

Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 8

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Abstract: <p>Course Description coming soon.</p>

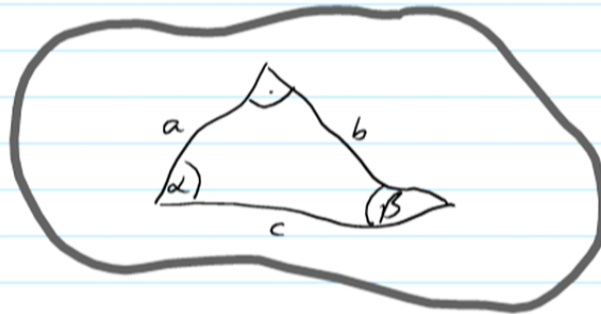
GR for Cosmology, Achim Kempf, Fall 15, Lecture 8

Note Title

How to describe the "shape" of a manifold?

Historically:

E.g., on a potato-shaped surface:



$$a^2 + b^2 \neq c^2$$

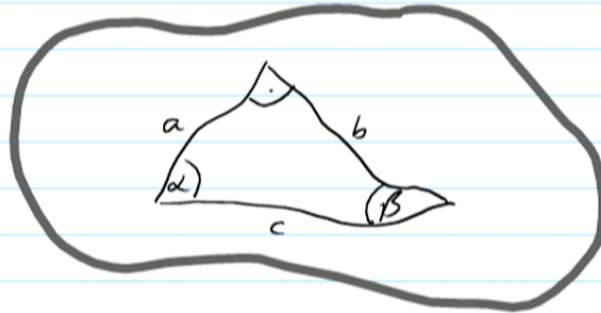
$$\alpha + \beta + 90^\circ \neq 180^\circ$$

Helmholtz & Gauss already considered checking for curvature of space this way.

Recall:

Defined α (α)

Historically:



E.g., on a potato-shaped surface:

$$a^2 + b^2 \neq c^2$$

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Helmholtz & Gauss already considered checking for curvature of space this way.

Recall:

Defined $g_{\mu\nu}(x)$

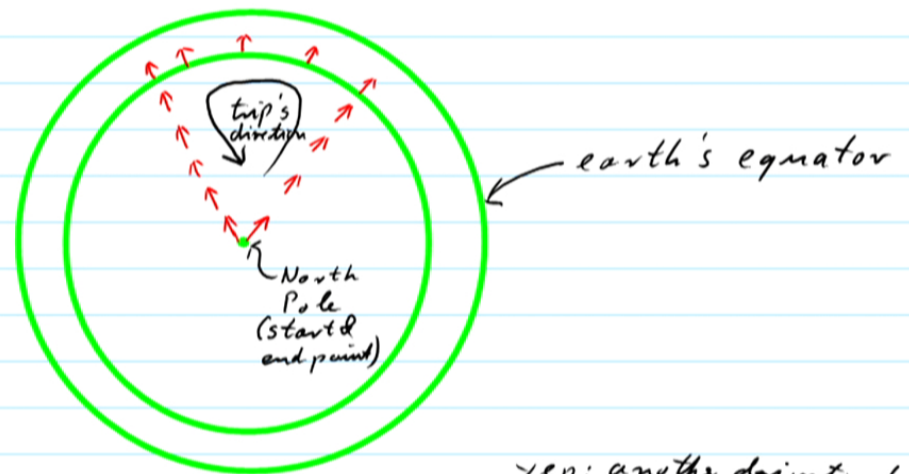
\Rightarrow infinitesimal distances \Rightarrow finite distances \Rightarrow shape

Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

Example:

- Start with a vector at North Pole.
- parallel transport down to some lower latitude, along that latitude and then back to pole.
- vector will arrive at pole rotated, in spite of having only been parallel transported!



This motivates:

Try to define local shape through "derivative" of vectors with respect to parallel transport!

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↙ yep: another derivative!

