

Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 7

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Abstract: <p>Course Description coming soon.</p>

GR for Cosmology, Achim Kempf, Fall 15, Lecture 7

Note Title

Recall: Physical motivation for the "Metric Tensor"

□ In Minkowski space, in inertial and cartesian coordinates:

$$\left[\overset{\substack{\text{4-dim spa-c-time} \\ \text{distance!}}}{\text{distance}(x, \hat{x})} \right]^2 = -(x^0 - \hat{x}^0)^2 + (x^1 - \hat{x}^1)^2 + (x^2 - \hat{x}^2)^2 + (x^3 - \hat{x}^3)^2$$

↑
indep. of choice
of inertial cds.

$$= \eta_{\mu\nu} (x^\mu - \hat{x}^\mu) (x^\nu - \hat{x}^\nu)$$

$$\text{with } \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

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$$\text{with } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

□ In Minkowski space, in an arbitrary coordinate system:

$$\left[\text{distance}(x, \hat{x}) \right]^2 = g_{\mu\nu}(x) (x^\mu - \hat{x}^\mu) (x^\nu - \hat{x}^\nu) + \mathcal{O}^3$$

(e.g. polar cds, or
accelerated cds)

with $g_{\mu\nu}(x) \neq \eta_{\mu\nu}$

↑
complicated higher
order terms

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□ Generalization to curved space-time, historically:

Allow over each point a local inertial frame

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Allow even such $g_{\mu\nu}(x)$ which in no coordinate system obey:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \text{ for all } x \in \mathcal{M}$$

$\Rightarrow g_{\mu\nu}(x)$ is not simply $\eta_{\mu\nu}$ in non-inertial coordinates

\Rightarrow Such $g_{\mu\nu}(x)$ take us beyond special relativity!

□ Enforce Einstein's equivalence principle:

Require $g_{\mu\nu}$ to be such that

for each $x \in \mathcal{M}$ there exists a coordinate

system such that $g_{\mu\nu} = \eta_{\mu\nu}$.

Recall equiv. principles (EP):

If freely falling small masses

coordinate system obey:

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△ Enforce Einstein's equivalence principle:

Require $g_{\mu\nu}$ to be such that

for each $x \in \mathcal{M}$ there exists a coordinate system so that at least at x :

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \left(\begin{array}{l} \text{i.e., locally, special relativity holds} \\ \text{dist}(x, \tilde{x})^2 = \eta_{\mu\nu}(x-\tilde{x})^\mu(x-\tilde{x})^\nu + \mathcal{O}^3 \\ \text{to lowest nontrivial order.} \end{array} \right)$$

Recall equiv. principles (EP):

If freely falling small masses

fall equally \Rightarrow "weak EP"

+ same internal non-grav. physics \Rightarrow "Einst. EP"

+ same internal grav. physics \Rightarrow "strong EP"

Recall: Math. definition of the metric tensor:

□ g is covariant tensor of rank $(0,2)$
 (because η is in special relativity)



e.g. $\theta^\mu(x) = dx^\mu$

□ Thus, if n cotangent vector fields $\theta^\mu(x)$
 form bases at each point x , then
 g is of the form:

$$g(x) = g_{\mu\nu}(x) \theta^\mu(x) \otimes \theta^\nu(x)$$

↑ recall: $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ and $g_{\mu\nu}$ is invertible (since nondegenerate)

□ $g_{\mu\nu}(x)$ invertible \Rightarrow there exists a tensor g^{-1} of rank $(2,0)$:

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□ $g_{\mu\nu}(x)$ invertible \Rightarrow there exists a tensor g^{-1} of rank $(2,0)$:

$$g^{-1}(x) = g^{\mu\nu}(x) \overset{\text{dual basis}}{e_\mu(x)} \otimes e_\nu(x) \text{ with } g^{\mu\nu}(x) g_{\nu\sigma}(x) = \delta^\mu_\sigma$$

→ Modern view of the Einsteinian equivalence principle:

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→ Modern view of the Einsteinian equivalence principle:

Recall: We asked that for each point $p \in M$ there is a coordinate system so that, when choosing the basis $\{dx^\mu\}, \{\frac{\partial}{\partial x^\nu}\}$

then $g(x) \equiv g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$, $g_{\mu\nu}(x) \equiv g(x)(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu})$

obs: $g_{\mu\nu}(p) \equiv \eta_{\mu\nu}$ (in general only at p)

Modern formulation of the Einsteinian equivalence principle:

Independently of any choice of coordinate system:

Now, knowing distances through $g_{\mu\nu}$, what else follows?

