

Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 11

Date: Oct 23, 2015 01:30 PM

URL: <http://pirsa.org/15100022>

Abstract: <p>Course Description coming soon.</p>

This makes it hard to identify the true degrees of freedom, so that they can be quantized.

Observe: Such local descriptions carry redundant information!

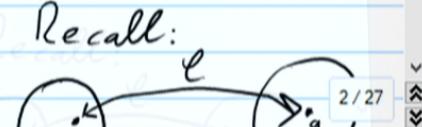
Why? Two (pseudo-)Riemannian mflds  $(M, g), (M, g')$  must be considered equivalent, i.e., they are describing the same space(-time), if there exists an isometric, i.e., metric-preserving, isomorphism:

$$\varphi: (M, g) \rightarrow (M, g')$$

Here:  $\varphi$  is called metric-preserving if, under the pull-back map

$$T\varphi^*: T_p(M)_2 \rightarrow T_{\varphi(p)}(M')_2$$

the metric obeys:



Why!

Two (pseudo-)Riemannian mflds  $(M, g), (M, g')$   
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 describing the same space(-time), if there exists  
 an isometric, i.e., metric-preserving, isomorphism:

$$\varrho: (M, g) \rightarrow (M, g')$$

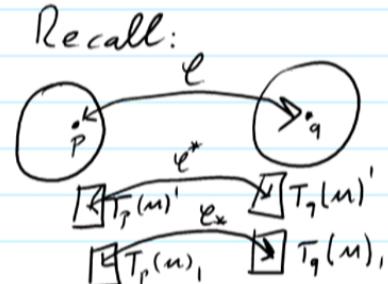
Here:  $\varrho$  is called metric-preserving if, under the pull-back map

$$T\varrho^*: T_p(M)_2 \rightarrow T_{\varrho(p)}(M)_2$$

the metric obeys:

$$T\varrho^*(g) = g'$$

$\Rightarrow \varrho$  can then be considered to be a mere change of chart.



Intuition:  $(M, g), (M', g')$  that are related by an isometric diffeomorphism are mere cd changes of another, i.e., have the same "shape".

Definition: A (pseudo-)Riemannian structure, say  $\Sigma$ , is an equivalence class of (pseudo-)Riemannian manifolds which can be mapped into each other via metric-preserving diffeomorphisms, i.e., via changes of coordinates.



Space(time) will need to be modelled as a (pseudo-)Riemannian structure,  $\Sigma$ , i.e., as an equivalence class of pairs  $(M, g)$ .

Problem: These equiv. classes are hard to handle

Definition: A (pseudo-)Riemannian structure, say  $\tilde{\Sigma}$ , is an equivalence class of (pseudo-)Riemannian manifolds which can be mapped into each other via metric-preserving diffeomorphisms, i.e., via changes of coordinates.



Space(time) will need to be modelled as a (pseudo-)Riemannian structure,  $\tilde{\Sigma}$ , i.e., as an equivalence class of pairs  $(M, g)$ .



Problem: These equiv. classes are hard to handle because absence or existence of  $\mathcal{C}$  is hard to check!

⇒ One would like to be able to reliably identify exactly one representative  $(M, g)$  per class  $\Sigma$ .

- This would be called a "fixing of gauge".
- Why would this be useful?

A key example of when gauge fixing needed: **Quantum gravity**

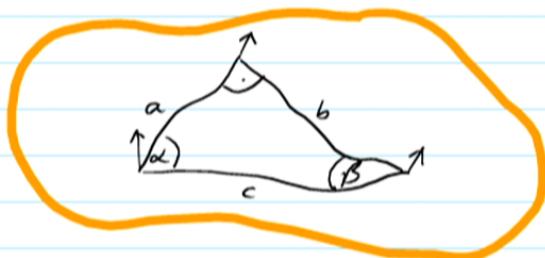
We discussed detecting and describing shape through



- deficiency angles
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A key example of when gauge fixing needed: *Quantum gravity*

We discussed detecting and describing shape through



- deficiency angles
- nontrivial metric distances  $(M, g)$
- nontrivial parallel transport  $(M, \Gamma)$

Recall: Quantum theory can be formulated in path integral form.