Recall: Lie derivative insensitive to g . Indeed, for $\xi_1 = \frac{\partial}{\partial x^1}, \xi_2 = \frac{\partial}{\partial x^2}$, we have $[L_{\xi_1}, L_{\xi_2}] = L_{[\xi_1, \xi_2]} = 0 \Rightarrow$ No shape info from L_{ξ} !

The Covariant Differentiation, ∇ :

Definition: Any linear map of tangent vector fields

$$\nabla : T'(M) \times T'(M) \rightarrow T'(M)$$

$$\nabla : \gamma, \xi \rightarrow \nabla_{\xi} \gamma$$

obeying (I) $\nabla_{\alpha} \eta = [\nabla_{\alpha} \eta, \forall] \subset \mathcal{F}(M)$

The Covariant Differentiation, ∇ :

Definition: Any linear map of tangent vector fields

$$\nabla : T'(M) \times T'(M) \rightarrow T'(M)$$

$$\nabla : \quad \text{hand} \quad \eta, \xi \rightarrow \nabla_{\xi} \eta$$

obeying

$$(I) \quad \nabla_{f\xi} \eta = f \nabla_{\xi} \eta, \quad \forall f \in \mathcal{F}(M)$$

$$(II) \quad \nabla_{\xi}(f\eta) = \underbrace{\xi(f)}_{\nabla_{\xi} f} \eta + f \nabla_{\xi} \eta \quad (\text{Leibniz rule})$$

is called a covariant derivative or affine connection.

Note:

For now, let us assume a metric has not (yet) been

$$(I) \quad \nabla_{f\xi} \eta = f \nabla_{\xi} \eta, \quad \forall f \in C^{\infty}(M)$$

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is called a covariant derivative or affine connection.

Note:

For now, let us assume a metric has not (yet) been specified, so we are free to choose ∇ , and this choice defines the shape of M !

∇ in a chart: \square Choose as bases for $T_x(M)$, e.g.: $\left\{ \frac{\partial}{\partial x^i} \right\}$

\square Given a covariant derivative ∇ , consider its

∇ in a chart: \square Choose as bases for $T_x(M)$, e.g.: $\left\{ \frac{\partial}{\partial x^i} \right\}$

\square Given a covariant derivative ∇ , consider its action on basis vectors, such as, e.g.: $\xi = \frac{\partial}{\partial x^i}$, $\eta = \frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k}$$

The Γ^k_{ij} are called "Christoffel symbol" or "Connection coefficients".

Thus, via the axioms:

$$\begin{aligned} \nabla_{\xi} \eta &= \nabla_{\xi} \frac{\partial}{\partial x^i} \left(\eta^j \frac{\partial}{\partial x^j} \right) \stackrel{(I)}{=} \xi^i \nabla_{\frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j} \right) \\ &\stackrel{(II)}{=} \xi^i \left(\eta^j_{,i} \frac{\partial}{\partial x^j} + \eta^j \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k} \right) \\ &= (\xi^i \eta^k_{,i} + \xi^i \eta^j \Gamma^k_{ij}) \frac{\partial}{\partial x^k} \end{aligned}$$

function
vector

← recall: $\nabla_{\alpha} f = \alpha(f) \quad \forall \alpha \in T'_p(M), f \in F_p(M)$

Notation:

$$\eta^k_{,i} := \eta^k_{,i} + \eta^j \Gamma^k_{ij}$$

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$$\eta^k_{;i} := \eta^k_{,i} + \eta^j \Gamma^k_{ij}$$

↑ semi-colon for covariant derivatives

Thus:

$$\nabla_S \eta = S^i \eta^k_{;i} \frac{\partial}{\partial x^k} \quad (*)$$

Important: the Γ^k_{ij} transform non-tensorially when $x \rightarrow \bar{x}$:

