Title: Renormalization group flow of entanglement entropy on spheres

Date: Oct 20, 2015 02:00 PM

URL: http://pirsa.org/15100030

Abstract: <p>I will describe entanglement entropy of a cap-like region for a generic quantum field theory residing in the Bunch-Davies vacuum on de Sitter space. Entanglement entropy in this setup is identical with the thermal entropy in the static patch of de Sitter, and it is possible to derive a simple relation between the vacuum expectation value of the energy-momentum tensor trace and the RG flow of entanglement entropy. In particular, renormalization of the cosmological constant and logarithmic divergence of the entanglement entropy are interrelated in this setup. These findings are confirmed by recovering known universal contributions for a free field theory deformed by a mass operator as well as correct universal behaviour at the fixed points. In three dimensions the renormalized entanglement entropy is stationary at the fixed points but not monotonic. Computational evidence that the universal `area law' for a conformally coupled scalar is different from the known result in the literature will be given, and I will argue that this difference survives in the limit of flat space.</p>

RG flow of entanglement entropy on spheres

OB, Dean Carmi, Michael Smolkin. arXiv:1504.00913

Chris Akers, OB, Shimon Yankielowicz, Michael Smolkin to appear soon

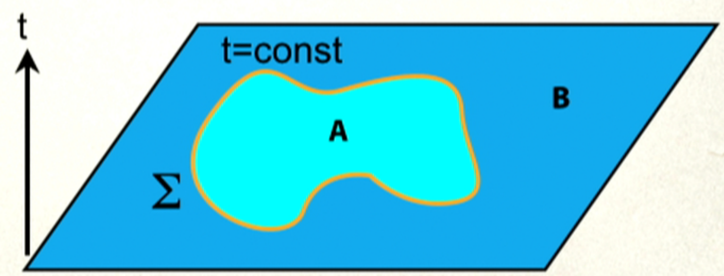
10.20.15

Outline

- Introduction and motivation
- Local Modular Hamiltonians for general QFTs and a perturbative calculation of EE
- Our setup - cap like region (sphere) inside a Sphere
- Massive scalar in even and odd dimensions and comparing to known results
- RG flows - Renormalized EE in 3d, Renormalization of EE and the partition function in 4d
- Minimally VS. Conformally coupled Scalar (free and interacting)

Introduction - Entanglement Entropy

- A system is in some state $|\psi\rangle$
- Write the state as $|\psi\rangle \in H_A \otimes H_B$
- Where Σ is the entangling surface which separates the regions
- Trace out degrees of freedom in B $\rho_A = \mathrm{Tr}_B \, |\psi\rangle\langle\psi|$
- von Neumann Entropy: $S_{\mathrm{EE}} = -\mathrm{Tr}\left[\rho_A \log \rho_A\right]$
- Diverges and dominated by short distance (UV) correlations near the entangling surface.
- Introducing a short distance cutoff - Leading divergence gives an <u>Area law</u>



Path Integral rep. of the reduced density matrix

- The ground state wave functional is:

$$\Psi\left(\phi_0\left(x\right)\right) = \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi_0(x)} D\phi e^{-S(\phi)}$$

To get the complex conjugate integrate to $t_E = \infty$

- The density matrix is written as

$$[\rho]_{\phi_0\phi_0'} = \Psi\left(\phi_0\left(x\right)\right)\bar{\Psi}\left(\phi_0'\left(x\right)\right)$$

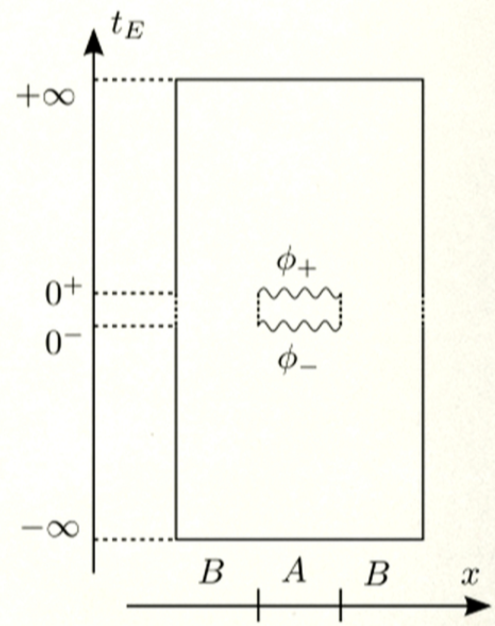
- The find the reduced density matrix of some region A:

- Integrate freely in the complement region B by stitching

$$\phi_0(x) = \phi_0'(x)$$

in region B.

