Title: Lifshitz Hydrodynamics

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Abstract: <p>We derive the constitutive relations of first order charged hydrodynamics for theories with Lifshitz scaling and broken parity in 2+1 and 3+1 spacetime dimensions. In addition to the anomalous (in 3+1) or Hall (in 2+1) transport of relativistic hydrodynamics, there is an additional non-dissipative transport allowed by the absence of boost invariance. We analyze the non-relativistic limit and use a phenomenological model of a strange metal</p>

<p>to argue that these effects can be measured in principle by using electromagnetic fields with non-zero gradients.</p>

Lifshitz Hydrodynamics

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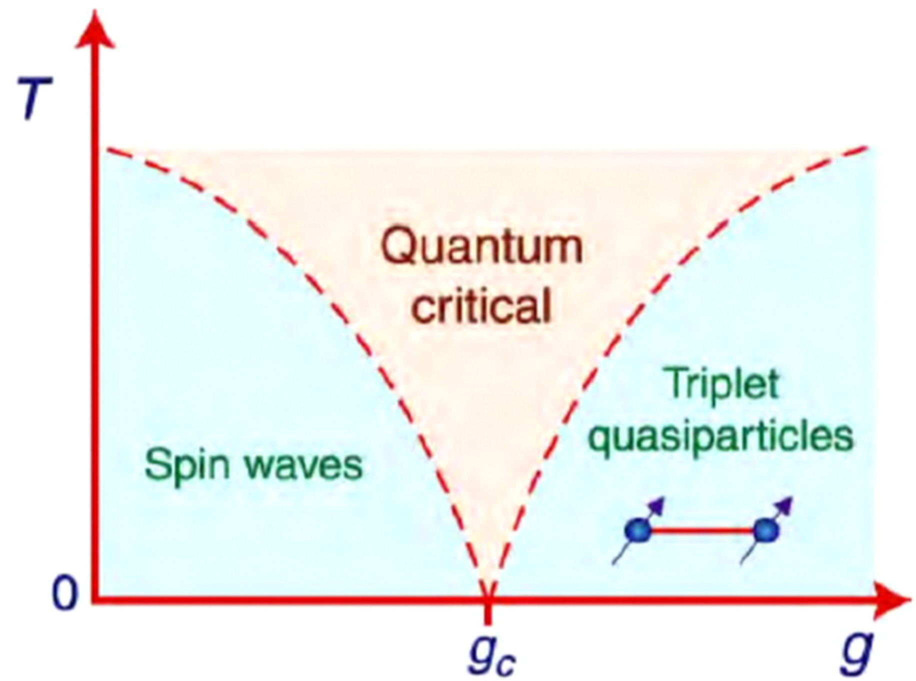
Overview

Motivations - Strange Metal

- A wide variety of metallic ferromagnets and antiferromagnets have a metallic phase (dubbed as strange metal) whose properties cannot be explained within the ordinary Landau-Fermi liquid theory.
- In this phase some quantities exhibit universal behaviour such as the resistivity, which is linear in the temperature T.
- Such universal properties are believed to be the consequence of quantum criticality (Coleman:2005,Sachdev:2011).
- At the quantum critical point there is a Lifshitz scaling (Hornreich:1975,Grinstein:1981) symmetry.

Motivations

Phase transitions at zero temperature are driven by quantum fluctuations.