Applied to gravity:

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Expect to have to handle path integrals of the type:

$$\int e^{iS[\Sigma]} D\Sigma$$

"all Riemannian  
structures  $\Sigma$ "

But what we initially have is, roughly of the form:

$$\int e^{iS(g)} \delta(\text{?}) Dg \text{ or } \int e^{iS(\Gamma)} \delta(\text{?}) D\Gamma$$

"all  $g$ "                                  "all  $\Gamma$ "

Here,  $\delta(\text{?})$  should be such that from each equivalence class of the  $g$ 's or the  $\Gamma$ 's only exactly one contributes to the path integral.

Much of Quantum Gravity research is concerned with working out suitable  $S(?)$  for  $g$ 's or  $\Gamma$ 's or other variables formed from them, such as the frame fields (see "Loop quantum gravity").

**Q:** Can one detect and describe a (pseudo-) Riemannian structure  $\Sigma$  directly?

**A:** Possibly yes, using "Spectral Geometry":

Independent of coordinate systems!

**Idea:** A manifold's vibration spectrum  $\{\lambda_n\}$  depends only on  $\Sigma$ !

Key question of the field of spectral geometry: (Weyl 1911)

Idea: A manifold's vibration spectrum  $\{\lambda_n\}$  depends only on  $\Sigma$ !

Key questions of the field of spectral geometry: (Weyl 1911)

Does the spectrum  $\{\lambda_n\}$  encode all about the shape, i.e.,  $\Sigma$ ?

Remarks:

- It cannot, if  $M$  has infinite volume, because then the spectrum of  $\Delta$  will become (almost) completely continuous.

## Remarks:

- It cannot, if  $M$  has infinite volume, because then the spectrum of  $\Delta$  will become (almost) completely continuous.
- The spectral geometry of pseudo-Riemannian manifolds is still very little developed.

## Theorem:

- Assume  $(M, g)$  is a compact Riemannian manifold without boundary,  $\partial M = \emptyset$ .  
implies finite volume
- Then, each  $\text{spec}(\Delta_0)$  is discrete with finite degeneracy.

Theorem:

□ Assume  $(M, g)$  is a compact Riemannian manifold without boundary,  $\partial M = \emptyset$ . implies finite volume

□ Then, each  $\text{spec}(\Delta_p)$  is discrete, with finite degeneracies and without accumulation points.

In practice: We can describe any arbitrarily large part of the universe by a compact Riemannian manifold,  $(M, g)$  without boundary,  $\partial M = \emptyset$ .

This allows us to describe, e.g. 3-dim. space at any fixed

## Types of waves (incl. sounds) on $M$ :

assumed compact, no boundary

(consider  $p$ -form fields  $w(x)$  on  $M$ , with time evolution, e.g.:

1. Schrödinger equation:  $i\hbar \partial_t w(x,t) = -\frac{\hbar^2}{2m} \Delta_p w(x,t)$

2. Heat equation:  $\partial_t w(x,t) = -d \Delta_p w(x,t)$

3. Klein Gordon (and acoustic) eqn:  $-\partial_t^2 w(x,t) = \beta \Delta_p w(x,t)$

- Each of them can be solved via separation of variables:
- Assume we find an eigenform  $\tilde{w}(x)$  of  $\Delta$  on  $M$ :

$$\Delta_p \tilde{w}(x) = \lambda \tilde{w}(x)$$

□ They exist: Each  $\Delta$  is self-adjoint w.r.t. the inner product  $(w, v) = \int_M w \star v$ .

Then: Schrödinger eqn solved by:  $w(x, t) := e^{\frac{i\hbar}{2m} \lambda t} \tilde{w}(x)$

Heat eqn solved by:  $w(x, t) := e^{-\frac{d}{2} \lambda t} \tilde{w}(x)$

Klein Gordon eqn solved by:  $w(x, t) := e^{\pm i \sqrt{B} \lambda t} \tilde{w}(x)$

$\Rightarrow$  The spectrum  $\text{spec}(\Delta_p)$  is the overtone spectrum of p-form type waves on the manifold  $M$ .

## Properties of $\text{spec}(\Delta_p)$ :

### □ Expectations:

The spectra  $\text{spec}(\Delta_p)$  for different  $p$  carry different information about  $M$ :

E.g., scalar and vector seismic waves travel (and reflect) differently.