- Boundary conditions in region A give the matrix element of the reduced density matrix



$$[\rho_A]_{\phi+\phi-} = \int_{t_E=-\infty}^{t_E=\infty} D\phi e^{-S(\phi)} \prod_{x\in A} \delta(\phi(+0,x) - \phi_+(x))\cdot\delta(\phi(-0,x) - \phi_-(x))$$

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- To calculate EE we need $S_{\rm EE} = -{\rm Tr}\left[\rho_A \log \rho_A\right]$
- Usually people do the replica trick:

$$S_A = -\frac{\partial}{\partial n} \log {\rm tr}_A \rho_A^n |_{n=1}$$

- This amounts to the calculation of:

$${\rm tr}_A \rho_A^n = \frac{Z_n}{Z_1^n}$$

where Z_n is the partition function on the n-sheeted Riemann surface
-> Conical singularity at the entangling surface

- **Subtleties** for the scalar.. $\mathcal{R} = 4\pi(1-n)\delta_\Sigma + \mathcal{O}(1-n)^2$

some comments towards the end!

Wave-guide geometries

(Hertzberg,Wilczek &Hung,Lewkowycz,Myers,Smolkin)

- The manifold is $R^2 \times \Sigma$
- The entangling surface Σ is then located at the origin
- Replica trick - replace R^2 with C_α a two dimensional cone angular excess $2\pi(\alpha - 1)$

$$ds^2 = (dr)^2 + r^2(d\theta)^2 \qquad 0 \leq r \leq \infty \qquad 0 \leq \theta \leq 2\pi\alpha$$

- O(2) symmetry - continue to any real value of α , find EE $\alpha \to 1$

- For a fermion in 4d and $R^2 \times S^2$ (entangling surface is a sphere):

$$S = \frac{\mathcal{A}_\Sigma}{24\pi\delta^2} + \mathcal{A}_\Sigma \left(2m^2 + \frac{1}{3R^2}\right) \log\left(m\delta\right) + \cdots$$

- The first term is the Area law, it's coefficient depends on the cutoff, hence it is not universal

- Inside the coefficient of the log divergence:
  The first term is the **universal area law**.

  The second term is purely geometric. This what we expect to get in the massless limit (CFT)

- In 5d we now have $R^2 \times S^3$

- This time the universal part will be finite, and in the large radius limit:

$$S = \frac{\mathcal{A}_\Sigma}{3(4\pi)^{3/2}}[\frac{1}{3\delta^3} - \left(m^2 + \frac{1}{2R^2}\right)\frac{1}{\delta}] + \frac{1}{36\pi}\mathcal{A}_\Sigma\left(m^3 + \frac{3}{4}\frac{m}{R^2} - \frac{3}{2\pi^2}\frac{m}{R^2}e^{-2\pi m R} + \cdots\right)$$

- Leading divergent term is again the non universal area law

- Now the finite term is universal (in 4d it was the coefficient of the log divergence!)

- Again we see a universal area law, but this time also some other effects not just from the entangling surface

- (I did not forget about the scalar..)

Generalize for CFTs and deformations

- For a scalable surface ~ any surface that changes with one scale R and for CFTs we can generalize what we just saw(Liu & Mezei)

$$S_{\text{odd}} = \frac{R^{d-2}}{\delta^{d-2}} + \cdots + \frac{R}{\delta} + (-\mathbf{1})^{\frac{\mathbf{d-1}}{\mathbf{2}}} \mathbf{s}_{\mathbf{d}}^{(\mathbf{\Sigma})} + O\left(\frac{\delta}{R}\right)$$

$$S_{\text{even}} = \frac{R^{d-2}}{\delta^{d-2}} + \cdots + \frac{R^2}{\delta^2} + (-\mathbf{1})^{\frac{\mathbf{d-2}}{\mathbf{2}}} \mathbf{s}_{\mathbf{d}}^{(\mathbf{\Sigma})} \log\frac{R}{\delta} + \text{const} + O\left(\frac{\delta^2}{R^2}\right)$$

- Polynomial divergences depend on choice of the cutoff as before

- In <u>even</u> d the coefficient of <u>log divergence</u> (does not depend on the state of the system)

  In <u>odd</u> d the <u>finite</u> term is the universal term

- In odd d the universal term in a CFT is finite, and no general structure is known for it

- However in even d the universal term is the coefficient of a log divergence that comes from the cutoff near the entangling surface:

In 4d Solodukhin's formula

$$S^{(\text{universal})} = \frac{a_4}{180} \int_\Sigma d^2\sigma \sqrt{\gamma} E_2 + \frac{c_4}{240\pi} \int_\Sigma d^2\sigma \sqrt{\gamma} I_2$$

$$\langle T(x) \rangle = \sum_n b_n I_n(x) - 2(-1)^{\frac{d}{2}} \mathbf{a} E_d(x) + B' \nabla_\mu J^\mu(x)$$

- In higher dimensions we expect the same story to hold

- As we saw in the examples before there are other universal terms when going outside the fixed point! (universal area law..)