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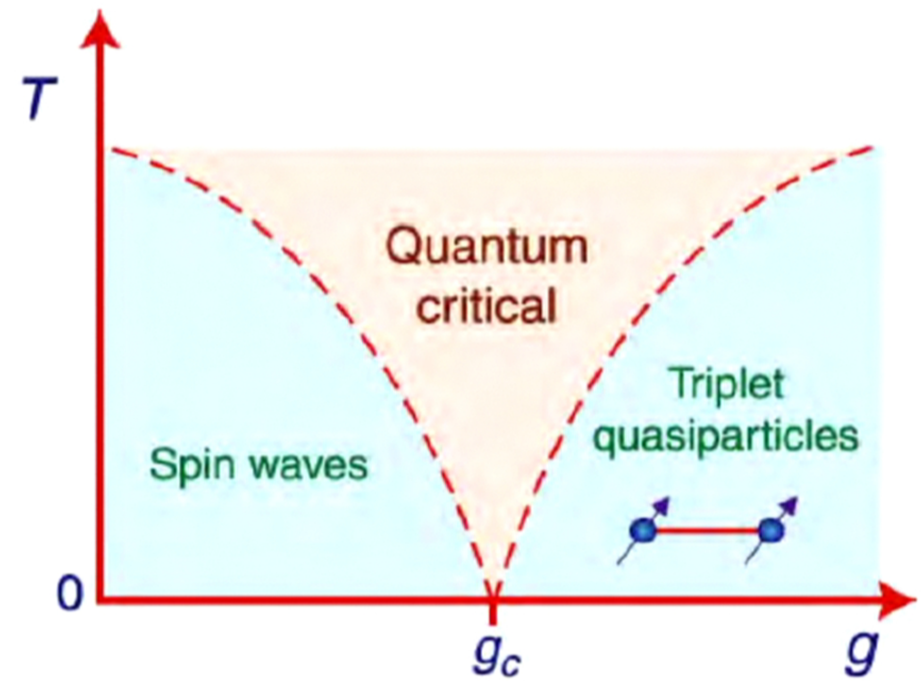

Motivations

- Systems with ordinary critical points have a hydrodynamic description with transport coefficients whose temperature dependence is determined by the scaling at the critical point (Hohenberg:1977).
- Quantum critical systems also have a hydrodynamic description, e.g. conformal field theories at finite temperature.
- At quantum critical regime the hydrodynamic description will be appropriate if the characteristic length of thermal fluctuations $\ell_T \sim 1/T^{1/z}$ is much smaller than the size of the system $L >> \ell_T$ and both are smaller than the correlation length of quantum fluctuations $\xi >> L >> \ell_T$.

Relativistic Hydrodynamics characterization

Hydrodynamics is an effective theory of low energy dynamics of conserved charges, which remain after integrating out high energy degrees of freedom.

Its characterizations are,

- Energy density ϵ
- Pressure p
- Temperature T
- Relativistic fluid velocity $u^{\mu} = \gamma\left(1, v^{i}\right)$, where $\gamma = \left(1 - v^{i} v_{i}\right)^{-1/2}$ and $u^{\mu} u_{\mu} = -1$
- Entropy density s
- Chemical potential μ
- Particle number density/charge density q
- The equation of state $\epsilon = f\left(p\right)$
- Conservation laws $\partial_{\nu} T^{\mu\nu} = 0, \quad \partial_{\mu} J^{\mu} = 0$
- First law $d\epsilon = Tds + \mu dn$

Relativistic Symmetries - Stress Energy Tensor

The Lorentz symmetry has 6 generators, 3 rotations J_i and 3 boosts K_i,

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [K_i, J_j] = \epsilon_{ijk} K_k, \quad [K_i, K_j] = -\epsilon_{ijk} J_k$$

In a relativistic theory the 6 Lorentz generators give rise to six conserved currents,

$$\left(\mathcal{I}^\lambda\right)^{\mu\nu} = X^\mu T^{\lambda\nu} - X^\nu T^{\lambda\mu}, \quad \partial_\lambda \mathcal{I}^\lambda = 0,$$

resulting in a **symmetric** stress tensor,

$$0 = \partial_\lambda \mathcal{I}^\lambda = T^{\mu\nu} - T^{\nu\mu} \Rightarrow \boxed{T^{\mu\nu} = T^{\nu\mu}}.$$

Relativistic Hydrodynamics - Stress Energy Tensor

We divide space into two directions:

- Parallel to the fluid velocity: u^μ
- Perpendicular to the fluid velocity: $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$

The thermodynamic stress energy tensor can be defined,

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + pP^{\mu\nu}$$

At the rest frame we have: $T^0_{\ 0} = -\epsilon$ and $T^i_{\ j} = p\delta^i_{\ j}$
In order to get hydrodynamics, we let the thermodynamic variables be x^μ dependent, $\epsilon\left(x^\nu\right)$, $p\left(x^\nu\right)$, $u^\mu\left(x^\nu\right)$.
The ideal (zeroth order) stress energy tensor is,

$$T^{\mu\nu}_0 = \epsilon(x^\lambda)u^\mu(x^\lambda)u^\nu(x^\lambda) + p(x^\lambda)P^{\mu\nu}(x^\lambda)$$