### □ But recall also: a) $[\Delta, *] = 0$

$$\text{b) } [\Delta, d] = 0$$

$$\text{c) } [\Delta, \delta] = 0$$

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□ But recall also:

$$a) [\Delta, *] = 0$$

$$b) [\Delta, d] = 0$$

$$c) [\Delta, \delta] = 0$$

This will relate  $\text{spec}(\Delta_p)$  to  $\text{spec}(\Delta_{n-p})$ ,  $\text{spec}(\Delta_{p+1})$  and  $\text{spec}(\Delta_{p-1})$ :

Use  $[\Delta, *] = 0$ :



Use  $[\Delta, *] = 0$ :

Assume:  $w \in \Lambda_p$  and  $\Delta w = \lambda w$ .

Define:  $v := *w \in \Lambda_{n-p}$

Then:



$$\Delta v = \Delta *w = * \Delta w = * \lambda w = \lambda v$$

$$\Rightarrow \text{spec}(\Delta_p) = \text{spec}(\Delta_{n-p})$$

Next:

Careful utilization of  $\int \Delta d\tau = 0$  and  $\int \Delta \delta \tau = 0$  yields

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Careful utilization of  $[\Delta, d] = 0$  and  $[\Delta, \delta] = 0$  yields much more information about these spectra!

□ Notice that:  $\Delta$  maps exact forms  $\omega = d\nu$  into exact forms:

$$\Delta\omega = \Delta d\nu = \underbrace{d}_{\text{an exact form}} \Delta\nu$$

i.e.:

$$\boxed{\Delta: d\Lambda_r \rightarrow d\Lambda_r}$$

$d\Lambda_r$  = image of  $\Lambda_r$  under  $d$ .

□ Analogously:  $\Delta$  maps co-exact forms  $\omega = \delta\beta$  into co-exact forms:

$$\Delta\omega = \Delta\delta\beta = \underbrace{\delta}_{\text{a co-exact form}} \Delta\beta$$

i.e.:

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$$\Delta\omega = \Delta\delta\beta = \delta\overbrace{\Delta\beta}^{\text{a co-exact form}}$$

i.e.:

$$\Delta : \delta\Lambda_r \rightarrow \delta\Lambda_r$$

□ Also:  $\Delta$  can map forms into 0, namely its eigenspace with eigenvalue 0, denoted  $\Lambda_r^0$ .  
 $\Lambda_r^0$  is called the space of "harmonic" p-forms.

$$\Delta : \Lambda_r^0 \rightarrow 0$$

Thus:

$\Delta$  maps  $d\Lambda_r$  and  $\delta\Lambda_r$  and  $\Lambda_r^\circ$  into themselves.

Are there any other forms that  $\Delta$  could act on? No!

Proposition ("Hodge decomposition"):

$$\Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^\circ$$



(Recall that  $\oplus$  implies that the three spaces are orthogonal!)

**Q:** Why useful?

**A:** It means that every eigenvector of  $\Delta_p$  is either

## Proposition ("Hodge decomposition"):

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**Q:** Why useful?



**A:** It means that every eigenvector of  $\Delta_p$  is either in  $d\Lambda_{p-1}$ , or in  $\delta\Lambda_{p+1}$ , or in  $\Lambda_p^\circ$  but is never a linear combination of vectors in those spaces.

Proof: It is clear that  $d\Lambda_{p-1} \subset \Lambda_p$  and  $\delta\Lambda_{p+1} \subset \Lambda_p$ .

We need to show the orthogonalities and completeness:

□ Show that  $d\Lambda_{p-1} \perp \delta\Lambda_{p+1}$ :

Indeed, assume  $w = d\nu \in \Lambda_p$  and  $\alpha = \delta\beta \in \Lambda_p$ .

$$\text{Then: } (w, \alpha) = (d\nu, \delta\beta) \stackrel{\text{use } \overset{\circ}{\underset{-d+\delta}{\sim}}}{=} (\overset{\circ}{dd}\nu, \beta) = 0 \quad \checkmark$$

Exercise:  
study the  
remainder  
of the proof.