- There is evidence (weak and strong coupling) that the general terms can appear (Hertzberg,Wilczek & Hung,Lewkowycz,Myers,Smolkin):

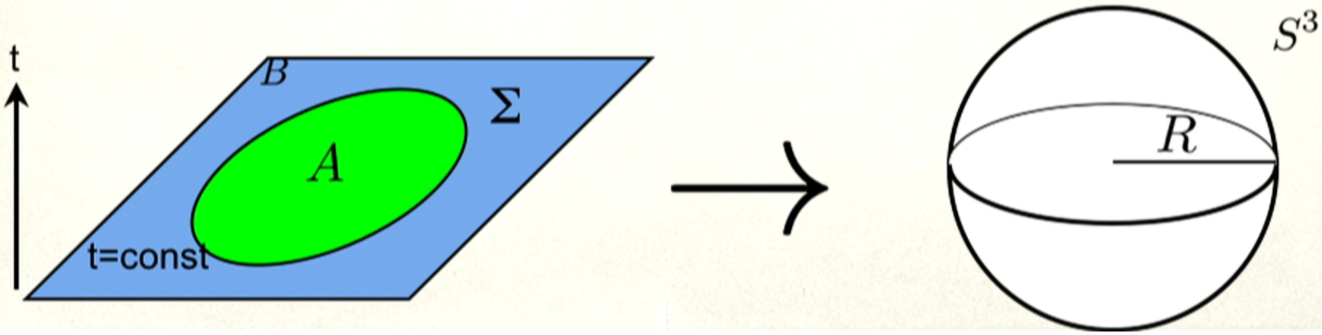
$$S_{\text{univ}} = \gamma\,(D,n)\int_\Sigma d^{D-2}\sigma\sqrt{h}[\text{“curvature”}]^n \times \begin{cases} m^{D-2-2n}\log m\delta & \text{for even D} \\ m^{D-2-2n} & \text{for odd D} \end{cases}$$

- Where for example for $n = 0$ we get the universal area law

- $n = \dfrac{D-2}{2}$ is again the purely geometric term that gives the universal contribution in the fixed point (CFT)

CFTs and Sphere entangling surfaces

- Casini, Huerta and Myers have shown that for CFTs: EE for a sphere entangling surface in flat space can be mapped to thermal entropy of the static patch of de Sitter.
- This mapping gives the <u>universal</u> terms of sphere entangling surfaces
- Wick rotating the static patch of de Sitter we get a sphere S^d



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For theses special cases the universal term corresponds to universal term in the free energy of the theory when we put it on a sphere

In even d the coefficient of the log divergence is given by the A-type (trace) anomaly

In odd d the universal term is the finite term of the free energy.

c-theorems and EE

- c-theorems (weak version):
  For a QFT that flows between fixed points, exists some c function such that:

$$c_{UV} \geq c_{IR}$$

- In 2, 3 and 4d there are proofs for c-theorems.
- In 3d this is the F-theorem:

  Conjectured:
  Free Energy on a 3-sphere decreases monotonically along RG-flows.

  Proved:
  Universal term in the EE of a disk in flat space
  (CHM to the free energy == EE of a disk at the fixed points)

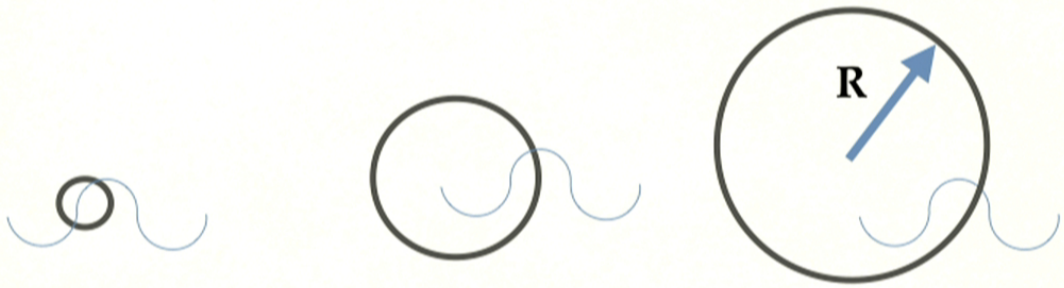
- In 3d proof <u>only</u> using EE
- In 4d no proof using EE
- Sphere in flat space / Sphere free energy isolates correct universal term in 3d and 4d
- Is there a proof in 3d without EE?

  Is there a proof in 4d using EE?

  If yes, can we generalize to higher d?

Relation between RG flows and EE

- Some EE region with one scale R, and a mass scale m
- We change the scale R
- $\frac{1}{m}$



- For example in 3d for a massive scalar for mR>>1:

$$S_{\text{scalar}}(mR) = a\frac{R}{\delta} - \frac{\pi}{6}mR - \frac{\pi}{240}\frac{1}{mR}$$

The first law of EE

- Define the Entanglement/Modular Hamiltonian in region A:

$$K_A = -\log \rho_A$$

- Usually the modular Hamiltonian is not local and not known explicitly..