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Relativistic Hydrodynamics - The Stress Energy Tensor

The ideal equation of motion $\partial_\mu T^{\mu\nu} = 0$ can be divided into two,

- $u_\nu \partial_\mu T^{\mu\nu} = 0 \Rightarrow (\epsilon + p)\,\theta + u^\mu \partial_\mu \epsilon = 0$
- $P^\mu{}_\nu \partial_\lambda T^{\lambda\nu} = 0 \Rightarrow (\epsilon + p)\, a^\mu + P^{\mu\nu} \partial_\nu p = 0$

where $\theta = \partial_\mu u^\mu$, $a^\mu = u^\nu \partial_\nu u^\mu$.

Relativistic Hydrodynamics - The Stress Energy Tensor

We make a derivative expansion of the stress energy tensor,

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}$$

where $\Pi^{\mu\nu}$ conatains derivatives of the thermodynamic variables and of the velocity. How to build $\Pi^{\mu\nu}$:

- In order to avoid redundancies we use the ideal equations of motion to remove the derivatives of the thermodynamic variables.
- To remove unnecessary ambiguities in the definition of the velocity flow, we choose the Landau frame,

$$T^{\mu\nu} u_\nu = -\epsilon u^\mu .$$

This It defines the velocity flow as a flow of energy density, in this frame there are vanishing energy dissipations $T^{\mu\nu} P^\lambda_\mu u_\nu = 0$.

Relativistic Hydrodynamics - The Stress Energy Tensor

For instance, the first order in gradient expansion called the viscous term can be built out of all possible single derivative terms,

$$\Pi^{\mu\nu}_{(1)} = -\eta\sigma^{\mu\nu} - \frac{\zeta}{d}\theta P^{\mu\nu}$$

where $\sigma^{\mu\nu} = \frac{1}{2}\left(P^{\mu\alpha}\partial_\alpha u^\nu + P^{\nu\alpha}\partial_\alpha u^\mu\right) - \frac{1}{d}\theta P^{\mu\nu}$ and d is the spatial dimension.

Relativistic Hydrodynamics - Entropy Current

We can define an entropy density current,

$$S^\mu = su^\mu$$

which we demand it to locally satisfy the second law of thermodynamics,

$$\boxed{\partial_\mu S^\mu \geq 0}.$$

From this we get to kinds of constrains.

- Inequality constraints $\partial_\mu S^\mu \geq 0$
- Equality constraints $\partial_\mu S^\mu = 0$

Relativistic Hydrodynamics - Entropy Current

For instance, by using $d\epsilon = Tds$ and $\epsilon + p = Ts$, we get the first order correction,

$$T\partial_\mu S^\mu = \eta\sigma^2 + \frac{\zeta}{d}\theta^2 \geq 0.$$

Thus it restricts the allowed values of the transport coefficients: $\eta \geq 0,\ \zeta \geq 0$ (Inequality constraint).
Another method to obtain the entropy current is by calculating $u_\nu \partial_\mu T^{\mu\nu} = 0$ and identifing from it the entropy current. For example,

$$u_\nu \partial_\mu T^{\mu\nu} = -T\partial_\mu \left(su^\mu\right) - \partial_\mu u_\nu \Pi^{\mu\nu} = 0 \Rightarrow \partial_\mu \left(su^\mu\right) = -\frac{1}{T}\partial_\mu u_\nu \Pi^{\mu\nu}$$

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Charged Relativistic Hydrodynamics

We can add other conserved currents (particle currents) to the theory,

$$J^{\mu}_{(a)} = q_a u^{\mu} + \nu^{\mu}_{a}, \quad u_{\mu}\nu^{\mu}_{a} = 0, \quad a = 1, ..., N$$

where N is the number of conserved currents, and ν^{μ}_{a} is a higher order term.
To every current we associate a chemical potential μ_a.

