Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 14

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Abstract: <p>Course Description coming soon.</p>

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GR for Cosmology, Achim Kempf, Fall 2015, Lecture 14

On $T^{\mu\nu}$, continued:

Recall: ▫ We defined $T^{\mu\nu}$ as that tensor which obeys for all $\delta g_{\mu\nu}(\lambda,x)$:

$$\frac{dS}{d\lambda}\bigg|_{\lambda=0} = \frac{1}{2}\int_B T^{\mu\nu}\, \underbrace{\delta g_{\mu\nu}}_{= \frac{d g_{\mu\nu}(\lambda,x)}{d\lambda}\big|_{\lambda=0}} \sqrt{g}\; d^4x$$

(we choose $T^{\mu\nu}$ symmetric because $g_{\mu\nu}$ is symmetric)

▫ The above is meant when writing:

$$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}} S$$

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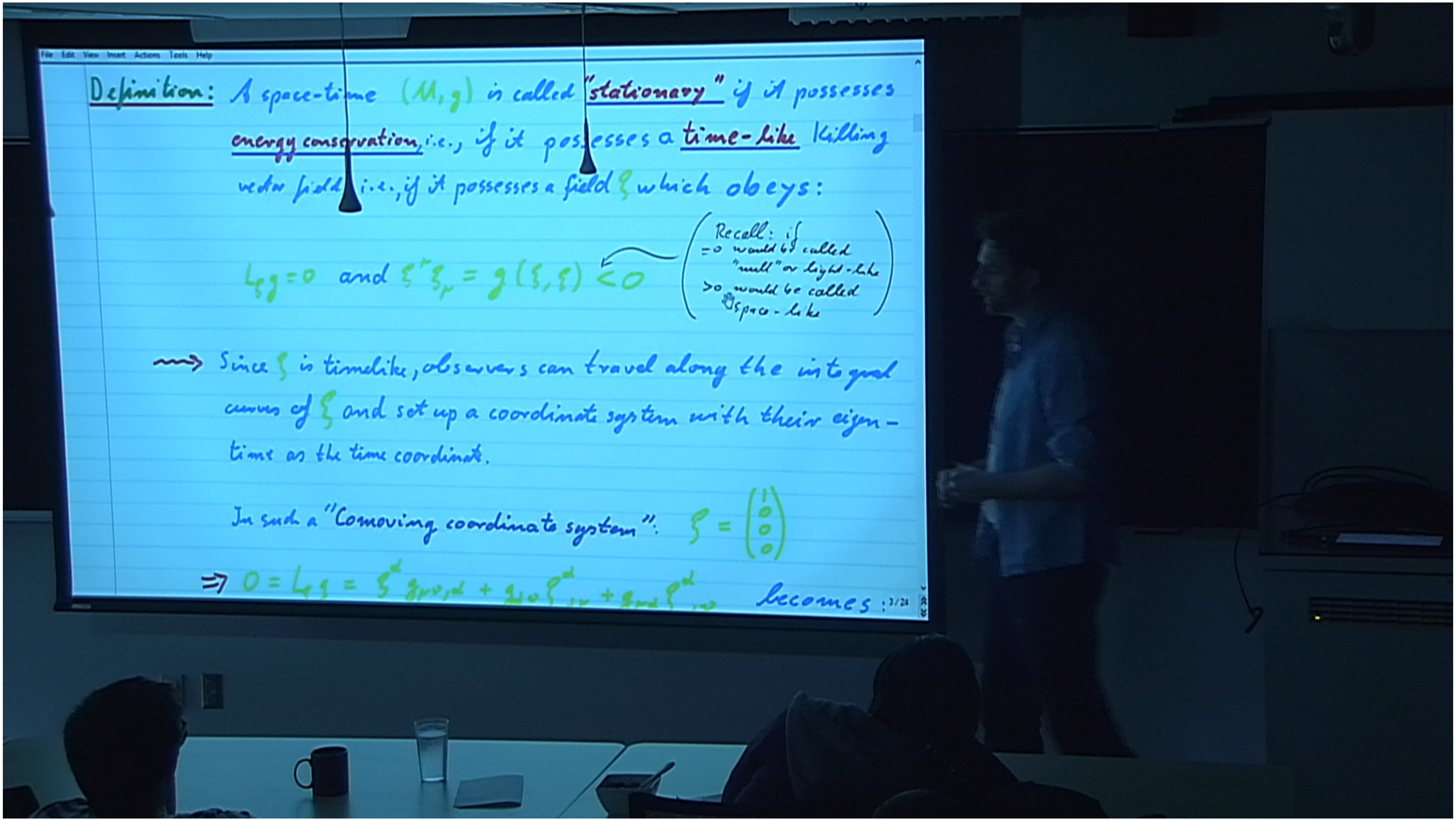