On one hand:

$$\nabla_{\frac{\partial}{\partial \bar{x}^a}} \frac{\partial}{\partial \bar{x}^b} = \bar{\Gamma}^c_{ab} \frac{\partial}{\partial \bar{x}^c} = \bar{\Gamma}^c_{ab} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial}{\partial x^k} \quad (\text{I})$$

because $\frac{\partial}{\partial \bar{x}}$ is tangent vector

On the other hand:

$$\nabla_{\frac{\partial}{\partial \bar{x}^a}} \frac{\partial}{\partial \bar{x}^b} = \nabla_{\frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right) \quad \text{use axiom (b)} \Rightarrow$$

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On the other hand:

$$\begin{aligned} \nabla_{\partial \bar{x}^a} \frac{\partial}{\partial \bar{x}^b} &= \underbrace{\nabla_{\frac{\partial x^i}{\partial \bar{x}^a}}}_{\in \mathcal{F}(M)} \left(\frac{\partial x^i}{\partial \bar{x}^b} \frac{\partial}{\partial x^i} \right) && \text{use axiom (b)} \Rightarrow \\ &= \frac{\partial x^i}{\partial \bar{x}^a} \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial x^i}{\partial \bar{x}^b} \frac{\partial}{\partial x^i} \right) && \text{use Leibniz rule (c)} \Rightarrow \\ &= \frac{\partial x^i}{\partial \bar{x}^a} \left[\left(\frac{\partial}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^i} + \frac{\partial x^i}{\partial \bar{x}^b} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right] \end{aligned}$$

On the other hand:

$$\nabla_{\partial \bar{x}^a} \frac{\partial}{\partial \bar{x}^b} = \underbrace{\nabla_{\frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial}{\partial x^i}}}_{\in \mathcal{F}(M)} \left(\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right) \quad \text{use axiom (b)} \Rightarrow$$

$$\stackrel{\text{hand}}{=} \frac{\partial x^i}{\partial \bar{x}^a} \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right) \quad \text{use Leibniz rule (c)} \Rightarrow$$

$$= \frac{\partial x^i}{\partial \bar{x}^a} \left[\left(\frac{\partial}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right]$$

$$= \left(\frac{\partial}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad (\text{II})$$

Compare I, II \Rightarrow

$$\bar{\Gamma}^c \cdot \frac{\partial x^k}{\partial \bar{x}^a} - \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} \cdot \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij} \cdot \left(\frac{\partial \bar{x}^r}{\partial x^k} \right) \quad 5/20$$

Compare I, II \Rightarrow

$$\bar{\Gamma}_{ab}^c \frac{\partial x^c}{\partial \bar{x}^a} = \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma_{ij}^k \quad \left(\frac{\partial \bar{x}^r}{\partial x^k} \Rightarrow \right)$$

\Rightarrow

$$\bar{\Gamma}_{ab}^r = \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b}}_{\text{This term is indep. of } \Gamma} + \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma_{ij}^k}_{\text{Only this term would be there, if the } \Gamma_{ij}^k \text{ were tensor coefficients in the } \frac{\partial}{\partial x^i}, dx^i \text{ bases.}}$$

This term is indep. of Γ
 $\Rightarrow \Gamma$ can be zero in one coordinate system and nonzero in another!

Only this term would be there, if the Γ_{ij}^k were tensor coefficients in the $\frac{\partial}{\partial x^i}, dx^i$ bases.

(Can be shown to be equivalent)

\downarrow

Physicists' definition of ∇ : Any set of n^3 functions $\Gamma_{ij}^k(x)$ where

$$| \quad a^b \frac{\partial \bar{x}^c}{\partial x^k} - \frac{\partial \bar{x}^a \partial \bar{x}^b}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial \bar{x}^a \partial \bar{x}^b}{\partial \bar{x}^a \partial \bar{x}^b} | \quad ij \quad \left(\frac{\partial \bar{x}^k}{\partial x^k} \right)$$



$$\bar{\Gamma}^r_{ab} = \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b}}_{\text{This term is indep. of } \Gamma} + \underbrace{\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k_{ij}}_{\text{Only this term would be there, if the } \Gamma^k_{ij} \text{ were tensor coefficients in the } \frac{\partial}{\partial x^i}, dx^i \text{ bases.}}$$

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(Can be shown to be equivalent)



Physicists' definition of ∇ : Any set of n^3 functions $\Gamma^r_{ab}(x)$ which transform this way are defining a cov. derivative ∇ .