□ Show that if  $w \in \Lambda_p$  and  $w \perp d\Lambda_{p-1}$  and  $w \perp \delta\Lambda_{p+1}$ , then:  $w \in \Lambda_p^0$ .

Indeed, assume  $w \perp d\Lambda_{p-1}$  and  $w \perp \delta\Lambda_{p+1}$ . Then:

$$\forall \alpha: (dd, w) = 0 \quad i.e. -(\alpha, \delta w) = 0 \Rightarrow \delta w = 0 \quad \begin{matrix} i.e. w \text{ is "co-exact"} \\ \end{matrix}$$

$$H.R.: (\delta \alpha) = 0 \quad i.e. (d \delta \alpha) = 0 \Rightarrow d \alpha = 0$$

□ Show that  $d\Lambda_{p-1} \perp \delta\Lambda_{p+1}$ :

Indeed, assume  $w = dw \in \Lambda_p$  and  $\alpha = \delta\beta \in \Lambda_p$ .

$$\text{Then: } (w, \alpha) = (dw, \delta\beta) \stackrel{\substack{\text{use } d^2=0 \\ -d^*=\delta}}{=} (ddw, \beta) = 0 \quad \checkmark$$

Exercise:

study the remainder  
of the proof.

□ Show that if  $w \in \Lambda_p$  and  $w \perp d\Lambda_p$ , and  $w \perp \delta\Lambda_{p+1}$ , then:  $w \in \Lambda_p^\circ$ .

Indeed, assume  $w \perp d\Lambda_p$ , and  $w \perp \delta\Lambda_{p+1}$ . Then:

$$\forall \alpha: (dd\alpha, w) = 0 \quad \text{i.e. } -(\alpha, \delta w) = 0 \Rightarrow \delta w = 0 \quad \text{i.e. } w \text{ is "co-exact"}$$

$$\forall \beta: (\delta\beta, w) = 0 \quad \text{i.e. } -(\beta, dw) = 0 \Rightarrow dw = 0 \quad \text{i.e. } w \text{ is exact}$$

$$\Rightarrow \Delta w = (d\delta + \delta d) w = 0 \Rightarrow w \in \Lambda_p^\circ \quad \checkmark$$

Conclusion so far:

In the Hodge decomposition,

$\Delta$  maps every term into itself, i.e.,  $\Delta$  can be diagonalized in each  $d\Lambda_r$ ,  $\delta\Lambda_r$ ,  $\Lambda^{\circ}_r$  separately.

$$\left\{ \begin{array}{l} \Lambda_{p-1} = d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \Lambda^{\circ}_{p-1} \\ \Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda^{\circ}_p \\ \Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda^{\circ}_{p+1} \\ \vdots \end{array} \right.$$

$\Rightarrow \Delta$  has eigenvectors and -values on each of these subspaces, for all  $r$ :

$$\text{spec}(\Delta|_{d\Lambda_r}), \text{ spec}(\Delta|_{\delta\Lambda_r}), \text{ spec}(\Delta|_{\Lambda^{\circ}_r}) = \{0\} \dots$$

$$\Rightarrow (\delta\omega, \delta\omega) + (d\omega, d\omega) = 0 \Rightarrow \delta\omega = 0 \text{ and } d\omega = 0.$$

I.e., harmonic forms are closed and co-closed but not exact or co-exact.

Thus,  $B_p := \dim(\Lambda_p^0)$  measures topological nontriviality.

The  $B_p$  are called the "Betti numbers".

$$\Rightarrow \forall d \in \Lambda_{p-1}: (d, \delta\omega) = 0, \text{ i.e., } (dd, \omega) = 0.$$

$$\Rightarrow \omega \perp d\Lambda_{p-1} \quad \checkmark$$

$$\text{Also: } \forall \beta \in \Lambda_{p+1}: (\beta, dw) = 0, \text{ i.e., } (\delta\beta, \omega) = 0.$$

$$\Rightarrow \omega \perp \delta\Lambda_{p+1} \quad \checkmark$$

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$$\left\{ \begin{array}{l} \vdots \\ \Lambda_{p-1} = d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \Lambda_{p-1}^{\circ} \\ \Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^{\circ} \\ \Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda_{p+1}^{\circ} \\ \vdots \end{array} \right.$$