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Definition: A space-time (M,g) is called "stationary" if it possesses
energy conservation, i.e., if it possesses a time-like Killing
vector field, i.e., if it possesses a field ξ which obeys:
$L_\xi g = 0$ and $\xi^\mu \xi_\mu = g(\xi,\xi) < 0$
Recall: if
$=0$ would be called "null" or light-like
>0 would be called space-like
⟶ Since ξ is timelike, observers can travel along the integral
curves of ξ and set up a coordinate system with their eigen-
time as the time coordinate.
In such a "Comoving coordinate system": $\xi = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$
$\Rightarrow 0 = L_\xi g = \xi^\alpha g_{\mu\nu,\alpha} + g_{\alpha\nu}\xi^\alpha_{\ ,\mu} + g_{\mu\alpha}\xi^\alpha_{\ ,\nu}$ becomes:
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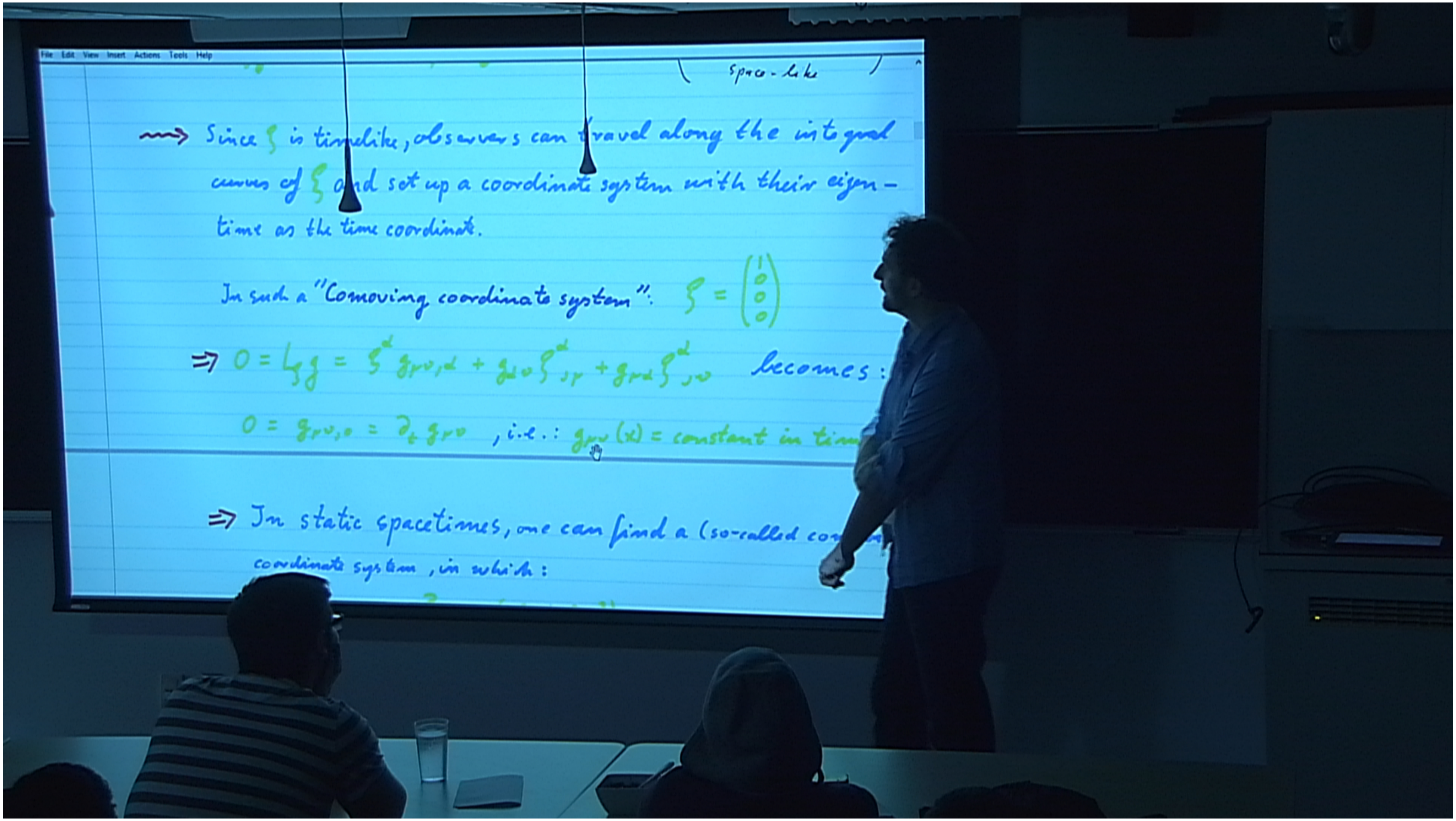