Charged Relativistic Hydrodynamics

The entropy current is obtained by identifing the correct term in $u_\nu \partial_\mu T^{\mu\nu} + \mu \partial_\mu J^\mu = 0$, which gives,

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T} \nu^\mu \right) = -\frac{1}{T} \partial_\mu u_\nu \Pi^{\mu\nu} - \nu^\mu \partial_\mu \frac{\mu}{T}$$

From it we have,

- $S^\mu = s u^\mu - \frac{\mu}{T} \nu^\mu$.
- From the 2^{nd} law, we get to first order $\nu^\mu = -\sigma T P^{\mu\nu} \partial_\mu \frac{\mu}{T}$.

Notice that the 2^{nd} law doesn't allow all possible terms, for instance it prohibit the term a^μ in ν^μ.

Charged Relativistic Hydrodynamics with External Sources - Anomaly 3 + 1

If we turn on external U(1) sources the conserved currents are no longer conserved,

$$\partial_\mu T^{\mu\nu} = F^{\mu\lambda} J_\lambda, \quad \partial_\mu J^\mu = C E^\mu B_\mu$$

where $E^\mu = F^{\mu\nu} u_\nu$, $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$ are the external fields, $F^{\mu\nu}$ is the field strength and C is the coefficient of the triangle anomaly.
In addition, we can add odd terms to the charge and entropy currents, namely the vorticity, $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho$ and get,

$$\nu^\mu = -\sigma T P^{\mu\nu} \partial_\mu \frac{\mu}{T} + \sigma E^\mu + \xi_\omega \boldsymbol{\omega}^\mu + \xi_B \mathbf{B}^\mu$$

$$S^\mu = s u^\mu - \frac{\mu}{T}\nu^\mu + D_\omega \boldsymbol{\omega}^\mu + D_B \mathbf{B}^\mu$$

Charged Relativistic Hydrodynamics with External Sources - Anomaly 3 + 1

To find what are the allowed terms which satisfy the 2^{nd} law, we calculate $\partial_\mu S^\mu$ and demand it be non negative.

$$\partial_\mu S^\mu_{\not{p}} = \omega^\mu \left(\partial_\mu D_\omega - 2\frac{\partial_\mu p}{\varepsilon + p} D_\omega - \xi_\omega \partial_\mu \left(\frac{\mu}{T}\right)\right)$$
$$+ \omega^\mu E_\mu \left(\frac{2qD_\omega}{\varepsilon + p} - 2D_B + \frac{\xi_\omega}{T} \right)$$
$$+ B^\mu \left(\partial_\mu D_B - \frac{\partial_\mu p}{\varepsilon + p} D_B - \xi_B \partial_\mu \frac{\mu}{T} \right)$$
$$+ B^\mu E_\mu \left(\frac{qD_B}{\varepsilon + p} + \frac{\xi_B}{T} - C\frac{\mu}{T} \right) = 0.$$

We get an equality constraints. Therefore, all brackets should vanish separately.

Charged Relativistic Hydrodynamics with External Sources - Anomaly 3 + 1

Changing variables from (μ, T) to $(\bar{\mu} = \frac{\mu}{T}, p)$ and using the relations,

$$\left(\frac{\partial T}{\partial p}\right)_{\bar{\mu}} = \frac{T}{\epsilon + p}, \ \left(\frac{\partial T}{\partial \bar{\mu}}\right)_{p} = -\frac{qT^2}{\epsilon + p}$$

we find,

$$\boxed{\xi_\omega = C\left(\mu^2 - \frac{2}{3}\frac{q\mu^3}{\epsilon + p}\right) + 2\beta T^2 - \frac{2q}{\epsilon + p}\left(2\beta\mu T^2 + \gamma T^3\right)}$$

$$\boxed{\xi_B = C\left(\mu - \frac{1}{2}\frac{q\mu^2}{\epsilon + p}\right) - \beta\frac{qT^2}{\epsilon + p}}$$

Thus, the parity odd transport coefficients ξ_ω, ξ_B are determined by the anomaly coefficient C and by numerical integration constants β, γ. β can appear in axial conserved charges and γ can appear in parity breaking theories.