- Small change of the state: $\tilde{\rho}_A = \rho_A + \delta\rho_A$

- Assuming the new density matrix is normalized:

$$\mathrm{Tr}\left[\tilde{\rho}_A\right] = 1 \Rightarrow \ \mathrm{Tr}\left[\delta\rho_A\right] = 0$$

$$\delta S_{EE} = -\mathrm{Tr}\left[\delta\rho_A\right] - \mathrm{Tr}\left[\delta\rho_A \log\rho_A\right] = \mathrm{Tr}\left[\delta\rho_A K_A\right]$$

The first law:

$$\boxed{\delta S_{EE} = \delta \left\langle K_A \right\rangle}$$

The first law in differential form (Rosenhaus, Smolkin)

$$S_{\mathrm{EE}} = -\mathrm{Tr}\left(\rho_A \log \rho_A\right) = \mathrm{Tr}\left(\rho_A K_A\right) = \mathrm{Tr}\left(\rho K_A\right) = \langle 0|K_A|0\rangle$$

$$S_{EE} = \langle 0\,|K_A|\,0\rangle = \frac{1}{Z}\int D\phi K_A e^{-I(\lambda, g_{\mu\nu}, \phi)}$$

$$\mathrm{Tr}\left(\rho_A\right) = \mathrm{Tr}\left(e^{-K_A}\right) = 1 \Rightarrow \frac{\delta}{\delta g_{\mu\nu}}\mathrm{Tr}\left(e^{-K_A}\right) = 0 \Rightarrow \langle\frac{\delta K_A}{\delta g^{\mu\nu}}\rangle = 0$$

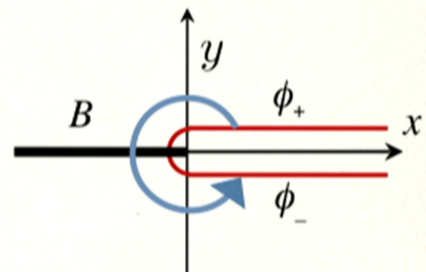
$$\frac{\delta S_{\mathrm{EE}}}{\delta g^{\mu\nu(x)}} = \cancel{\langle\frac{\delta K_A}{\delta g^{\mu\nu}}\rangle} + \frac{\sqrt{g(x)}}{2}\langle T_{\mu\nu}(x)K\rangle$$

- Geometric perturbation: $$\boxed{\frac{\delta S_{EE}}{\delta g^{\mu\nu(x)}} = \frac{\sqrt{g(x)}}{2}\langle T_{\mu\nu}(x)K\rangle}$$

Local modular Hamiltonian

- Write again the reduced density matrix:
  $$\langle \phi_- | \rho_A | \phi_+ \rangle = \langle \phi_- | Tr_B(|0\rangle\langle 0|) | \phi_+ \rangle = \int_{\substack{\phi(0^+, n^+) = \phi_+ \\ \phi(0^-, n^-) = \phi_-}} D\phi \exp[-I(\phi)]$$
-
- For a plane entangling surface it will be local for ANY QFT (Kabbat & Strassler `94):

$$K_A = -2\pi \int_A T_{\mu\nu} \, \xi^\mu \, n^\nu = -2\pi \int_\Sigma \int_0^\infty dx \, x \, T_{yy} \Leftarrow$$



- K is now the integrated generator ("Hamiltonian") of rotation in the plane $x - y$. The 2π is the inverse temperature, or the euclidean time.
- Important - O(2) symmetry around the entangling surface

Our setup

- Goal: RG flows of EE where we can do explicit calculations.
  At the fixed points of the flow correct universal terms in 3d and 4d

- What do we need:
  We want a setup with O(2) symmetry such that the modular Hamiltonian is local and known for all QFTs (not just CFTs) as it was for a plane.

  Half a plane is local but there is no built in scale (no anomaly in even d).

  Wave guides do have a scale but we also want the correct universal terms in 3d and 4d

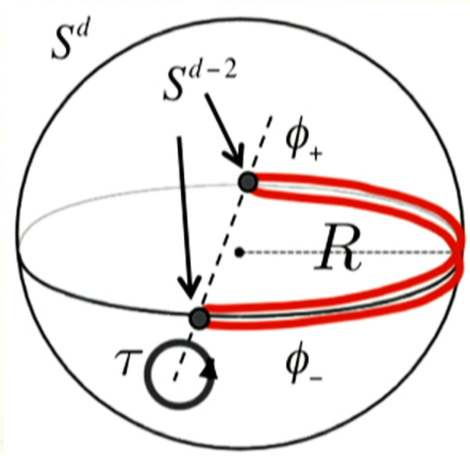
- Start with the static patch of de Sitter and Wick rotate to get S^d with radius R.
  The metric is: $ds^2 = R^2\left(\cos^2\theta d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2_{d-2}\right)$

- Entangling region is half a sphere (one scale)