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$0 = g_{\mu\nu,0} = \partial_t g_{\mu\nu}$, i.e.: $g_{\mu\nu}(x)$ = constant in
$\Rightarrow$ In static spacetimes, one can find a (so-called
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$\Rightarrow$ In static spacetimes, one can find a (so-called comoving) coordinate system, in which:

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□ **However:** Stationarity does not imply that there is a cds in which

$$g = \begin{pmatrix} g_{00}, 0, 0, 0 \\ 0 ; & \\ 0 ; & \times\!\!\times \\ 0 ; & \end{pmatrix} \qquad (\times)$$

Example: The g of a stationary black hole that is rotating, given by the "Kerr metric".

⇝

Definition: A space-time is called "static", if the time-like Killing field ξ, viewed as a 1-form,

$$\xi = \xi_\mu \, dx^\mu$$

also obeys the "Frobenius condition":

$$\xi \wedge d\xi = 0 \qquad (F)$$

The "dominant energy condition":

$T_{\mu\nu} v^\mu v^\nu \geq 0$ for all timelike v (i.e., weak energy condition)

and

$K_\mu := T_{\mu\nu} v^\nu$ obeys $K_\mu K^\mu \leq 0$ (i.e., $T_{\mu\nu} V^\nu$ is non-space-like)

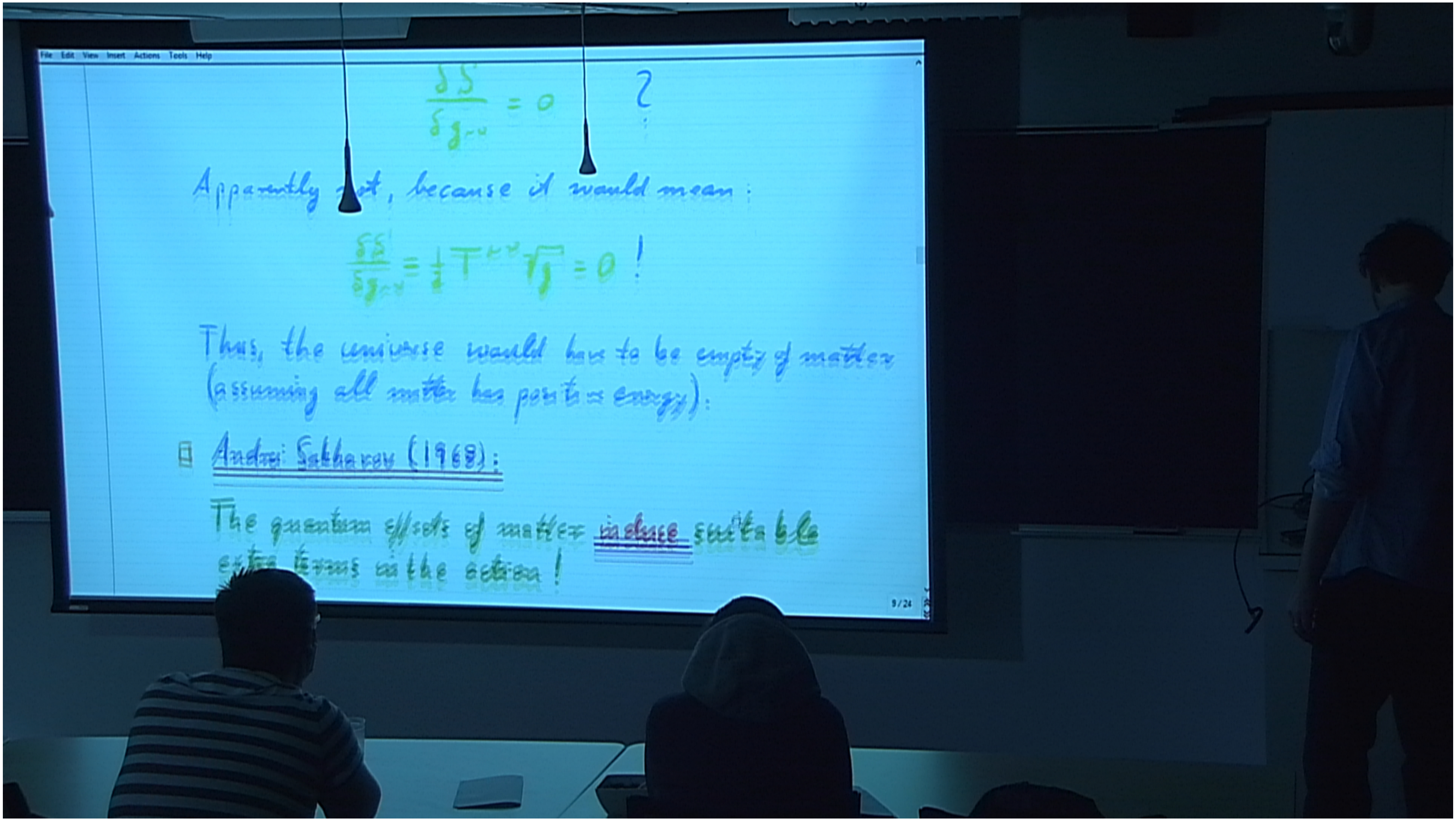
Why assume it?