Lifshitz Scaling

- Lifshitz symmetry treats time and space differently:

$$t \to \lambda^z t, \quad x^i \to \lambda x^i$$

- The scale of the thermodynamics quantities: $[\epsilon] = [p] \sim z + d$.
- Since the only scale is the temperature $[T] \sim z$ we get,

$$\epsilon = p \propto T^{\frac{z+d}{z}}$$

- From the thermodynamic relation $d\epsilon = Tds$ we get the dependence of the entropy density on the temperature, $s \sim T^{d/z}$.
- From the thermodynamic relations $\epsilon + p = Ts$ and $s = \frac{\partial p}{\partial T}$ we find the **equation of state**,

$$z\epsilon = dp$$

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Lifshitz Symmetries - The Stress Energy Tensor

In a Lifshitz theory there are 3 rotational generators J_i, 4 translational generators P_μ and one dilation generator D,

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, P_j] = \epsilon_{ijk} P_k, \quad [D, P_t] = zP_t, \quad [D, P_i] = P_i.$$

Because Lifshitz symmetry treats time and space different, it breaks Lorentz boosts, resulting in the breaking of the symmetric stress tensor,

$$\boxed{T^{0i} \neq T^{i0}}$$

We still maintain a rotational symmetry $T^{ij} - T^{ji} = 0$.
We also get a different trace equation for the stress energy tensor:

$$zT^0{}_0 + \delta^j{}_i T^i{}_j = 0$$

Lifshitz Symmetries - Generalization

The Lifshitz algebra can be generalized for constant velocities u^μ with scaling dimension $[u^\mu] = 0$. We define the generators,

$$P^{\parallel} = u^\mu \partial_\mu, \quad P_\mu^{\perp} = P_\mu^\nu \partial_\nu, \quad D = z x^\mu u_\mu P^{\parallel} - x^\mu P_\mu^{\perp}.$$

Then, the momentum operators commute among themselves and

$$\left[D, P^{\parallel}\right] = zP^{\parallel}, \quad \left[D, P_\mu^{\perp}\right] = P_\mu^{\perp}$$

The condition that rotational invariance is not broken with respect to the rest frame of the fluid imposes the condition,

$$P_\mu^\alpha T^{\mu\nu} P_\nu^\beta = P_\nu^\beta T^{\nu\mu} P_\mu^\alpha$$

The trace identity associated to D becomes,

$$z T^\mu{}_\nu u_\mu u^\nu - T^\mu{}_\nu P_\mu{}^\nu = 0.$$

Lifshitz Hydrodynamics

The hydrodynamic equations are,

$$\partial_\mu T^{\mu\nu} = 0.$$

Imposing the Landau frame for unnecessary ambiguities,

$$T^{\mu\nu} u_\nu = -\epsilon u^\mu.$$

The most general stress energy tensor,

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + pP^{\mu\nu} + \pi_S^{(\mu\nu)} + \boldsymbol{\pi}_{\mathbf{A}}^{[\mu\nu]} + \left(u^\mu \pi_A^{[\nu\sigma]} + u^\nu \pi_A^{[\mu\sigma]}\right) u_\sigma.$$

$\pi_S^{\mu\nu}$ is the same as in the relativistic case, for instance the first order part is,

$$\pi_S^{\mu\nu} = -\eta\sigma^{\mu\nu} - \frac{\xi}{d}\theta P^{\mu\nu}.$$

Lifshitz Hydrodynamics

- The new transport coefficients that cannot appear in the relativistic theory can be found in the antisymmetric part of the stress energy tensor, $\pi_A^{\mu\nu}$.
- The condition that boost but not rotational invariance is broken with respect to the rest frame of the fluid imposes the condition,

$$P_{\alpha\mu}\pi_A^{[\mu\nu]}P_{\nu\beta} = 0$$

- This implies that the antisymmetric term should take the form,

$$\boxed{\pi_A^{[\mu\nu]} = u^{[\mu}V^{\nu]}}$$

where $V^{\mu}u_{\mu} = 0$.