- Myers & Sinha have shown that for a CFT in even d:

$$R\frac{dS_{\rm EE}}{dR} = 4(-1)^{\frac{d}{2}-1}a$$

and also in odd d we get the free energy



- The modular Hamiltonian is:

$$K = -2\pi \int_A T_{\mu\nu}\xi^\mu n^\nu + c'$$

$n^\nu = (R\cos\theta)^{-1}\partial_\tau$

$\xi^\mu = \partial_\tau$

- The sphere is maximally symmetric and from this we get that for any QFT:

$$\langle T_{\mu\nu}(x)\rangle = -\frac{1}{d}\frac{C}{R^d}\, g_{\mu\nu}(x)$$

$$C \sim \langle T\rangle$$

- C is a function of the theory (coupling mass scales etc.)
- Taking a specific variation (scaling of the sphere) we get:

$$2\int d^dx g^{\mu\nu}(x)\frac{\delta}{\delta g^{\mu\nu}(x)} = -R\frac{d}{dR} \Rightarrow R\frac{dS_{EE}}{dR} = \int d^dx\,\sqrt{g(x)}\langle T(x)K\rangle_\lambda$$

- We can now use the following relation (Osborn & Shore):

$$-\int d^dy\sqrt{g(y)}\langle T(y)T_{\mu\nu}(x)\rangle_{\text{conserved}} = \frac{g_{\mu\nu}(x)}{d}\, R\frac{d}{dR}\left(\frac{C}{R^d}\right)$$

- We contract with n^ν ,ξ^μ and integrate to get K in the correlator

- Finally we get the simple relation:

$$R\frac{dS_{\mathrm{EE}}}{dR} = -\frac{\Omega_d R^{d+1}}{d}\frac{d}{dR}\langle T\rangle$$

- Holds for any QFT (not just a CFT!).

- In these RG flows we don't have some external scale, so the theory flows between fixed points as we <u>change the radius of the sphere</u>

- In this setup it is enough to calculate a one-point function to get the exact flow in our case.

- It is also easy to show that in this setup EE equals the thermal entropy of the static patch of de Sitter for all QFTs:

$$S_{\mathrm{EE}} = S_{\mathrm{Th}} = \beta(U - F)$$

Example -Conformally Coupled Free Scalar

$$I = \int_{S^d} \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}\xi_c \mathcal{R}\,\phi^2 \right) \qquad \xi_c = \frac{d-2}{4(d-1)} \quad \mathcal{R} = \frac{d(d-1)}{R^2}$$

$$T = g^{\mu\nu}T_{\mu\nu} = -m^2\phi^2 - \frac{d-2}{2}\phi(-\nabla^2 + \xi_c\mathcal{R} + m^2)\phi$$

$$C_\phi = m^2\, R^d \langle\phi^2\rangle \qquad \langle\phi^2\rangle = \frac{\Gamma(1-d/2)\Gamma(\lambda)\Gamma(d-1-\lambda)}{\pi(4\pi)^{d/2}R^{d-2}} \sin\left(\frac{\pi}{2}(d-2\lambda)\right)$$

$$\lambda = \tfrac{d-1}{2} + i\sqrt{(mR)^2 - \tfrac{1}{4}}$$

- The VEV is calculated as the coincident point limit of the Green's function on a sphere
- Divergent for even d, finite for odd d
- Dimensional Regularization - pole for even d is a log divergence, which will correspond to universal terms.

Conformal Scalar in odd d

- Plugging odd we get a finite result, and thermal factor in bold:

$$C_\phi = \frac{\pi (mR)^2 \mathbf{coth}(\boldsymbol{\pi}\sqrt{\mathbf{m^2R^2 - 1/4}})}{\sqrt{m^2R^2 - 1/4}} \frac{(-)^{\frac{d-1}{2}}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} \prod_{j=1}^{\frac{d-1}{2}} \left((d/2 - 1/2 - j)^2 - 1/4 + m^2R^2\right)$$

- Expanding in mR>>1 (large radius, low temp.).

$$R\frac{dS^{\rm scalar}_{\rm univ}}{dR} \underset{mR \gg 1}{=} \frac{(d-2)(d-4)}{24(d-1)} \frac{(-)^{\frac{d+1}{2}}\pi}{(4\pi)^{\frac{d-2}{2}}\Gamma\left(\frac{d}{2}\right)} A_\Sigma m^{d-2} + \dots$$

- This result matches with the universal area law of half a plane in flat space for a conformal scalar

- Not a good c-function candidate - diverges in the flat space/IR limit.

Therefore some subtraction is needed to get a reasonable flow - REE?