- The local energy-momentum flow vector, K, may not be conserved but should be non-space-like: "All flow should be into the future."

- In an orthonormal basis, the dominant energy condition takes the form:

$$T^{00} \geq |T^{ab}|$$

(Note: This is all intuition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$\frac{\delta S}{\delta g_{\mu\nu}} = 0$?
Apparently not, because it would mean:
$\frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} T^{\mu\nu} \sqrt{g} = 0$!
Thus, the universe would have to be empty of matter
(assuming all matter has positive energy).
Andrei Sakharov (1968):
The quantum effects of matter induce suitable
extra terms in the action!
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Sakharov's reasoning: (modernized version)

- Classical deterministic evolution of matter obeys:

$$\frac{\delta S}{\delta \psi_{(i)}} = 0$$

- But quantum theory allows every evolution $\psi_{(i)}(x,t)$ to happen "virtually", with probability amplitudes:

a normalization constant.

$$\text{prob.ampl.}\left[\psi_{(i)}\right] = N\, e^{\frac{iS'[g,\psi]}{\hbar}}$$

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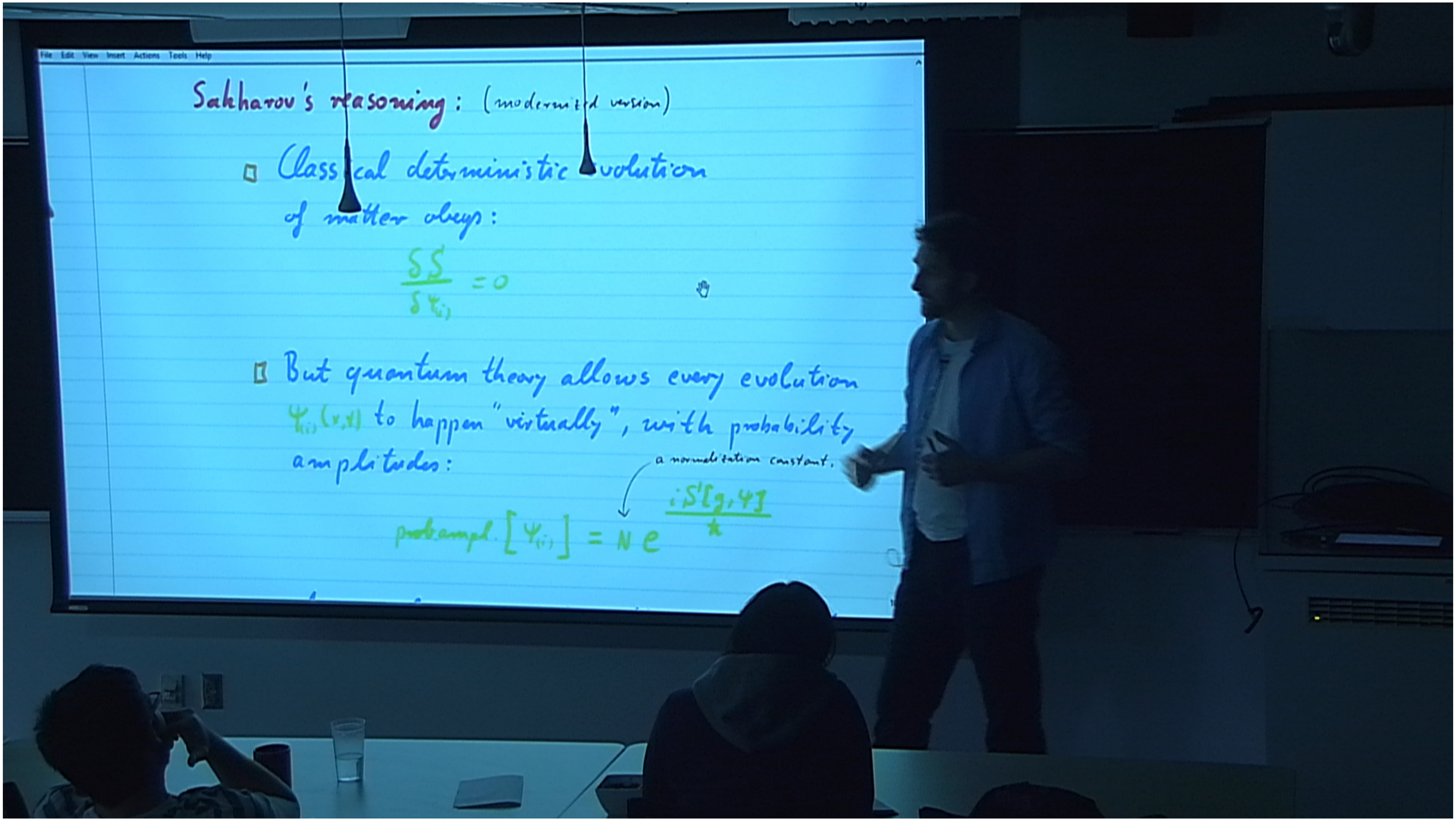
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matter fields when matter quantum effects are taken into account.

Problems:

- These calculations are very difficult.

  $\Rightarrow$ Use perturbative methods.

- There occur integrals that are divergent at short distances.

  $\Rightarrow$ Introduce a cutoff at some minimum length ℓ_c (or maximum momentum $\frac{\hbar}{\ell_c}$).

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- But as always in quantum theory, the effective action will contain terms of all possible forms that

The only question is)