Lifshitz Hydrodynamics

To find a $\pi_A^{\mu\nu}$ which doesn't violate the 2^{nd} law, we examine the divergence of the entropy current. Using the conservation equation, $u_\nu \partial_\mu T^{\mu\nu} = 0$, we find,

$$T\partial_\mu (su^\mu) = -\pi_A^{[\mu\sigma]} \left(\partial_{[\mu} u_{\sigma]} - u_{[\mu} u^\alpha \partial_\alpha u_{\sigma]}\right) + ... = -V^\mu a_\mu + ...$$

We get, to first order, **one** new transport coefficient,

$$\pi_A^{[\mu\nu]} = -\alpha u^{[\mu} a^{\nu]}.$$

The scaling dimension of the transport coefficients is $[\eta] = [\zeta] = [\alpha] = d + z - 1$, which determines their temperature dependence to be

$$\eta \sim \zeta \sim \alpha \sim T^{(d+z-1)/z}.$$

Charged Lifshitz Hydrodynamics

If we consider a charged fluid we have an additional new transport coefficient,

$$V^\mu = -\alpha a^\mu - T\alpha' P^{\mu\nu}\partial_\nu \frac{\mu}{T}$$

$$\nu^\mu = -\alpha' a^\mu - T\sigma P^{\mu\nu}\partial_\nu \frac{\mu}{T}$$

The new transport coefficient is α'.
In order to satisfy the 2^{nd} law the following inequalities should hold,

$$\alpha\sigma \geq (\alpha')^2,\ \alpha \geq 0,\ \sigma \geq 0.$$

Lifshitz Hydrodynamics 3+1

Consider external fields E^μ and B^μ in 3+1 dimensions, and allow an odd terms in the currents. As in the relativistic case there are only two such pseudovectors, the vorticity ω^μ and the magnetic field B^μ,

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma,$$

$$B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}.$$

The difference between the relativistic case and Lifshitz case is in the antisymmetric part of the stress energy tensor, $\pi_A^{[\mu\nu]} = u^{[\mu} V^{\nu]}$,

$$V^\mu_{A,\not{P}} = -T\beta_\omega \omega^\mu - T\beta_B B^\mu$$

$$\nu^\mu_{\not{P}} = \xi_\omega \omega^\mu + \xi_B B^\mu.$$

Lifshitz Hydrodynamics 3+1

We impose the 2^{nd} law $\partial_\mu S^\mu \geq 0$,

$$\partial_\mu S^\mu_{\not{p}} = \omega^\mu \left(\partial_\mu D_\omega - (2D_\omega + \beta_\omega) \frac{\partial_\mu p}{\varepsilon + p} - \xi_\omega \partial_\mu \left(\frac{\mu}{T} \right) \right)$$
$$+ \omega^\mu E_\mu \left((2D_\omega + \beta_\omega) \frac{q}{\varepsilon + p} - 2D_B + \frac{\xi_\omega}{T} \right)$$
$$+ B^\mu \left(\partial_\mu D_B - \frac{\partial_\mu p}{\varepsilon + p} (D_B + \beta_B) - \xi_B \partial_\mu \frac{\mu}{T} \right)$$
$$+ B^\mu E_\mu \left(\frac{q\,(D_B + \beta_B)}{\varepsilon + p} + \frac{\xi_B}{T} - C\frac{\mu}{T} \right) = 0.$$

We find equality constraints $\partial_\mu S^\mu = 0$, which mean that all brackets should vanish separately.

Lifshitz Hydrodynamics 3+1

It is possible to solve these constraints by changing variables to $(p, \bar{\mu} = \frac{\mu}{T})$ and then to $(T, \bar{\mu})$ using,

$$\left(\frac{\partial T}{\partial p}\right)_{\bar{\mu}} = \frac{T}{\varepsilon + p}, \qquad \left(\frac{\partial T}{\partial \bar{\mu}}\right)_{p} = -\frac{qT^2}{\varepsilon + p},$$

we find for $z \neq 1$,

$$\beta_B = c_B T^{\frac{2-z}{z}}$$

$$\beta_\omega = (2c_B\bar{\mu} + c_\omega)\, T^{\frac{2}{z}}$$

$$\xi_B = C\left(\mu - \frac{1}{2}\frac{q\mu^2}{\varepsilon + p}\right) - c_B\frac{2-z}{2-2z}\frac{q}{\varepsilon + p}T^{\frac{2}{z}},$$

$$\xi_\omega = C\left(\mu^2 - \frac{2}{3}\frac{q\mu^3}{\varepsilon + p}\right) + c_B\frac{z}{1-z}\left(1 - \frac{2\mu q}{\varepsilon + p}\right)T^{\frac{2}{z}}$$

$$-\frac{qT}{\varepsilon + p}\left(\frac{c_\omega}{1-z} + 2c_B\bar{\mu}\right)T^{\frac{2+z}{z}}.$$