$\Rightarrow \Delta$  has eigenvectors and -values on each of these subspaces, for all +:

$$\text{spec}(\Delta|_{d\Lambda_r}), \text{ spec}(\Delta|_{\delta\Lambda_r}), \text{ spec}(\Delta|_{\Lambda_r^{\circ}}) = \{0\} \dots$$

⇒  $\Delta$  has eigenvectors and -values on each of these subspaces, for all  $i$ :

$$\text{spec}(\Delta|_{d\Lambda_i}), \text{ spec}(\Delta|_{S\Lambda_i}), \text{ spec}(\Delta|_{\lambda_i}) = \{0\} \dots$$

These spectra are related!



Proposition:

$$\text{spec}(\Delta|_{d\Lambda_i}) = \text{spec}(\Delta|_{S\Lambda_{i+1}})$$

and for each eigenvector in one there is one in the other.

Proposition:

$$\text{spec}(\Delta|_{d\Lambda_r}) = \text{spec}(\Delta|_{\delta\Lambda_{r+1}})$$

and for each eigenvector in one there is one in the other.

This means:

⋮

$$\Lambda_{p-1} = \overbrace{d\Lambda_{p-2}}^{\text{same spectrum}} \oplus \underbrace{\delta\Lambda_p}_{\text{same spectrum}} \oplus \Lambda^{\circ}_{p-1}$$

$$\Lambda_p = \overbrace{d\Lambda_{p-1}}^{\text{same spectrum}} \oplus \underbrace{\delta\Lambda_{p+1}}_{\text{same spectrum}} \oplus \Lambda^{\circ}_p$$

$$\Lambda_{p+1} = \overbrace{d\Lambda_p}^{\text{same spectrum}} \oplus \underbrace{\delta\Lambda_{p+2}}_{\text{same spectrum}} \oplus \Lambda^{\circ}_{p+1}$$

⋮

Proof:

Assume:  $\lambda \in \text{spec}(\Delta|_{d\Lambda_r})$  with eigenvector  $w \in d\Lambda_r$ .

Define:  $v := \delta w \in \delta\Lambda_{r+1}$

Then:  $\Delta v = \Delta \delta w = \delta \Delta w = \lambda \delta w = \lambda v$

$\Rightarrow \lambda \in \text{spec}(\Delta|_{\delta\Lambda_{r+1}})$  and  $v$  is the eigenvector.

Conversely:

Assume:  $\lambda \in \text{spec}(\Delta|_{\delta\Lambda_{r+1}})$  with eigenvector  $w \in \delta\Lambda_{r+1}$ .

Define:  $v := dw \in d\Lambda_r$

Then:  $\Delta v = \Delta dw = d\Delta w = \lambda dw = \lambda v$

-      1      1      -

Re-use  $[\Delta, \ast] = 0$ :

$$\Lambda_{r+1} \quad \Lambda_{n-r-1}$$

□ Proposition:  $\ast : d\Lambda_r \rightarrow \delta\Lambda_{n-r}$

i.e.:  $\ast : \underline{\text{exact } r+1 \text{ forms}} \rightarrow \underline{\text{co-exact } n-r-1 \text{ forms}}$



Proof: Assume  $w = d\varphi \in d\Lambda_r$

Define  $v := \ast w$

$$\begin{aligned} \Rightarrow v &= \ast d\varphi = (-1)^{r(n-r)} \overset{\delta}{\underset{\parallel}{\ast}} d \ast \ast \varphi \\ &= \delta \omega \in \delta\Lambda_{n-r} \text{ for } \omega = (-1)^{r(n-r)} \ast \varphi \end{aligned}$$

□ Proposition:  $\ast : \delta\Lambda_r \rightarrow d\Lambda_{n-r}$

Recall:  $*$  preserves the spectrum of  $\Delta$  as we showed already.