Renormalized EE (REE) in 3d in flat space

- EE for a disk in flat space at fixed points $S_{\rm EE} = c_1 \frac{R}{\delta} - c_0$
- To extract the universal terms, REE was proposed (Liu & Mezei), in 3d:

$$\mathrm{REE} \equiv R\frac{dS_{EE}}{dR} - S_{EE}$$

- At a fixed points this gives the free energy $(\mathrm{REE} = c_0)$
- Gets rid of terms like mR are are that pollute the constant term.
- Casini and Huerta have shown (using strong sub-additivity): $\boxed{S''_{\rm EE} \leq 0}$
- Integrating this we see explicitly the F-theorem: $(\mathrm{REE}' = RS''_{EE})$

$$\boxed{\Delta c_0 = c_0^{\rm UV} - c_0^{\rm IR} = -\int_0^\infty dR\, R\, S''_{\rm EE} \geq 0}$$

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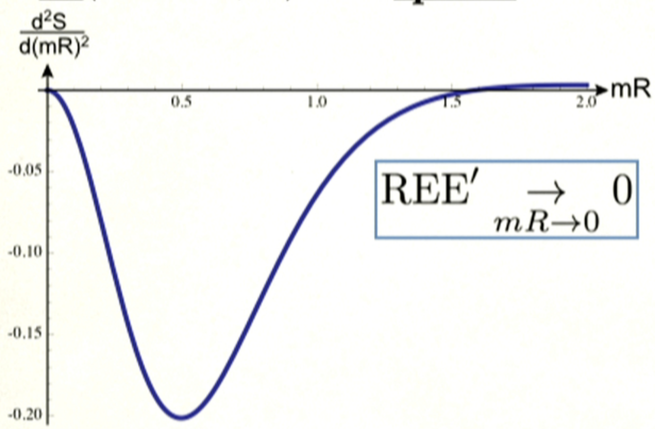
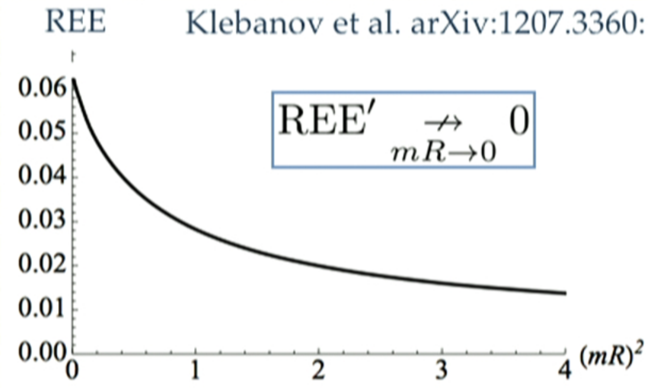
REE for a massive scalar in 3d

S″(or REE′ / R) on a **sphere**:



REE for a disk in **flat space**:



- S″ changes signs, hence REE is not monotonic!
- It is also easy to see that in our setup REE is stationary, meaning: $S''_{\rm EE} \to 0$
- In flat space numerical calculations show that REE is not stationary (right figure).

Conformal Scalar in even d

- Divergent log term, UV physics does not care about the (thermal) state:

$$R\frac{dS^{\text{scalar}}_{\text{univ}}}{dR} = \frac{(d_0-2)(d_0-4)}{12(d-1)} \frac{(-)^{d_0/2+1} A_\Sigma}{(4\pi)^{(d_0-2)/2}\Gamma\left(\frac{d_0}{2}\right)}$$
$$\times \left(m^{d_0-2} + \frac{(d_0-6)\left(5d_0^2 - 18d_0 + 4\right)}{120} \frac{m^{d_0-4}}{R^2} + \ldots + 24 \frac{\Gamma^2\left(\frac{d_0}{2}\right)}{d_0(d_0-2)^2} \frac{m^4}{R^{d_0-6}} \right) \log(m\delta)$$

- Leading term in mR>>1 is the universal area law that does not care about curvature, this result again matches known results for half a plane in flat space for a conformal scalar
- For example in 4d the area law **vanishes** (important for the rest of the talks) for the conformal scalar

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- We can check also the leading curvature correction.
- We can see that very close to the surface the metric is $R^2 \times S^{d-2}$ and perturbation:

$r \equiv R(\theta - \pi/2)$

$$ds^2 = (1 - \frac{r^2}{3\,R^2} + \ldots)r^2\, d\tau^2 + R^2 d\theta^2 + (1 - \frac{r^2}{R^2} + \ldots)R^2\, d\Omega^2_{d-2}$$

- The leading perturbation of the wave guide geometry is important
- This amounts to the following integral:

$$\delta S_{\rm EE} = \frac{1}{2}\int d^2\, x \int d^{d-2} y \sqrt{\gamma}\, \langle T^{\mu\nu}(x,y) K \rangle h_{\mu\nu}(x,y) + \mathcal{O}(h^2)$$

- We again find full agreement between the area law and the first curvature correction and our results, for the conformal scalar.

- So far we discussed the log divergent term but eventually our equation relates the vev of renormalized stress energy tensor and a renormalized EE.

- In fact renormalization is needed to see the RG flow and the anomaly at the fixed points (massless scalar in the UV to empty theory in the IR).