The "absolute" covariant derivative ∇ :

Consider the covariant derivative but:
without choosing a direction vector ξ :

$$\nabla : T_x(M)^1 \rightarrow T_x(M)^1$$

$$\nabla : \eta = \eta^i(x) \frac{\partial}{\partial x^i} \rightarrow \nabla \eta(x) = \eta^k{}_{;i}(x) dx^i \otimes \frac{\partial}{\partial x^k}$$

(I.e. feed the open covariant slot
of $\nabla \eta$ with contravariant ξ .)

Indeed: The contraction of $\nabla \eta$ with ξ yields:

$$\nabla \eta(\xi) = \eta^k{}_{;i} dx^i(\xi) \frac{\partial}{\partial x^k} = \eta^k{}_{;i} \xi^i \frac{\partial}{\partial x^k} = \nabla_\xi \eta \quad \text{ok with } \dots$$

without choosing a direction vector ξ :

$$\nabla : T_x(M)^i \rightarrow T_x(M)_i$$

$$\nabla : \eta = \eta^{i(x)} \frac{\partial}{\partial x^i} \rightarrow \nabla \eta(x) = \eta^{k}_{;i}(x) dx^i \otimes \frac{\partial}{\partial x^k}$$

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$$dx^i(\xi) = \xi(x^i) = \xi^j \frac{\partial}{\partial x^j} x^i = \xi^j \delta^i_j = \xi^i$$

We defined ∇ algebraically. Now, extract the

Geometric meaning of ∇ :

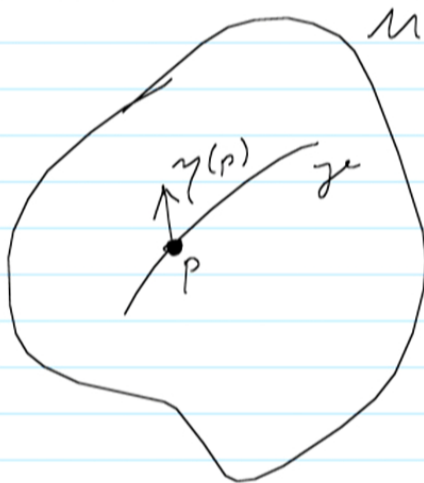
(\int) ∇ describes infinitesimal parallel transport
It should also describe finite parallel transport

Definition: Assume ∇ is given. Choose a path $\gamma: \mathbb{R} \rightarrow M$.

Then, a tangent vector field η is called auto-parallel along γ , if

$$\nabla_{\dot{\gamma}} \eta = 0$$

i.e. if η doesn't change under parallel transport along the path γ .



Geometric meaning of ∇ :

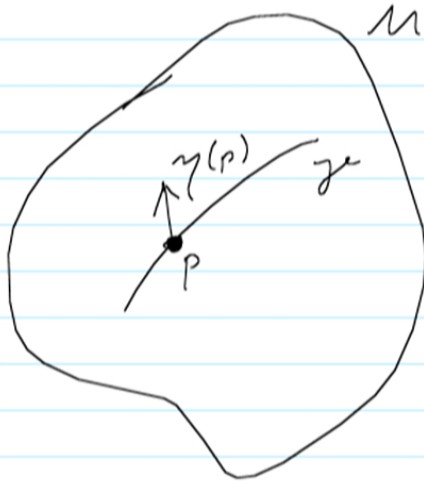
(J) ∇ describes infinitesimal parallel transport
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Then, a tangent vector field η is called auto-parallel along γ , if

$$\nabla_{\dot{\gamma}} \eta = 0$$

i.e. if η doesn't change under parallel transport along the path γ . 🖐



Note: We'll see that they always exist, i.e., we can always parallel transport a vector η finite distances.

