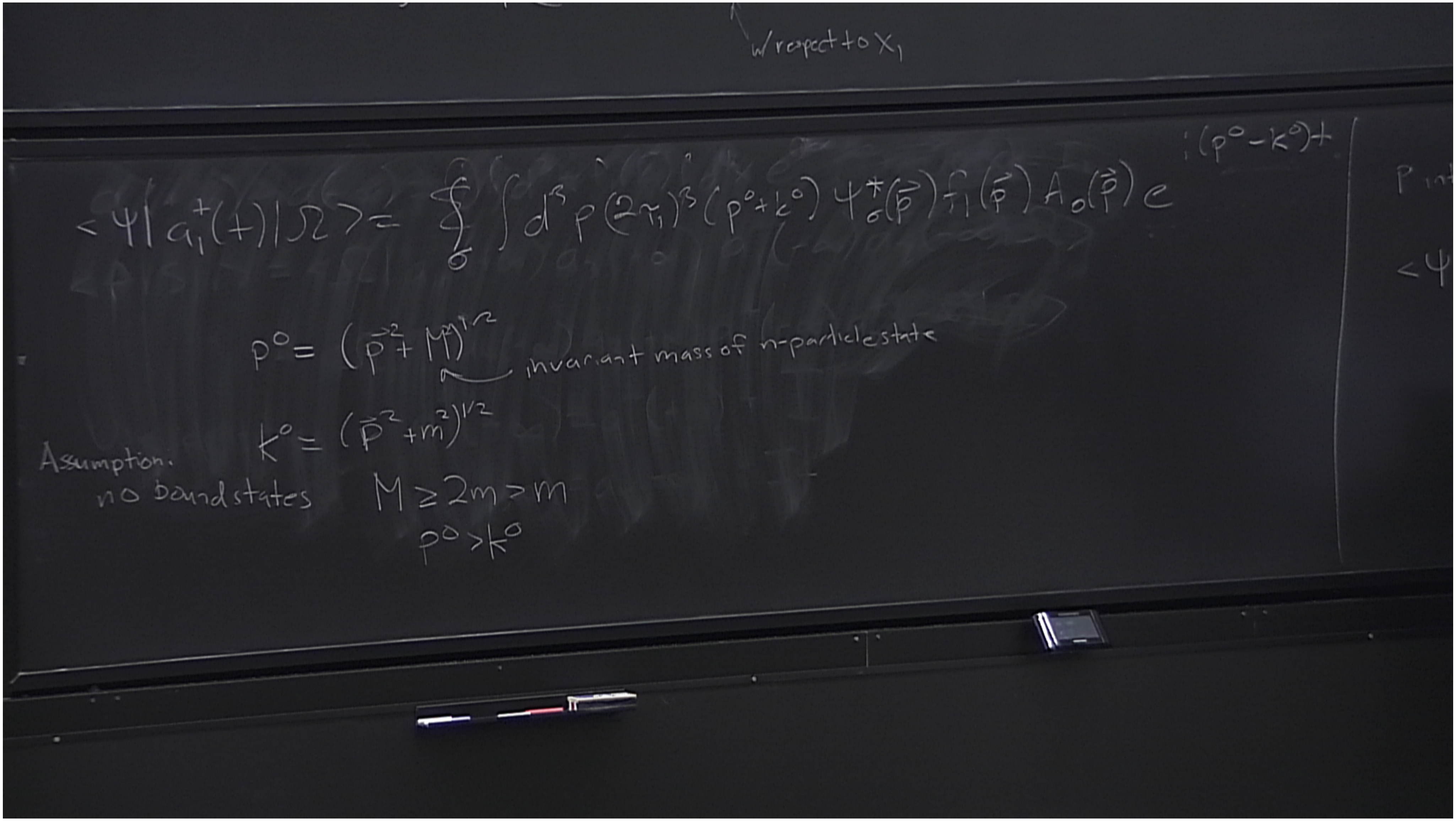
$$p^0 = (\vec{p}^{\,2} + M^2)^{1/2}$$

M — invariant mass of n-particle state

$$k^0 = (\vec{p}^{\,2} + m^2)^{1/2}$$

Assumption.
no bound states $\quad M \geq 2m > m$

$$p^0 > k^0$$


w/respect to x_1
$\langle \Psi | a_1^\dagger(t) | \Omega \rangle = \sum_\sigma \int d^3p \, (2\pi)^3 (p^0 + k^0) \, \Psi_\sigma^*(\vec{p}) f_1(\vec{p}) A_\sigma(\vec{p}) e^{i(p^0 - k^0)t}$
$p^0 = (\vec{p}^2 + M^2)^{1/2}$
invariant mass of n-particle state
$k^0 = (\vec{p}^2 + m^2)^{1/2}$
Assumption:
no bound states
$M \geq 2m > m$