Lifshitz Hydrodynamics 3+1

For $z = 1$,

$$\beta_B = \beta_\omega = 0,$$

$$\xi_B = C\left(\mu - \frac{1}{2}\frac{q\mu^2}{\varepsilon + p}\right) - \gamma_B \frac{qT^2}{\varepsilon + p},$$

$$\xi_\omega = C\left(\mu^2 - \frac{2}{3}\frac{q\mu^3}{\varepsilon + p}\right) + 2\gamma_B T^2 - \frac{2q}{\varepsilon + p}\left(2\gamma_B \mu T^2 + \gamma_\omega T^3\right).$$

The same as in the relativistic case! for $\gamma_B = \beta, \gamma_\omega = \gamma$.

Lifshitz Hydrodynamics 3+1 Non-Relativistic limit $c \to \infty$

- We look at fluids with broken Galilean boosts.
- We derive the constitutive relations by taking the non-relativistic limit of the Lifshitz hydrodynamic equations.
- We group together terms proportional to factors of c and take the limit where $v << c$.
- The pressure is not affected while the relativistic energy is expanded in terms of the mass density ρ and internal energy U as

$$\epsilon = c^2\rho - \frac{\rho v^2}{2} + U$$

- From the thermodynamic relation $\varepsilon + p = Ts + \mu q$, we get that

$$q = \rho c - \rho\frac{v^2}{2c}, \quad \mu = c + \frac{\mu_{NR}}{c}$$

- The electromagnetic fields scale as $A_i \to A_i$, $A_0 \to A_t/c$.

Lifshitz Hydrodynamics 3+1 Non-Relativistic limit $c \to \infty$

- We take the non-relativistic limit of the hydrodynamic equations.
- The current conservation equation, $u_\nu \partial_\mu T^{\mu\nu} + \mu \partial_\mu J^\mu = 0$, gives the usual continuity equation,

$$\partial_t \rho + \partial_i(\rho v^i) = 0.$$

- By taking the non-relativistic limit of $\partial_\mu S^\mu \geq 0$ we get constraints on the non-relativistic transport coefficients,

$$\partial_i D_\omega - \frac{\beta_\omega}{\rho}\partial_i p + \frac{\xi_\omega}{T^2}\partial_i T = 0$$

$$\boxed{\beta_\omega = 0}$$

$$\partial_i D_B - \frac{\beta_B}{\rho}\partial_i p + \frac{\xi_B}{T^2}\partial_i T = 0$$

$$\boxed{\beta_B - \frac{C}{T} = 0} \rightarrow \textbf{defined by the relativistic anomaly alone!}$$

Lifshitz Hydrodynamics 3+1 Non-Relativistic limit $c \to \infty$

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Lifshitz Hydrodynamics 3+1 Non-Relativistic limit $c \to \infty$

- Navier-Stokes equations become:

$$\partial_t P^i + \partial_j \left(P^i v^j\right) + \partial^i p =$$
$$\rho\left(E^i + \epsilon^{ijk} v_j B_k\right) + \partial_j\left(\eta\sigma^{ij} + \delta^{ij}\zeta\partial_k v^k\right)$$

where $\sigma_{ij} = \partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v^k$ is the shear tensor.

- The momentum density is

$$P^i = \rho v^i - \alpha_a a^i - \alpha_T \partial^i T - T\beta_{\mathbf{B}}\mathbf{B}^{\mathbf{i}},$$

where we define the acceleration as $a^i = D_t v^i = \partial_t v^i + v^j \partial_j v^i$ and the magnetic field as $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$.

- The term β_B allows a Chiral Magnetic Effect in a non-relativistic theory.

Lifshitz Hydrodynamics 3+1 Non-Relativistic limit $c \to \infty$

- Navier-Stokes equations become:

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- The momentum density is

$$P^i = \rho v^i - \alpha_a a^i - \alpha_T \partial^i T - T\beta_{\mathbf{B}}\mathbf{B^i},$$

where we define the acceleration as $a^i = D_t v^i = \partial_t v^i + v^j \partial_j v^i$ and the magnetic field as $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$.