□ In a chart,

$$\eta = \eta^i(x) \frac{\partial}{\partial x^i}$$

and

$$\gamma: [a, b] \rightarrow \mathcal{M}$$

$$\gamma: t \rightarrow x^i(t)$$

and the tangent vector:

$$\dot{\gamma}(x(t)) = \frac{dx^k}{dt} \frac{\partial}{\partial x^k}$$

Thus:
$$\nabla_{\dot{\gamma}} \eta = \nabla_{\frac{dx^k}{dt} \frac{\partial}{\partial x^k}} \left(\eta^i \frac{\partial}{\partial x^i} \right) = \frac{dx^k}{dt} \nabla_{\frac{\partial}{\partial x^k}} \left(\eta^i \frac{\partial}{\partial x^i} \right)$$

$$= \frac{dx^k}{dt} \left(\frac{\partial \eta^i}{\partial x^k} \frac{\partial}{\partial x^i} + \eta^i \Gamma^j_{ki} \frac{\partial}{\partial x^j} \right)$$

$$\begin{aligned}
 & \frac{dx^k}{dt} \left(\frac{\partial \eta^i}{\partial x^k} \frac{\partial}{\partial x^i} + \eta^j \Gamma^i_{kj} \frac{\partial}{\partial x^i} \right) \\
 & = \left(\frac{d\eta^i}{dt} + \frac{dx^k}{dt} \eta^j \Gamma^i_{kj} \right) \frac{\partial}{\partial x^i} \stackrel{!}{=} 0
 \end{aligned}$$

\Rightarrow η auto parallel to γ means:

$$\frac{d\eta^i}{dt} + \eta^j \frac{dx^k}{dt} \Gamma^i_{kj} = 0$$

i.e. this is the condition for the vectors of η being parallel translates of each other, along γ .

Conclusion:

This is 1st order ODEs for η . Thus:

□ Conclusion:

This is 1st order ODEs for η . Thus:

Initial condition $\eta(y(0)) \Rightarrow$ solution $\eta(y(t))$ exists
at least locally

\Rightarrow □ Proposition:

Given a path $\gamma: [t, s] \rightarrow M$, the
autoparallel transport of a tangent vector η
at $\gamma(t)$ to $\gamma(s)$ is unique.

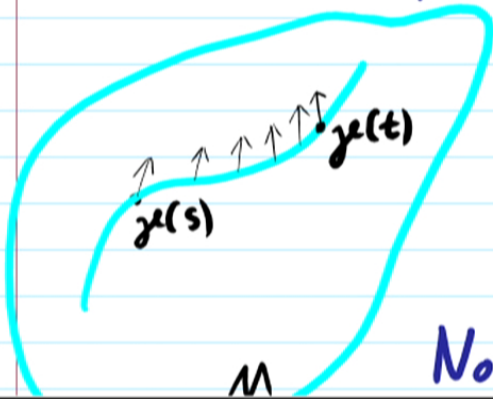
I.e., the path γ defines a parallel transport map τ :

$$\tau(t,s): T_{\gamma(t)} \rightarrow T_{\gamma(s)}$$

$$\tau(t,s): \eta(\gamma(t)) \rightarrow \eta(\gamma(s))$$

□ Q: Can one use τ to obtain ∇ as a Newton-Leibniz limit?