Renormalization of EE in 4d

- Up to now we've written EE using bare parameters.
- We should account for counter terms such that the partition function is finite.
- The most general action we can write for a massive scalar in a curved background (for a renormalizable theory) is (Brown&Collins):

$$S_{tot}^{d=4} = \int_{S^d} \left(\frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi_c \mathcal{R}\, \phi^2 + \Lambda_0 + \kappa_0 \mathcal{R} + \frac{b_0}{16\pi^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + 2a_0 E_4 + c_0 \mathcal{R}^2 \right)$$

- We write the relation between bare and renormalized params in minimal subtraction scheme - we only subtract poles:

$$\Lambda_0 = \mu^{d-4}\left(\Lambda + \frac{\Lambda_p}{d-4}m^4\right) \qquad \kappa_0 = \mu^{d-4}\left(\kappa + \frac{\kappa_p}{d-4}m^2\right)$$

- We fix Λ_p knowing that the following is finite

$$-m^2\frac{\partial Z}{\partial m^2} = \int_{S^d}\left(\frac{m^2}{2}\langle\phi^2\rangle + \mu^{d-4}m^2\mathcal{R}\frac{\kappa_p}{d-4} + 2\mu^{d-4}m^4\frac{\Lambda_p}{d-4}\right)$$

since LHS is the renormalized vev

- Plugging the bare one-point function:

$$\langle \phi^2 \rangle = \frac{m^2}{8\pi^2(d-4)} - \frac{m^2}{16\pi^2}\left(\log(4\pi R^2) - \psi(\lambda) - \psi(3-\lambda) + 1 - \gamma\right) + \mathcal{O}(d-4)$$

we get $\Lambda_p = \frac{-1}{32\pi^2}$, $\kappa_p = 0$

- We can now impose a physical condition:

The decoupling of the massive theory in the IR , $mR \to \infty$
where the massive scalar is integrated out meaning $\langle T(x) \rangle \to 0$

- This sets the rest of the renormalized coupling such that we get:

$$\langle T(x) \rangle = \frac{m^4}{16\pi^2}\left(2\log(mR) - \psi(\lambda) - \psi(3-\lambda) + \frac{2}{3(mR)^2}\right) - \frac{1}{240\pi^2 R^4}$$

- And finally in 4d: $R\frac{dS_{\mathrm{EE}}}{dR} = \frac{(mR)^2}{18} - \frac{(mR)^4}{12} - \frac{i(mR)^6\left(\psi'(3-\lambda)-\psi'(\lambda)\right)}{24\sqrt{(mR)^2-1/4}} - \frac{1}{90}$
  => Renormalization of the partition function corresponds to renormalization of the EE.

- It is interesting to see what we get for 8d:

$$R\frac{dS_{EE}}{dR} = \cdots + 1260R^6\left(m^6 + \frac{3m^4}{R^2}\right)\log(mR) + \cdots$$

  So even after renormalization we can identify the universal terms which are now finite

- Works in 4d and higher, reproducing the correct anomaly in the UV.

- We have checked all of our results also for the Dirac fermion, in odd and even d.

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Conformally VS Minimally coupled Scalar

- For the scalar in flat space there is a family of conserved stress-energy tensors:

$$T^{(\xi)}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial_\alpha\phi\partial^\alpha\phi\right) - \xi\left(\partial_\mu\partial_\nu - g_{\mu\nu}\partial_\alpha\partial^\alpha\right)\phi^2$$

- When coupling to a background metric ξ is the coupling to curvature :

$$\xi\int\mathcal{R}\phi^2 \subset S$$

- where $\xi = 0$ is the minimal coupling

- and $\xi = \xi_c$ is the conformal coupling for which $T^\mu_\mu = 0$

- In a curved background different choices of the coupling correspond to different theories

- In flat space the choices correspond to different, arbitrary, choices of the stress-energy tensor for the same theory.

- The difference between the definition in flat space is just a total derivative term:

$$T^{(\xi)} - T^{(0)} \sim \xi \partial_\alpha \partial^\alpha \phi^2$$

- Debate -
  The modular hamiltonian does have a boundary at the entangling surface:
  Should the contribution from boundary term that come from the total derivative term be taken into account?

- How is this related to our results?
  We started with a curved background and chose the conformal coupling.

  Our results for the universal area law match with the results for half a plane in flat space

- For example in 4d we saw the area law vanishes, which agreed with the result when taking the boundary into account and choosing the conformal coupling!

- Moreover, if we repeat our calculation for different couplings to curvature (not conformal) we will get back the universal area law in 4d

- This means our results are in favour of a difference between the minimally coupled and conformally coupled also in flat space.