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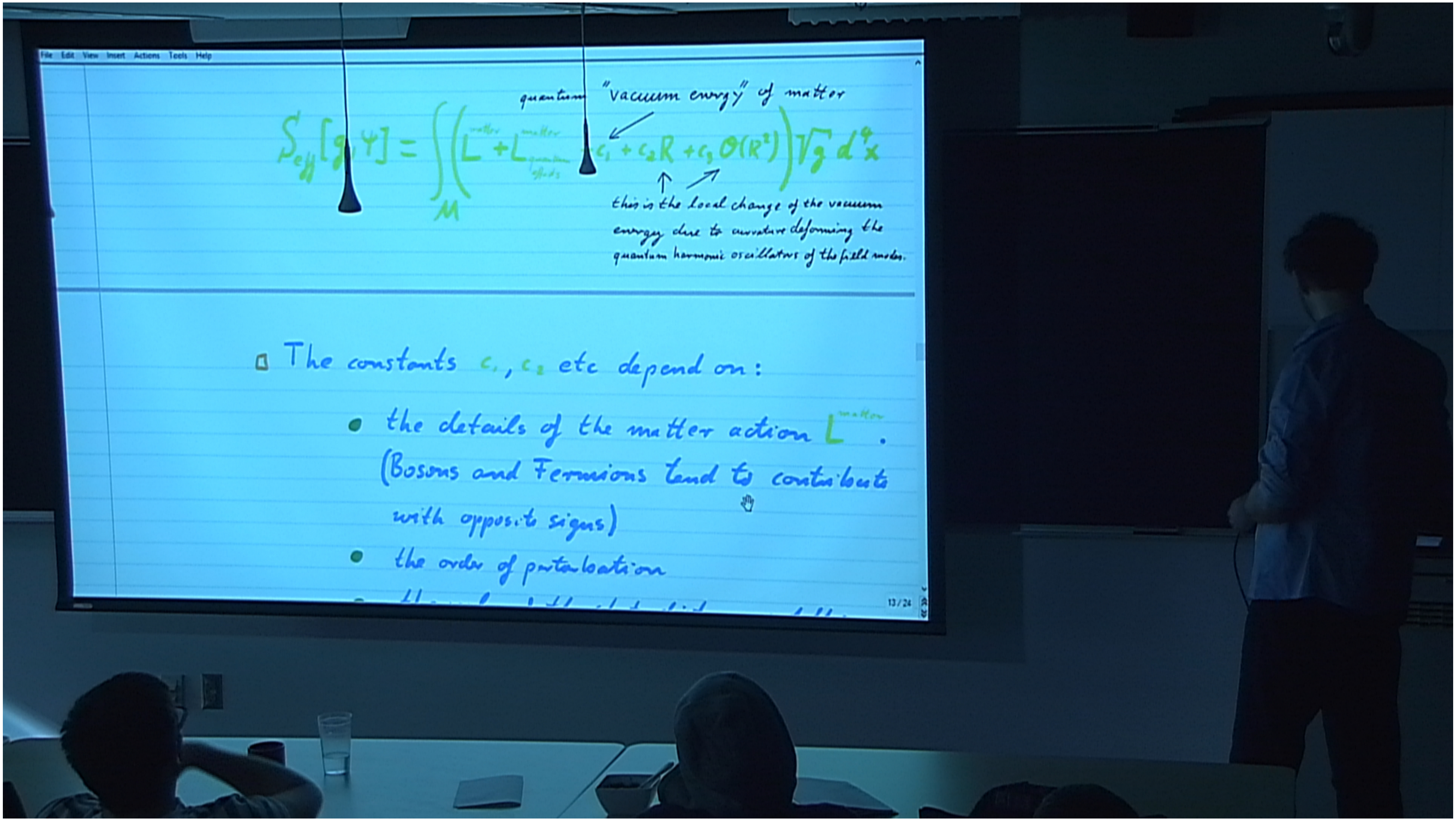
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quantum "vacuum energy" of matter
$S_{eff}[g, \psi] = \int_M \left(L^{matter} + L^{matter}_{quantum\ effects} + c_1 + c_2 R + c_3 O(R^2) \right) \sqrt{g}\, d^4x$
this is the local change of the vacuum energy due to curvature deforming the quantum harmonic oscillators of the field modes.
The constants c_1, c_2 etc depend on:
the details of the matter action L^{matter}.
(Bosons and Fermions tend to contribute with opposite signs)
the order of perturbation
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- The constants c_1, c_2 etc depend on:
  - the details of the matter action L^{matter}.
    (Bosons and Fermions tend to contribute with opposite signs)
  - the order of perturbation
  - the value of the short-distance cutoff:

The λ_i are unitless numbers that are roughly of order one, depending on the precise matter Lagrangian

$c_1 = \lambda_1 \ell_c^{-4}$ (must make up for $[\text{length}]^4$ from d^4x)

$c_2 = \lambda_2 \ell_c^{-2}$ (because R has units $[\text{length}]^{-2}$)

$c_3 = \lambda_3 \ell_c^{0}$ (terms R^2 or $R^{\mu\nu} R_{\mu\nu}$ etc have units $[\text{length}]^{-4}$))

$c_4 = \lambda_4 \ell_c^{2}$ (higher powers in R have prefactors $\sim \ell_c^{\text{positive power}}$)

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terms will have.)

are consistent with the symmetries of the theory i.e. here with general covariance (i.e. that are scalars):

quantum "vacuum energy" of matter

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$\Rightarrow$ For small ℓ_c, we have:

$$c_1 \gg c_2 \gg c_3 \gg c_4 \gg \dots$$

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The equations of motion for g:

The action principle, $\frac{\delta S_{eff}}{\delta g_{\mu\nu}} = 0$, yields:

$$\frac{\delta}{\delta g_{\mu\nu}} \int_B \left(c_1 + c_2 R_{\mu\nu} g^{\mu\nu}\right)\sqrt{g}\, d^4x = -\frac{1}{2}\sqrt{g}\, T^{\mu\nu}$$

in principle, it is the effective quantum expectation value

Evaluate the left hand side:

recall: $= \frac{1}{2} g^{\mu\nu}\sqrt{g}$

a.) $$\delta \int_B c_1 \sqrt{g}\, d^4x = \int_B c_1 \frac{\delta\sqrt{g}}{\delta g_{\mu\nu}} \delta g_{\mu\nu}\, d^4x$$

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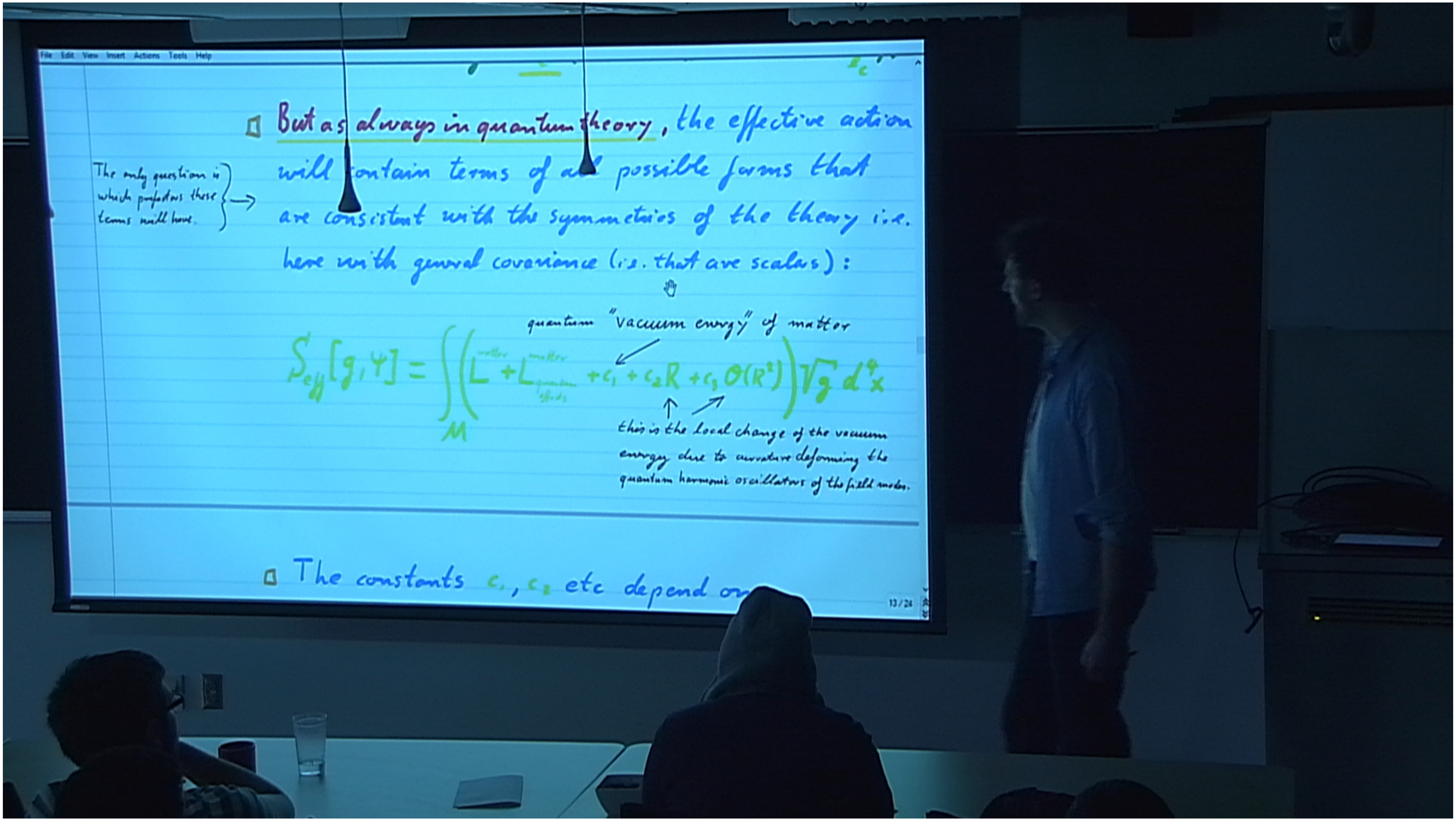