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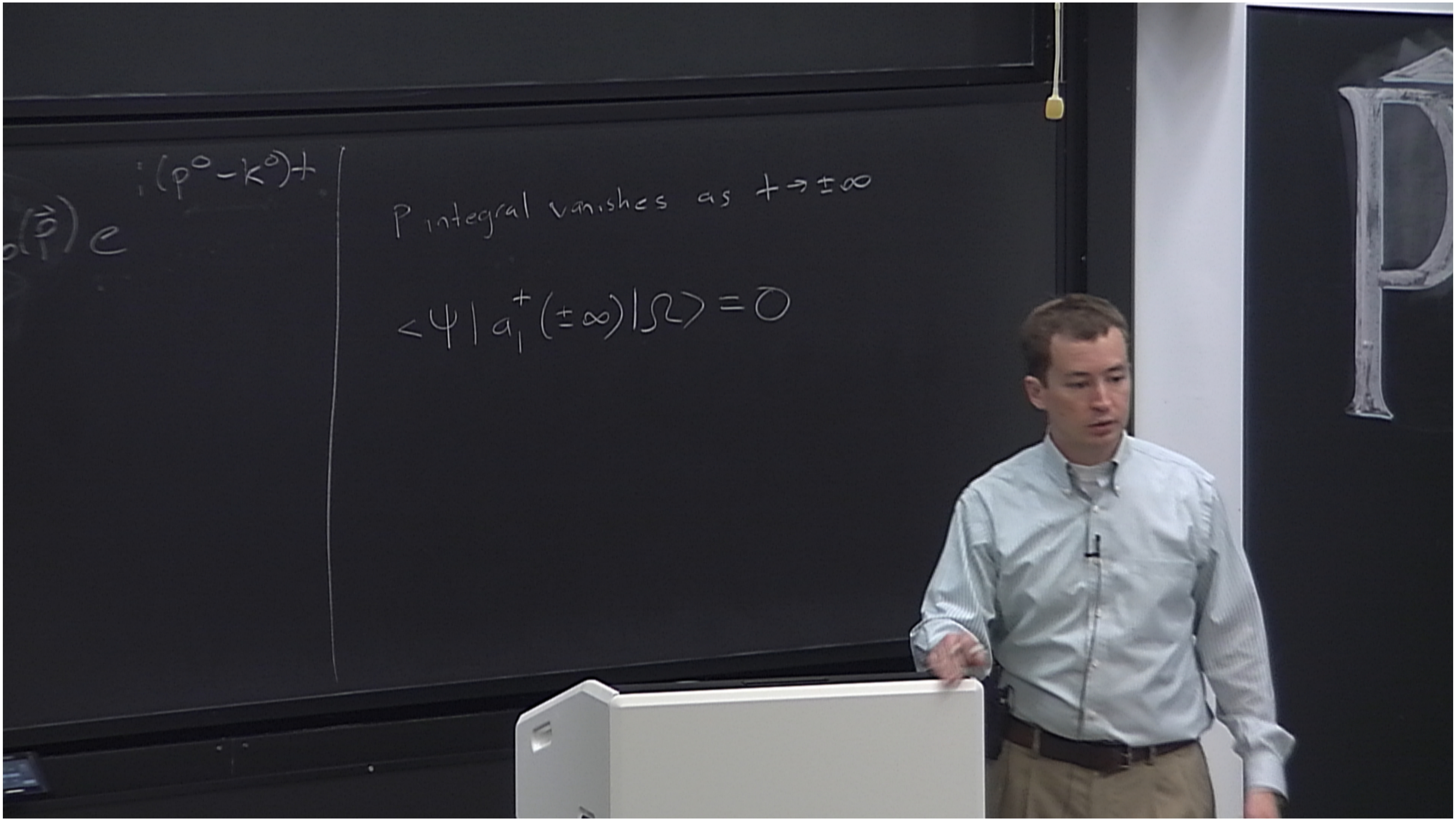
P integral vanishes as $t \to \pm\infty$
$\langle \psi | a_1^+(\pm\infty) | \Omega \rangle = 0$

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Suppose:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \frac{1}{3!}g\varphi^3$$

Have to shift + rescale

new term

$$\mathcal{L} = \frac{1}{2}Z_\varphi(\partial\varphi)^2 - \frac{1}{2}Z_m m^2\varphi^2 + \frac{1}{3!}Z_g\, g\varphi^3 + Y\varphi$$

Renormalization is required

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