- It's important to note that on the sphere the improvement term vanishes, because $\langle \phi^2 \rangle$ is constant! So why do we get different results?
  The vev does depend on the coupling and this results in different universal area law for different couplings:

$$\langle \phi^2 \rangle = \frac{\Gamma(1-d/2)\Gamma(\lambda)\Gamma(d-1-\lambda)}{\pi(4\pi)^{d/2}R^{d-2}} \sin\left(\frac{\pi}{2}(d-2\lambda)\right)$$

$$\lambda = \frac{d-1}{2} + i\sqrt{(mR)^2 - \frac{1}{4}} \to \lambda = \frac{d-1}{2} + i\sqrt{(mR)^2 - \frac{1}{4} + (\xi - 1)\frac{d(d-2)}{4}}$$

The interacting O(N) Model

(ongoing Chris Akers, O.B., Yankielowicz,Smolkin)

- For this model: $$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{t}{2}\phi^2 + \frac{u}{4}\phi^4$$

  Metlitski, Fuertes, Sachdev calculated EE for half a plane in flat space.

- They did an ϵ expansion around 4d to get a flow from the Gaussian (free) fixed point to the Wilson-Fisher (interacting) Fixed point.

- They used the replica trick and analyzed the partition function with a conical singularity.

- In free fixed point universal area law: $\delta S_{\text{Gaussian}} \sim \frac{1}{\epsilon} \nearrow^{\substack{\epsilon \to 0 \\ \log(m\delta)}}$

- In interacting fixed point no divergence: $\delta S_{\text{Wilson-Fisher}} \sim \mathrm{O}(1)$

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- But I just said in 4d the universal area law depends on the choice of coupling to gravity even in flat space, so did they make a choice?

- Yes!

- Before discussing our case - Casini et. al also used the perturbative method by Rosenhaus and Smolkin to explain the difference between the free and interacting fixed points:

$$\delta S = -\frac{\pi}{d(d-1)(d-2)} \int d^d x x^2 \langle 0|T(x)T(0)|0\rangle = \mu_s - 6\frac{\xi^2}{\xi_c}\mu_s$$

- μ_s is the result for the minimally coupled scalar

- Their explanation:
although the result is the same as for the free scalar for both fixed points we need to choose different couplings

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- Back to Metlitski et. al:

- We claim:

  In the Gaussian fixed point is the same as for the minimally coupled scalar.

  So in the limit of 4d their universal area law did not vanish..

- In the interacting fixed point they were forced to add an effective operator on the singular surface

  => In the IR fixed point (interacting) this flows to the conformal coupling, hence no area law!

- So in the UV they basically assumed minimally coupled scalar..

- We can do even better:

- The operator they were forced to introduce is a coupling to the entangling surface in the replica trick from loop calculations:

$$\delta I = \frac{c_r}{2} \int_\Sigma \phi^2$$

- Where they found that at the Wilson-Fisher fixed point the renormalized coupling is:

$$c_r^* = -\frac{2\pi}{3}(n-1)$$

- We can easily see that this is simply the conformal coupling(!):

$$\mathcal{R} = 4\pi(1-n)\delta_\Sigma + \mathcal{O}(1-n)^2$$

$$\delta I = \frac{\eta_c}{2} \int_{\mathcal{M}_n} \mathcal{R}\phi^2 = -\frac{\pi}{3}(n-1) \int_\Sigma \phi^2$$

- We can go back to their calculation and fix it from the outset:

$$I = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\eta}{2}R\phi^2 \right)$$

Now taking into account the delta function in the curvature

Results in:

$$\delta S = -\frac{N A_\Sigma m^2 \log \delta}{6(4\pi)} + \eta \frac{N m^2 A_\Sigma}{4\pi} \log \delta$$

- Which again vanishes for the conformal scalar..

- Our story (no replica trick) -
- Take the same theory and put it on a sphere (arbitrary ξ).
- Calculate $C \sim \langle T \rangle$

$$I = \int_{S^d} [\frac{1}{2}(\partial\vec{\phi})^2 + \frac{t_0}{2}\vec{\phi}^2 + \frac{1}{2}(\eta + \eta_0)R\vec{\phi}^2 + \frac{u_0}{4N}(\vec{\phi}^2)^2]$$

- We're doing large N, so we couple to some auxiliary fields and get:

$$Z = \int \mathcal{D}s\, e^{-N\left(\frac{1}{2}\mathrm{Tr}\ln\hat{O}_s - \int_{S^d}\frac{(s-\eta_0 R)^2}{4u_0}\right) - I_g}$$

- where:

$$\hat{O}_s \equiv -\Box + t_0 + \eta_c R + s$$

- Saddle point approximation in large N: $\bar{s} = u_0 \langle x|\hat{O}_{\bar{s}}^{-1}|x\rangle + \eta_0 R$

Summary

- Our main result was the simple, analytic, relation between the VEV of the stress-energy tensor and EE:

$$R\frac{dS_{\rm EE}}{dR} = -\frac{\Omega_d R^{d+1}}{d}\frac{d}{dR}\langle T\rangle$$

no replica trick!

- We checked explicitly that the results match known results in the literature for the free scalar and fermion

- Showed that our result indicate also flat space knows about curvature coupling!

- We showed that REE does not give a good c-function on a sphere. Is there some other c-function?