□ Proposition: (for the proof, see e.g. the text by Straumann)

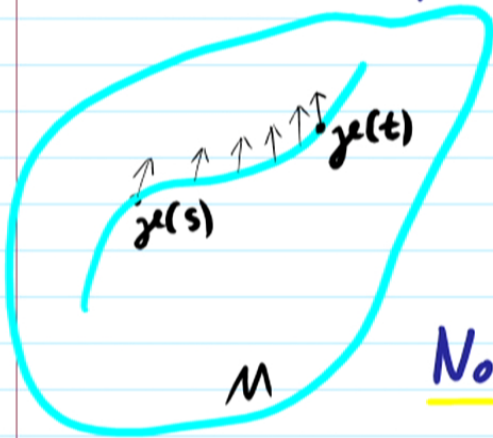


$$\nabla_{\dot{\gamma}} \eta(\gamma(t)) = \frac{d}{ds} \Big|_{s=t} \tau(s,t) (\eta(\gamma(s)))$$

Note: Since we can choose paths with arbitrary $\dot{\gamma}$

□ Q: Can one use τ to obtain ∇ as a Newton-Leibniz limit?

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$$\nabla_{\dot{\gamma}} \eta(\gamma(t)) = \frac{d}{ds} \Big|_{s=t} \tau(s,t)(\eta(\gamma(s)))$$

Note: Since we can choose paths with arbitrary $\dot{\gamma}$ this equation can be used as a geometric definition of ∇ .

∇ for arbitrary tensors:

▽ for arbitrary tensors:

□ The parallel transport map $\tau(s, t)$ transports tangent vectors η from $\gamma(s)$ to $\gamma(t)$.

□ Definition: $\tau(s, t)$ also parallel transports the dual vectors ω , namely so that contraction is conserved:

$$\underbrace{\tau(\omega)}_{\text{parallel transported } \omega} \left(\underbrace{\tau(\eta)}_{\text{parallel transported } \eta} \right) = \omega(\eta) \quad (C')$$

□ Extension of τ to tensor products:

Definition:

arbitrary tensor ∇_{ξ} arb. point $p \in M$

$$\nabla_{\xi} S'(p) := \nabla_{\dot{\gamma}} S'(\gamma(t)) \Big|_{t=0}$$

arb. tangent vector ξ

$$:= \frac{d}{dt} \Big|_{t=0} \tau(t,0)(S'(\gamma(t)))$$

here, γ is any path through p obeying:

$$\dot{\gamma}(0) = \xi(p), \quad \gamma(0) = p$$

Absolute covariant derivative:

led to ∇S which is (r, p) tensor

Exercise:

Show that when S is a scalar function $S \in F(M)$, then:

$$\nabla_{\xi} S = \xi(S) = \xi^i \frac{\partial}{\partial x^i} S$$

$$\nabla_{\xi} S'(p) := \nabla_{\dot{\gamma}} S'(\gamma(t)) \Big|_{t=0}$$

arb. tangent vector \nearrow

$$:= \frac{d}{dt} \Big|_{t=0} \tau(t,0)(S'(\gamma(t)))$$

here, γ is any path through p obeying:

$$\dot{\gamma}(0) = \xi(p), \quad \gamma(0) = p$$

a scalar function $S \in \mathcal{F}(M)$,
then:

$$\nabla_{\xi} S = \xi(S) = \xi^i \frac{\partial}{\partial x^i} S$$

□ Absolute covariant derivative:

(for abs. derivative one is not specifying the direction.) \nearrow

$$(\nabla S)(\eta_1, \dots, \eta_p, \omega_1, \dots, \omega_{r-p}) := \nabla_{\xi} S(\eta_1, \dots, \eta_p, \omega_1, \dots, \omega_{r-p})$$

led to ∇S which is (r,p) tensor \uparrow

Properties of ∇ :

* ∇ is a derivation: (because ∇ inherits the Leibniz rule from $\frac{d}{ds}$)

$$\nabla_{\xi}(S_1 \otimes S_2) = \frac{d}{ds} \Big|_{s=t} \tau(S_1 \otimes S_2) = \frac{d}{ds} \Big|_{s=t} \tau(S_1) \otimes \tau(S_2)$$