□ Distances yield volumes, namely $g_{\mu\nu}(x)$ induces an $\Omega(x)$.

□ g, g^{-1} yield duality of covariance and contravariance.

□ g yields "Hodge star" $*$: $\Lambda_p \rightarrow \Lambda_{n-p}$ duality.

□ $*$ yields $(,)$ making the Λ_p Hilbert spaces.
for Riemannian manifolds

□ g yields co-derivative $\delta: \Lambda_p \rightarrow \Lambda_{p-1}$

□ d, δ yield the Laplacian/d'Alembertian $\Delta: \Lambda_p \rightarrow \Lambda_p$

Proposition:

Given a notion of distance, i.e., a metric, g , this also

induces a volume form Ω . (i.e., a positive $\Omega \in \Lambda_n(M)$, i.e., that when integrated over any portion of M yields a positive number)

Namely:

□ Assume, as always, that M is oriented.

□ Consider a positive chart.

(i.e. has positive $\det(\text{Jacobian})$ with given atlas)

Then:

$$\square \quad \Omega := \sqrt{|\det(g_{ij}(x))|} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

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□ $\Omega := \sqrt{|g|} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$ is a well-defined volume form.

$:= |\det(g_{ij}(x))|$

Proof: □ Nonzero for all $p \in M$?

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Proof:

□ Nonzero for all $p \in M$?

Yes, because g is assumed non-degenerate.

□ Well-defined, i.e., is definition chart-independent?

Yes: To see this, change chart: $x \rightarrow \tilde{x}$

Then:

$$\tilde{g}_{ij}(\tilde{x}(x)) = g_{rs}(x) \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \quad \text{because covariant}$$

i.e., as matrices:

$$\tilde{g} = \left(\frac{\partial x}{\partial \tilde{x}} \right)^t g \left(\frac{\partial x}{\partial \tilde{x}} \right)$$

more take determinants

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$$\Rightarrow |\tilde{g}| = \left| \frac{\partial x}{\partial \tilde{x}} \right|^2 |g| \quad \text{i.e. } |\tilde{g}|^{1/2} = \left| \frac{\partial x}{\partial \tilde{x}} \right| |g|^{1/2}$$

$$\text{Also: } d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^m = \det \left(\frac{\partial \tilde{x}}{\partial x} \right) dx^1 \wedge \dots \wedge dx^m$$

... to see this, change basis. \hat{x} \hat{x}

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Notation: (Ω is an n -form. What are its coefficients, as a covariant $(0, n)$ tensor?)

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□ Define:

$$\epsilon_{i_1, \dots, i_n} := \begin{cases} +1 & \text{if } (i_1, \dots, i_n) \text{ is even permutation of } (1, 2, \dots, n) \\ -1 & \text{if } (i_1, \dots, i_n) \text{ is odd permutation of } (1, 2, \dots, n) \\ 0 & \text{else} \end{cases}$$

unlike in SRT, ϵ_{\dots} is not canonical, because

Ω is: →

□ Then, Ω also reads:

$$\begin{aligned} \Omega &= \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n \quad (n\text{-form}) \\ &= \sqrt{|g|} \underbrace{\epsilon_{i_1, \dots, i_n}}_{=: \Omega_{i_1, \dots, i_n}} dx^{i_1} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_n} \\ \Omega &= \Omega_{i_1, \dots, i_n} dx^{i_1} \otimes \dots \otimes dx^{i_n} \quad (\text{covariant tensor}) \end{aligned}$$

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$$\Omega = \underbrace{\epsilon_{i_1, \dots, i_n}}_{=: \Omega_{i_1, \dots, i_n}} dx^{i_1} \otimes \dots \otimes dx^{i_n} \quad (\text{covariant tensor})$$

□ Ω is called the "canonical", or "(pseudo)Riemannian", or "metric", volume form.