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But as always in quantum theory, the effective action will contain terms of all possible forms that are consistent with the symmetries of the theory i.e. here with general covariance (i.e. that are scalars):
The only question is which prefactors these terms will have.
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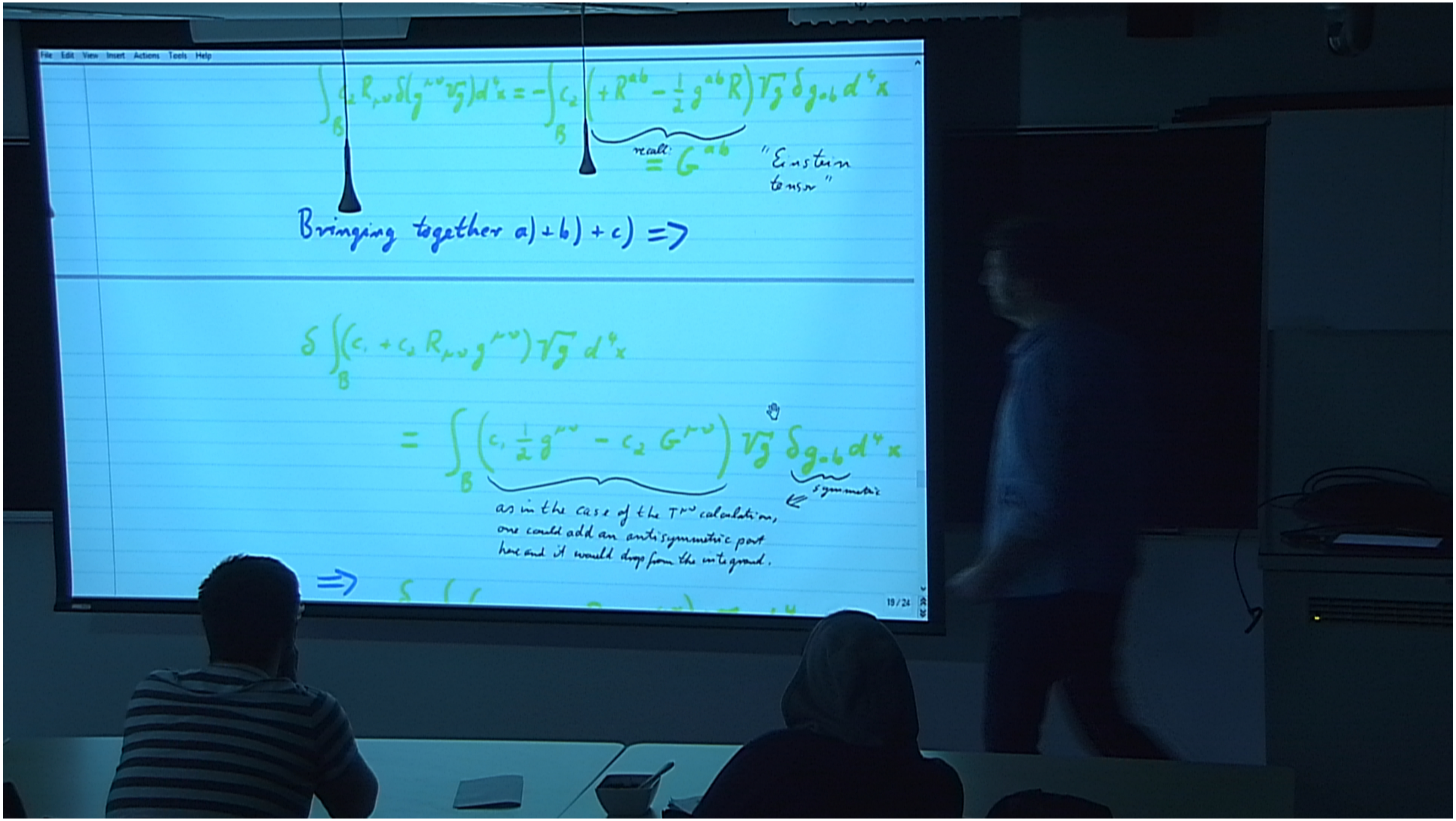
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recall:
"Einstein tensor"
Bringing together a) + b) + c) =>
symmetric
as in the case of the T calculation, one could add an antisymmetric part here and it would drop from the integrand.
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ΔL ≳ 100nm
ΔL