- The term β_B allows a Chiral Magnetic Effect in a non-relativistic theory.

Parity Breaking Lifshitz Hydrodynamics 3+1 Drude Model

- We model the collective motion of electrons in the strange metal as a charged fluid moving through a static medium, that produces a drag on the fluid.
- The hydrodynamic equations are

$$\partial_\mu J^\mu = \mathcal{A}, \quad \partial_\mu T^{\mu\nu} = F^{\nu\sigma} J_\sigma - \lambda c \delta^{\nu i} J_i .$$

- We describe a steady state, which implies that the external fields are constant in time.

Parity Breaking Lifshitz Hydrodynamics 3+1 Drude Model

There are two kind of contributions, pointing in different directions.

- One from the Lorenz force term,

$$J_y = -\frac{\rho}{\lambda^2} E_x B_z .$$

- The second is due to the Chiral Magnetic term and points in the direction of the magnetic field,

$$\boxed{J_z = \left[\frac{T\beta_B}{\lambda^2}\partial_x B_z\right] E_x} .$$

This new current would be forbidden in a Galilean-invariant theory. **It can be measured in the lab!**

Additional Work

- 2+1 Dimension
  A transverse (Hall) current is also generated,
  $$J_y = \frac{1}{\lambda^2}\left[\frac{\beta_E}{\rho}\partial_x^2 p - \beta_T \partial_x^2 T - \beta_\mu \partial_x^2 \mu_{NR}\right] E_x.$$
  This can be interpreted as an anomalous Hall effect. (A transverse current in the absence of magnetic fields).
  Other related Lifshitz topics:
- Holography - Zeroth order Navier Stokes equations was found,
  $$(\varepsilon + p)\, u^\alpha \partial_\alpha u_\nu + P_\nu^\alpha \partial_\alpha p = 0.$$
- Partiton function - Zeroth order stress energy tensor was found
  $$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu + p\eta^{\mu\nu},$$
  and also lifshtiz equation of state was found, $-z(\varepsilon - \mu\rho) + dp = 0$.
- Superfluid - 8 new parity even transport coefficients were found.

Summary

- We discussed constitutive relations of fluids for systems with Lifshitz symmetry and broken parity.
- When the condition of boost invariance is relaxed there can be new terms in the energy-momentum tensor that can be grouped in a vector V_A^μ

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + pP^{\mu\nu} + \pi_S^{(\mu\nu)} + u^\mu V_A^\nu,$$
$$J^\mu = qu^\mu + \nu^\mu.$$

- In $3+1$ dimensions the terms that break parity are proportional to the magnetic field or the vorticity

$$V^\mu_{A,\not{P}} = -T\beta_\omega\omega^\mu - T\beta_B B^\mu,$$
$$\nu^\mu_{\not{P}} = \xi_\omega\omega^\mu + \xi_B B^\mu.$$

- The chiral anomaly is present in ξ_ω and ξ_B.
- For $z=1$, $\beta_\omega = \beta_B = 0$ we find the same result as in the relativistic case.

Summary

- In the non-relativistic limit the momentum density receives corrections to first order. In $3+1$ dimensions there is a parity breaking term proportional to the magnetic field

$$P^i = \rho v^i - \alpha_a a^i - \alpha_T \partial^i T - T \beta_B B^i$$

- The second law forbids a term proportional to the vorticity, but $\beta_B \neq 0$
- Using a Drude model with drag coefficient λ for the strange metal, the parity breaking term is responsible for producing a current in the direction of the magnetic field,

$$J_z = \left[\frac{T \beta_B}{\lambda^2} \partial_x B_z \right] E_x .$$

Can be measured in the lab!

Future Research

- What are the higher order corrections to the constitutive relations? For instance, finding the allowed second order transport coefficients in the antisymmetric part of the stress energy tensor.
- Understand why are the transport coefficients in the odd sector of the theory the same in both the relativistic theory and the Lifshitz theory?

$$Lorentz = Lifshitz(z = 1).$$

- Find a Partition function that can recover Lifshitz hydrodynamics beyond the ideal order.
- Find a Holographic setup which produces Lifshitz hydrodynamics beyond the ideal order.