Q: Other use of g ?

A: One needs g to formulate d'Alembertian Δ , or \square , for wave equations.

Why? a) \square should be non-directional 2nd derivative, but $d^2 = 0$.

b.) need e.g. $\square: \Lambda^0 \rightarrow \Lambda^0$ for Klein Gordon, i.e., need degree of forms conserved by \square .

Strategy:

A) Use g for a covariant \leftrightarrow contravariant tensors relation

B) Define a map "Hodge" $*$: $\Lambda_r \rightarrow \Lambda_{n-r}$

C) Define the "Coderivative": $\delta: \Lambda_r \rightarrow \Lambda_{r-1}$

D) Define "Laplacian/d'Alembertian": $\square := d\delta + \delta d$

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 - D) Define "Laplacian/d'Alembertian": $\square := d\delta + \delta d$

Then, e.g., the Klein Gordon equation reads:

$$(\square + m^2) \phi = 0$$

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\square $g(x)$ can be used as a map: by evaluation of one tensor factor:

$$g(x): T_x(M)' \rightarrow T_x(M),$$

$$g(x): \xi^i(x) e_i(x) \rightarrow g_{\mu\nu}(x) \theta^\mu(x) \otimes \theta^\nu(x) (\xi^i(x) e_i(x))$$

$$= \underbrace{g_{\mu\nu}(x) \xi^i(x)}_{\in \mathcal{F}_x(M)} \underbrace{\theta^\mu(x)}_{\in T_x(M)} \theta^\nu(x) \in T_x(M),$$

$\theta^\nu(e_i) = \delta_i^\nu$

\Rightarrow For the coefficient

functions we have: $g: \xi^i(x) \rightarrow \omega_\nu(x) = g_{\nu\alpha}(x) \xi^\alpha(x)$ (relative to bases θ^i, e_j)

\square Conversely, g^{-1} acts as:

$$g^{-1}(x): T_x(M)_1 \rightarrow T_x(M)'$$

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\square In this way, g, g^{-1} can lower or raise any 

tensor index, e.g.: $g: t^{ij}_k \rightarrow t_i{}^j{}_k = g_{is} t^{sj}_k$

and: $g^{-1}: \tau^{ij}_k \rightarrow \tau^{i j k} = g^{sk} \tau^{i s j}$

B) The Hodge $*$ map: $\Lambda_p \rightarrow \Lambda_{n-p}$

Recall:

$$\dim(\Lambda_p) = \binom{n}{p} = \binom{n}{n-p} = \dim(\Lambda_{n-p})$$

- Idea:
- ▣ each $v \in \Lambda_p$ is a covariant tensor
 - ▣ through g it is equivalent to a contravariant tensor \tilde{v}
 - ▣ can feed Ω with \tilde{v} to obtain $*v \in \Lambda_{n-p}$.

concretely:

anything totally antisymmetric

one could choose other bases.

▣ Consider any $v := \frac{1}{p!} v_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \Lambda_p$

convenient because drops out here \rightarrow

coefficients as a covariant tensor \rightarrow

$$= v_{i_1, \dots, i_p} dx^{i_1} \otimes \dots \otimes dx^{i_p}$$

- ▣ Use g^{-1} to define a contravariant image of v :

$$\tilde{v} = \tilde{v}^{i_1, \dots, i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}}$$

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$$\binom{p}{1} \binom{n-p}{1} = \dim(\Lambda_{n-p})$$

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□ Apply Ω on \tilde{v} :