Summary:

$$\Lambda_{p-1} = \underbrace{d\Lambda_{p-2}}_{\text{same spectrum}} \oplus \underbrace{\delta\Lambda_p}_{\text{same spectrum}} \oplus \Lambda_{p-1}^o$$

$$\Lambda_p = \underbrace{d\Lambda_{p-1}}_{\text{same spectrum}} \oplus \underbrace{\delta\Lambda_{p+1}}_{\text{same spectrum}} \oplus \Lambda_p^o$$

$$\Lambda_{p+1} = \underbrace{d\Lambda_p}_{\text{same spectrum}} \oplus \underbrace{\delta\Lambda_{p+2}}_{\text{same spectrum}} \oplus \Lambda_{p+1}^o$$

:

Now we also found:

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$$\Lambda_p = \underbrace{d\Lambda_{p-1}}_{\substack{\text{same} \\ \vdots}} \oplus \underbrace{\delta\Lambda_{p+1}}_{\substack{\text{same spectrum} \\ \nearrow \searrow}} \oplus \Lambda_o^p$$

$$\Lambda_{n-p} = \underbrace{d\Lambda_{n-p-1}}_{\substack{\text{same} \\ \vdots}} \oplus \underbrace{\delta\Lambda_{n-p+1}}_{\substack{\text{spectrum} \\ \swarrow \nearrow}} \oplus \Lambda_o^{n-p}$$

Example:  $\dim(\mathcal{U})=3$

Exercise: do same for  $\dim(\mathcal{U})=4$

$$\Lambda_o = \delta\Lambda_1 \oplus \Lambda_o^1$$

Example:  $\dim(M) = 3$

Exercise: do same for  $\dim(M) = 4$

$$\Lambda_0 =$$

$$\delta\Lambda_1 \oplus \Lambda^{\circ}_0$$

$$\Lambda_1 = d\Lambda_0 \oplus \delta\Lambda_2 \oplus \Lambda^{\circ}_1$$

$$\Lambda_2 = d\Lambda_1 \oplus \delta\Lambda_3 \oplus \Lambda^{\circ}_2$$

$$\Lambda_3 = d\Lambda_2 \oplus \Lambda^{\circ}_3$$

Same color means same spectrum of  $\Delta$ .

Conclusion: There is relatively little independent information in the spectra of p-form waves on  $M$ !

E.g. when  $\dim(M) = 2$  then the spectrum of connection

Literature:

(neglecting literature on detecting boundary shapes from spectra)

Indeed: The spectra of  $\Delta$  do not contain sufficient information in general to uniquely identify the Riemannian structure from the spectra alone.Examples: Cases have been found of pairs  $(M, g)$ ,  $(\tilde{M}, \tilde{g})$  that are isospectral for  $\Delta$  on all  $\Lambda_p$  but that are not diffeomorphically isometric!Nevertheless: All examples are of limited significance:

- manifolds that are locally, if not globally isometric, or
- manifolds that are isospectral only w.r.t. respect to some  $\Delta$

Nevertheless: All examples are of limited significance:

- manifolds that are locally, if not globally isometric, or
- manifolds that are isospectral only w. respect to some  $\Delta$  or
- manifolds that are discrete pairs (e.g. mirror images).

Fresh approach to spectral geometry (AK)

Strategy: Iterate infinitesimal inverse spectral geometry

Assume both, the mfld and its spectra are given:

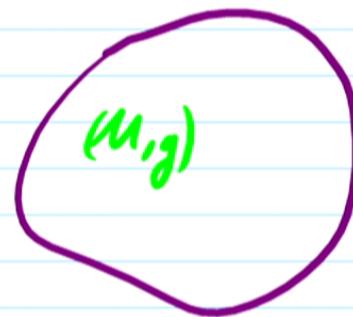


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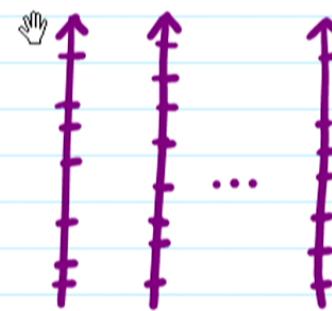
## Fresh approach to spectral geometry (AK)

Strategy: Iterate infinitesimal inverse spectral geometry

Assume both, the mfd and its spectra are given:



A compact Riemannian manifold  $(M, g)$  without boundary



The spectra  $\{\lambda_n^{(i)}\}$  of Laplacians  $\Delta^{(i)}$  on the manifold.