$\tau(s,t)$ $S_1(\gamma(s))$ $S_2(\gamma(s))$

$$= \left[\frac{d}{ds} \Big|_{s=t} \tau(S_1) \right] \otimes \tau(S_2) \Big|_{s=t} + \tau(S_1) \Big|_{s=t} \otimes \frac{d}{ds} \Big|_{s=t} \tau(S_2)$$

$$= (\nabla_{\xi} S_1) \otimes S_2 + S_1 \otimes \nabla_{\xi} S_2 \quad (A)$$

* Eq. (A) implies that ∇ and contractions do commute. 14/20

* ∇ is a derivation: (because ∇ inherits the Leibniz rule from $\frac{d}{ds}$)

$$\begin{aligned} \nabla_{\xi}(S_1 \otimes S_2) &= \frac{d}{ds} \Big|_{s=t} \tau(S_1 \otimes S_2) = \frac{d}{ds} \Big|_{s=t} \tau(S_1) \otimes \tau(S_2) \\ &= \left[\frac{d}{ds} \Big|_{s=t} \tau(S_1) \right] \otimes \tau(S_2) \Big|_{s=t} + \tau(S_1) \Big|_{s=t} \otimes \frac{d}{ds} \Big|_{s=t} \tau(S_2) \\ &= (\nabla_{\xi} S_1) \otimes S_2 + S_1 \otimes \nabla_{\xi} S_2 \quad (A) \end{aligned}$$

* Eq. (A) implies that ∇ and contractions do commute.

Action of ∇ on tensors in a chart?

▢ Recall: $\nabla_{\xi} \frac{\partial}{\partial x^i} = \xi^l \Gamma^k_{li} \frac{\partial}{\partial x^k}$

▢ Problem: Find $\nabla_{\xi} dx^i = ?$

→ • Consider $\eta \otimes \omega$.

\downarrow tangent vector field
 \downarrow cotangent vector field

• Differentiate:

$$\nabla_{\xi} (\eta \otimes \omega) \stackrel{(A)}{=} (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega$$

• Contract: (use that ∇ and contraction commute)

Same strategy will be used below for general

\Rightarrow • Consider $\eta \otimes \omega$.

\downarrow tangent vector field
 \downarrow cotangent vector field

• Differentiate:

$$\nabla_{\xi}(\eta \otimes \omega) \stackrel{(A)}{=} (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega$$

• Contract: (use that ∇_{ξ} and contraction commute)

(by exercise above)

$$\xi(\omega(\eta)) \stackrel{=}{=} \nabla_{\xi}(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$$

$\underbrace{\hspace{10em}}_{\text{scalar function}}$

$$(i.e. \quad \xi(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta))$$

\Rightarrow An expression for $\nabla_{\xi}(\omega)(\eta)$:

Same strategy will be used below for general tensors.

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$$\nabla_{\xi}(\eta \otimes \omega) = (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega$$

• **Contract:** (Use that ∇_{ξ} and contraction commute)

(by exercise above)

$$\xi(\omega(\eta)) \stackrel{=}{=} \nabla_{\xi}(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$$

(i.e. $\xi(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta)$)

\Rightarrow An expression for $\nabla_{\xi}(\omega)(\eta)$:

$$(\nabla_{\xi} \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_{\xi} \eta) \quad (*)$$

Now: Choose $\omega := dx^i$ and $\eta := \frac{\partial}{\partial x^i}$

$$\Rightarrow (\nabla_{\xi} dx^j) \left(\frac{\partial}{\partial x^i} \right) = \xi \left(\underbrace{\langle dx^j, \frac{\partial}{\partial x^i} \rangle}_{\delta_{ij} = \text{const.}} \right) - \langle dx^j, \nabla_{\xi} \frac{\partial}{\partial x^i} \rangle$$

(Notation: |

$$(\nabla_{\xi} \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_{\xi} \eta) \quad (*)$$

Now: Choose $\omega := dx^i$ and $\eta := \frac{\partial}{\partial x^i}$

$$\Rightarrow (\nabla_{\xi} dx^i) \left(\frac{\partial}{\partial x^i} \right) = \xi \left(\underbrace{\langle dx^i, \frac{\partial}{\partial x^i} \rangle}_{\delta^i_i = \text{const.}} \right) - \langle dx^i, \nabla_{\xi} \frac{\partial}{\partial x^i} \rangle$$