$n-p$ factors

□ Apply Ω on \tilde{v} :

$$\Omega(\tilde{v}) = \underbrace{\Omega_{i_1, \dots, i_n}}_{!!} \tilde{v}^{i_1, \dots, i_p} \overbrace{dx^{i_{p+1}} \otimes \dots \otimes dx^{i_n}}^{n-p \text{ factors}} \in \Lambda_{n-p}$$

$$(*v)_{i_{p+1}, i_{p+2}, \dots, i_n}$$

□ Define $*v := \Omega(\tilde{v})$, $\leftarrow \dots$:

$$*v = (*v)_{i_1, \dots, i_{n-p}} dx^{i_1} \otimes \dots \otimes dx^{i_{n-p}}$$

$$*v = \frac{1}{(n-p)!} (*v)_{i_1, \dots, i_{n-p}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-p}}$$

Proposition:

Assume $v \in \Lambda_p$. Then

$$v \wedge *v = (-1)^{p(n-p)+s} \dots$$

← E.g. $s=1$ for space-time

$$*\nu = (*\nu)_{i_1, \dots, i_{n-p}} dx^{i_1} \otimes \dots \otimes dx^{i_{n-p}}$$

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What is s ? The "signature" of g is $\text{sgn}(g) = (r, s)$, where in diagonal form: $g = \begin{pmatrix} \underbrace{1 \dots 1}_r \\ \underbrace{-1 \dots -1}_s \end{pmatrix}$

Use $*$ to turn $\Lambda(M)$ into an "Inner Product Space":

Definition:

The Hodge $*$ provides a "scalar" (or also

Use $*$ to turn $\Lambda(M)$ into an "Inner Product Space":

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The Hodge $*$ provides a "scalar" (or also called "inner") product for $\Lambda(M)$:

Exercises:

- 1.) Write (α, β) in coordinates
- 2.) Show that $(,)$ is always positive definite on Λ_0 , i.e., $(\alpha, \alpha) > 0 \forall \alpha \in \Lambda_0, \alpha \neq 0$.

$$(\alpha, \beta) := \int_M \underbrace{\alpha \wedge * \beta}_{n \text{ form}}$$

\uparrow \uparrow $\underbrace{\hspace{10em}}_{(n-p) \text{ form}}$
 p -form p -form

This definition is extended linearly to forms that are lin. comb. of forms of orb. degree, p .

Notes: \square If g is indefinite, then also $(,)$ is indefinite.

c) $(,)$ yields an adjoint for d , the co-derivative δ :

Recall: For any operator $A: D_A \subset \mathcal{X} \rightarrow \mathcal{X}$ (with D_A dense, i.e., $\overline{D_A} = \mathcal{X}$), its adjoint A^+ is defined to have the domain

$$D_{A^+} := \left\{ v \in \mathcal{X} \mid \exists w \in \mathcal{X} \ \forall z \in D_A: \langle v, Az \rangle = \langle w, z \rangle \right\}$$

and this action: $A^+v := w$. We then have:

$$\langle A^+v, z \rangle = \langle v, Az \rangle \quad \forall z \in D_A, v \in D_{A^+}$$

Definition:

The co-derivative, δ , is the (anti-)adjoint of d with respect to the inner product $(,)$ on $\Lambda(U)$:

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$$\boxed{(\delta \alpha, \beta) := -(\alpha, d\beta)} \quad \forall \alpha \in D_\delta, \beta \in D_d$$

Definition:

$$\langle Av, z \rangle = \langle v, Az \rangle \quad \forall z \in D_A, v \in D_{A^*}$$

The co-derivative, δ , is the (anti-)adjoint of d with respect to the inner product $(,)$ on $\Lambda(M)$:

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c) The Codifferential δ explicitly

Clearly: $\delta : \Lambda^p(M) \rightarrow \Lambda^{p-1}(M)$

Proposition: $\delta : \nu \rightarrow \left((-1)^{n-p+s} *d* \right) \nu$ (Some authors define δ as the negative of this)

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Clearly: $\delta : \Lambda^r(M) \rightarrow \Lambda^{r-1}(M)$

Proposition: $\delta : \nu \rightarrow \left((-1)^{n-p+r+s} *d* \right) \nu$ (Some authors define δ as the negative of this)