## Perturbation:

Now change the shape of  $(M, g)$  slightly, through:

$$g \rightarrow g + h$$

This will slightly change the spectra to

$$\{\lambda_n^{(i)}\} \rightarrow \{\lambda_n^{(i)} + \mu_n^{(i)}\}$$

## Why is this linearization useful?

- One can define a self-adjoint Laplacian  $\Delta^{(m)}$  on  $T_2(M)$ , with Hilbert basis  $\{b_m(x)\}$  and eigenvalues  $\{\lambda_m^{(m)}\}$

## Why is this linearization useful?

□ One can define a self-adjoint Laplacian  $\Delta^{(m)}$  on  $T_2(M)$ , with Hilbert basis  $\{b_m(x)\}$  and eigenvalues  $\{\lambda_m^{(m)}\}$ :

$$\Delta^{(m)} b_m(x) = \lambda_m b_m(x)$$

⇒ The metric's perturbation  $h \in T_2(M)$  can be expanded:

$$h = \sum_{n=1}^{\infty} h_n b_n(x)$$

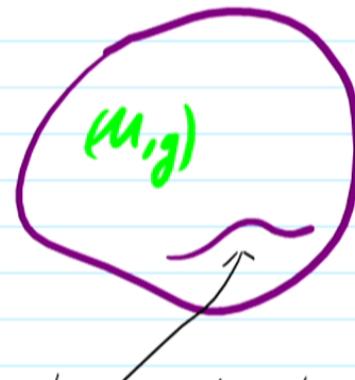
The perturbation of  $\text{spec}(\Delta^{(m)})$  is:

...  $\rightarrow$  ...  $\leftarrow$  ...  $\rightarrow$  ...  $\leftarrow$  ...

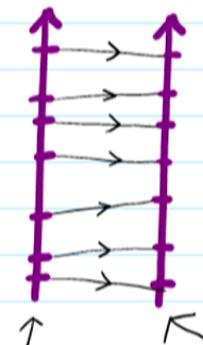
$n=1$ 

The perturbation of  $\text{spec}(\Delta^{(m)})$  is:

$$\{\lambda_n^{(m)}\} \rightarrow \{\lambda_n^{(m)} + \mu_n^{(m)}\}$$



New bump, described by  
the coefficients  $\{h_n\}_{n=1}^{\infty}$ , of  
 $g \rightarrow g + h$



Spectrum  
 $\{\lambda_n^{(m)}\}_{n=1}^{\infty}$

New spectrum  
 $\{\lambda_n^{(m)} + \mu_n^{(m)}\}_{n=1}^{\infty}$

$\Rightarrow$  We obtain a linear map  $S$ :

$$S: \{h_n\} \rightarrow \{\mu_n\}$$

$$S: h_n \rightarrow \mu_n = S_{nn} h_n$$

Notice:

Consider only eigenvectors and eigenvalues up to a cutoff scale.

Then, there are as many parameters  $\{h_n\}_{n=1}^N$  as  $\{\mu_n\}_{n=1}^N$ .

$\Rightarrow S$  is a square matrix.

If  $\det(S) \neq 0$ , then  $S^{-1}$  exists.

$\rightsquigarrow$  should be able to iterate the perturbation?

Remarks:  $\square$  Not all  $h$  actually change the shape:

If  $h = L_g g$  for some vector field  $\xi$ , then  $g \rightarrow g + h$  is merely the infinitesimal change of chart belonging to the flow  $\xi$ .

$\square$  Symmetric covariant 2-tensors such as  $h$  have a canonical decomposition similar to the Hodge decomposition. Thus,  $\Delta$  has three spectra on  $T_2(M)$ .

Reference: See also e.g. the video of my recent talk at PI: <http://pirsa.org/15090062>

Infinitesimal spectral geometry arose from my paper on

If  $h = L_g \xi$  for some vector field  $\xi$ , then  
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- ◻ Symmetric covariant 2-tensors such as  $h$  have a canonical decomposition similar to the Hodge decomposition. Thus,  $\Delta$  has three spectra on  $T_2(M)$ .

Reference: See also e.g. the video of my recent talk at PI: <http://pirsa.org/15090062>

Infinitesimal spectral geometry arose from my paper on how Spacetime could be simultaneously continuous and discrete, in the same way that information can.