Notation:
 $\langle \omega, \xi \rangle = \omega(\xi)$
 (inner product,
 contraction)

$$= - \langle dx^i, \xi^l \Gamma_{li}^k \frac{\partial}{\partial x^k} \rangle$$

$$= - \xi^l \Gamma_{li}^i$$

$$\Rightarrow \nabla_{\xi} dx^i = - \xi^l \Gamma_{li}^i dx^i$$

For general tensors: (by exactly same strategy as above but applied to multiple tensor products, we obtain:

$$\nabla_{\xi} S(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s) \quad (\text{as in Eq. (*) above})$$

$$= S'(\eta_1, \dots, \eta_r, \omega_1, \dots, \omega_s)$$

$$- S'(\nabla_{\xi} \eta_1, \eta_2, \dots, \eta_r, \omega_1, \dots, \omega_s) - \dots$$

$$- S'(\eta_1, \dots, \nabla_{\xi} \eta_r, \omega_1, \dots, \omega_s)$$

$$- S'(\eta_1, \dots, \eta_r, \nabla_{\xi} \omega_1, \omega_2, \dots, \omega_s) + \dots$$

Choosing the basis vectors dx^i and $\frac{\partial}{\partial x^i}$, we obtain

for

$$S = \sum_{j_1, \dots, j_q}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

that $\nabla_{\xi} S$ reads

$$\nabla_{\xi} S = \xi^k \sum_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

with:

$$\sum_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} := \sum_{j_1, \dots, j_q, k}^{i_1, \dots, i_p} + \Gamma_{k\ell}^{i_1} \sum_{j_1, \dots, j_q}^{\ell i_2, \dots, i_p} + \dots + \Gamma_{k\ell}^{i_p} \sum_{j_1, \dots, j_q}^{i_1, \dots, \ell}$$

for

$$S = \sum_{j_1, \dots, j_q}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

that $\nabla_{\xi} S$ reads

$$\nabla_{\xi} S = \xi^k \sum_{j_1, \dots, j_q, j_k}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

with:

$$\begin{aligned} \sum_{j_1, \dots, j_q, j_k}^{i_1, \dots, i_p} &:= \sum_{j_1, \dots, j_q, j_k}^{i_1, \dots, i_p} + \Gamma_{k\ell}^{i_1} \sum_{j_1, \dots, j_q}^{\ell i_2, \dots, i_p} \\ &+ \dots + \Gamma_{k\ell}^{i_p} \sum_{j_1, \dots, j_q}^{i_1, \dots, \ell} \\ &- \Gamma_{kj_1}^{\ell} \sum_{\ell, \dots, j_q}^{i_1, \dots, i_p} \\ &- \dots - \Gamma_{kj_q}^{\ell} \sum_{j_1, \dots, \ell}^{i_1, \dots, i_p} \end{aligned}$$

Special cases:

□ Tangent vector fields:

$$\xi^i{}_{;jk} = \xi^i{}_{,k} + \xi^j \Gamma^i{}_{jk}$$

□ Cotangent vector fields:

$$\omega_{ijk} = \omega_{i,k} - \omega_j \Gamma^l{}_{ki}$$

Recall: Specifying ∇ specifies parallel transport of vectors and this should specify the manifold's shape, but how?

□ Tangent vector fields:

$$\xi^i_{;jk} = \xi^i_{,jk} + \xi^j \Gamma^i_{kj}$$

□ Cotangent vector fields:

$$\omega_{i;jk} = \omega_{i,jk} - \omega_l \Gamma^l_{ki}$$

Recall: Specifying ∇ specifies parallel transport of vectors and this should specify the manifold's shape, but how?

→ Indeed, ∇ specifies Torsion & Curvature