Properties: $\square \delta^2 = 0$

\square In coordinates:

$$(\delta \omega)^{i_1 \dots i_{p-1}} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} \omega^{k i_1 \dots i_{p-1}} \right)_{,k}$$

\square If M is contractible (and in every contractible part):

$$\delta \nu = 0 \Rightarrow \exists \omega : \nu = \delta \omega$$

Proposition: $\delta : \nu \rightarrow \left((-1)^{n_p+n+s} *d* \right) \nu$ (Some authors define δ as the negative of this)

Properties: $\square \delta^2 = 0$

\square In coordinates:

$$(\delta \omega)^{i_1 \dots i_p} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} \omega^{k i_1 \dots i_p} \right)_{,k}$$

\square If M is contractible (and in every contractible part):

$$\delta \nu = 0 \Rightarrow \exists \omega : \nu = \delta \omega$$

Exercises: \square Show the above.

\square Determine whether or not δ is a derivation.

Use d and δ to obtain the Maxwell equations on M

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□ Define:

"Field strength": $F_{\mu\nu}(x) := \begin{pmatrix} 0, -E_1, -E_2, -E_3 \\ E_1, 0, B_3, -B_2 \\ E_2, -B_3, 0, B_1 \\ E_3, B_2, -B_1, 0 \end{pmatrix}$, $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$

↑ electric field
↑ magn. field
↑ "Field strength" 2-form

"Current" 3-form $\rightarrow j(x) := \frac{1}{3!} \epsilon_{\mu\nu\lambda\sigma} j^\sigma dx^\mu \wedge dx^\nu \wedge dx^\lambda$

↑ "current 4-vector"

□ Then: The Maxwell Eqns read:

"Homogeneous
Maxwell equations"
(indep. of metric)

$$dF = 0, \quad \delta F = *j$$

"Inhomogeneous
Maxwell equations"
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□ Define:

"Field strength":

$$F_{\mu\nu}(x) := \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \quad F = F_{\mu\nu} dx^\mu \wedge dx^\nu$$

↑ "Field strength" 2-form

"Current" 3-form $\rightarrow j(x) := \frac{1}{3!} \epsilon_{\mu\nu\lambda\sigma} j^\mu dx^\nu \wedge dx^\lambda \wedge dx^\sigma$

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□ Then: The Maxwell Eqns read:

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Current 1-form, i.e., cotangent vector $f_{16/23}$

Remarks:

□ F is assumed to be an exact 2-form, i.e.,:

$$F = dA$$

(the 1-form A is called the 4-potential)

□ This already implies the homogeneous Maxwell equations:

$$dF = d^2A = 0 \quad !$$

→ One calls them "structure equations".

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The gauge principle of electrodynamics is the observation that, for any $w \in \Lambda_0$:

$$A \text{ and } \tilde{A} := A + dw$$

describe the same physics.

The Aharonov-Bohm effect and topological phases in general, can make A itself visible when Poincaré lemma doesn't apply.

They do because the (classically) observable fields are only the E and B fields in F and since $d^2 = 0$:

D The Laplacian/d'Alembertian, Δ , \square :

□ Definition of the Laplacian:

$$\Delta := \delta d + d \delta$$



Some authors define Δ as the negative of this and let δ be the adjoint of d .

□ Clear: $\Delta : \Lambda^p(M) \rightarrow \Lambda^p(M)$

□ If signature $s=1$: Then also called d'Alembertian and denoted $\square := d\delta + \delta d$.

□ Action on, e.g., $f \in \Lambda_0(M)$ in a chart: Exercise: verify

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□ If signature $s=1$: Then also called d'Alembertian and denoted $\square := d\delta + \delta d$.

□ Action on, e.g., $f \in \Lambda_0(M)$ in a chart:

Exercise: verify

$$\square f = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} g^{\mu\nu} f_{,\mu} \right)_{,\nu}$$

$$= \left(-\frac{\partial^2}{\partial x^0^2} + \frac{\partial^2}{\partial x^1^2} + \frac{\partial^2}{\partial x^2^2} + \frac{\partial^2}{\partial x^3^2} \right) f$$

if $g = \eta$

, if Λ is a Hilbert space

Properties of the d'Alembertian, \square in the Hilbert space $\Lambda(M)$:

* Defined: $\square : \Lambda_r(M) \rightarrow \Lambda_r(M)$

$$\square : \omega \rightarrow (\delta d + d\delta)\omega$$

* In the Hilbert space $\Lambda(M)$:

$$\square = \delta d + d\delta \text{ obeys } (d, \square\beta) = (\square d, \beta)$$

* \square is self-adjoint, $\square = \square^*$, for suitable boundary conditions, or if $\partial M = \emptyset$, and assuming $(,)$ is positive definite.

* Exercises: \square Verify $\square = \square^*$ formally, using only $\delta = -d^*$.

\square Verify that $\square^* = *\square$, $\square d = d\square$, $\square\delta = \delta\square$.

Consequences of the self-adjointness of \square : - if Λ is a Hilbert space

- A) The operators Δ and \square can be diagonalized, with real spectrum.
- B) For Riemannian manifolds, $\text{spec}(\Delta) \subset [0, \infty)$.
- C) For compact Riem. manifolds (of finite volume): $\text{spec}(\Delta)$ is *discrete*.
- D) Then, $\text{spec}(\Delta)$ is carrying a lot of information about (M, g) !

Still the finite volume Riemannian case.

Remark: There exists a related mathematical discipline, called "Spectral Geometry", combining differential

of Riemannian

Application: Klein-Gordon "action":

$$g^{\alpha\beta} \overset{\in T^2}{e_\alpha \otimes e_\beta} (\overset{\in T_1}{\phi_{,\alpha} \theta^\alpha} \otimes \overset{\in T_1}{\phi_{,\beta} \theta^\beta})$$

$$S[\phi] := \frac{1}{2} \int_M \overset{\uparrow \Lambda_0}{g^{\mu\nu}} \overset{\uparrow \Lambda_0}{\phi_{,\mu}} \overset{\uparrow \Lambda_0}{\phi_{,\nu}} \overset{\uparrow \Lambda_m}{\Omega}$$

↑
Klein Gordon
field $\phi \in F(M)$

⤴
(Recall special relativity:
 $S[\phi] = \int_{\mathbb{R}^4} \eta^{\mu\nu} \phi_{,\mu} \phi_{,\nu} d^4x$)

$$= \frac{1}{2} \int_M g^{\mu\nu}(x) \left(\frac{\partial}{\partial x^\mu} \phi \right) \left(\frac{\partial}{\partial x^\nu} \phi \right) \overset{\uparrow \Omega}{\sqrt{|g(x)|}} d^m x$$

↑ next: integrate by parts!

$$= \frac{1}{2} \int_M -\phi \underbrace{\frac{\partial}{\partial x^\nu} \left(\sqrt{|g|} g^{\mu\nu} \frac{\partial}{\partial x^\mu} \phi \right)}_{= \square \phi} \frac{1}{\sqrt{|g|}} \underbrace{\sqrt{|g|}}_{= \Omega} d^m x$$

Obtain the Klein Gordon wave equation:

□ Recall: Euler Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}$

□ Here: $\mathcal{L} = -\frac{1}{2} \phi \square \phi$ (the 0-form that we are integrating: $S' = \int_m \mathcal{L} \Omega$)

□ Obtain Klein Gordon equation:

$$\square \phi = 0$$

(with "mass": $\mathcal{L} = -\frac{1}{2} \phi (\square + m^2) \phi$
yielding $(\square + m^2) \phi = 0$)

Q: Which physical fields are described by K-G fields?

A: □ Meson fields

↙ there are many sorts of mesons. Most important mesons: "Pions". They transmit the nuclear force among protons & neutrons

□ Higgs field (Gives all particles their mass. Found at LHC. Nobel to Higgs, Englert (Brout) in 2013)