Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 6

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Abstract: <p>Course Description coming soon.</p>

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GR for Cosmology, Achim Kempf, Fall 15, Lecture 6

Note Title

Integration!

Q: What is special about totally antisymmetric covariant tensors, i.e., about differential forms?

A: Antisymmetry $\Rightarrow$ special transformation property under chart changes:

$\sim$ det (Jacobian)

$\Rightarrow$ suitable for integration:

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A: Antisymmetry $\Rightarrow$ special transformation property under chart changes:

$\sim$ det (Jacobian)

$\Rightarrow$ suitable for integration:

s-forms have natural integrals in s-dimensional manifolds

Except: Depending on charts, sign of Jacobian may be wrong!

Thus: Before defining integration on mflds, must study notion of "Orientation" of the mfld.

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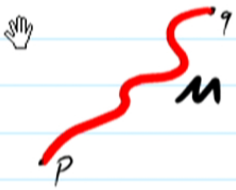
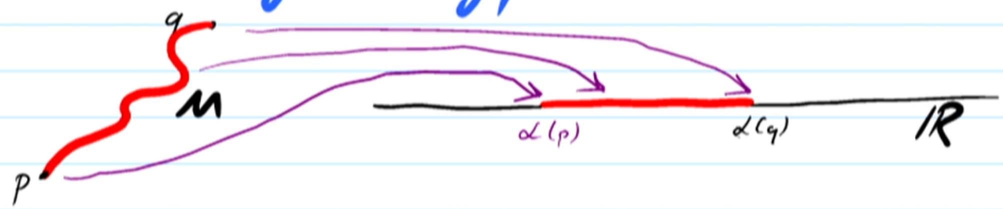
Namely: Consider e.g. 1-dim mfld:



- Could have charts of the type



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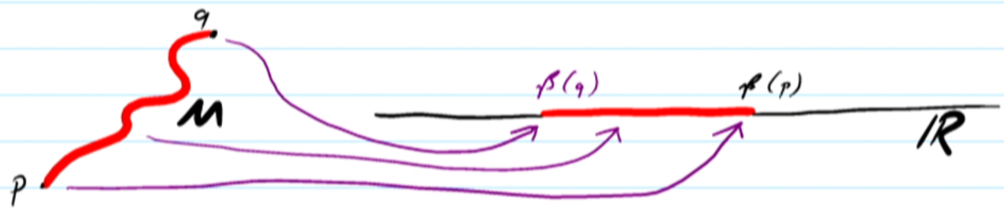

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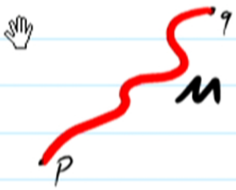

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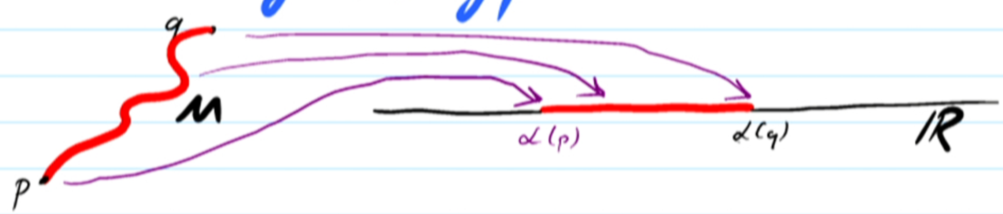

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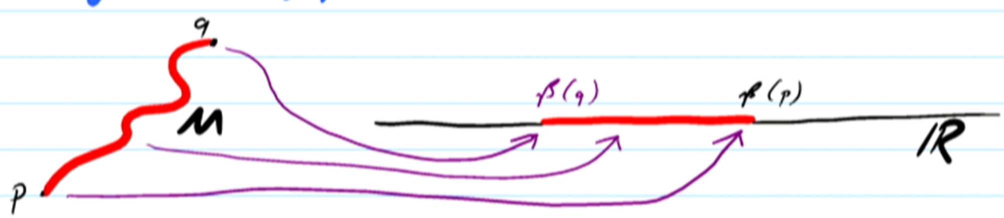

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Input: DVI - 1920x1080i@59.92Hz
Output: SDI - 1920x1080i@60Hz

For n-dim mflds, may need several charts.

Definitions:

- A complete collection of charts, i.e., an Atlas, A, is called oriented if among all overlapping charts with coordinates say $x, \tilde{x}$ the Jacobi determinants are positive:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) > 0$$

- A mfld M is called orientable

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are not orientable.

- A mfld, M, together with a choice of oriented atlas, A, is called an oriented manifold.

- Then, an <u>arbitrary</u> chart is called positive (or negative) if its Jacobian determinant with charts of the atlas A is positive (or negative).

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Definition:

An n-form $\Omega \in \Lambda_n(M)$ is called a volume form if it nowhere vanishes. (We will later find a preferred volume form for space-time)

Proposition:

M possesses a volume form

$\Updownarrow$

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Integration:

- Recall change of cds in integration in $\mathbb{R}^n$:

For $(x^1, \dots, x^n) \to (\tilde{x}^1, \dots, \tilde{x}^n)$:

$g: \mathbb{R}^n \to \mathbb{R}$

Riemann or Lebesgue integrals

$$\int_{\mathbb{R}^n} g(x^1, \dots, x^n)\, dx^1 \dots dx^n = \int_{\mathbb{R}^n} g(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \dots d\tilde{x}^n \quad (*)$$

Jacobian determinant is negative if coordinate systems change handedness.

- Now for a general n-dimensional diffable mfld M,

consider an n-form ω in a chart:

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- Now for a general n-dimensional diffable mfld M, consider an n-form ω in a chart:

$$\omega = f(x)\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

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Then what is ω in an overlapping, second chart?

$$\omega = f(x(\tilde{x}))\, \frac{\partial x^1}{\partial \tilde{x}^{i_1}} \frac{\partial x^2}{\partial \tilde{x}^{i_2}} \cdots \frac{\partial x^n}{\partial \tilde{x}^{i_n}}\, \underbrace{d\tilde{x}^{i_1} \wedge d\tilde{x}^{i_2} \wedge \ldots \wedge d\tilde{x}^{i_n}}_{\text{totally antisymmetric!}}$$

- terms are nonzero only if contain each number $1, \ldots, n$ exactly once, e.g. $d\tilde{x}^1 \wedge d\tilde{x}^3 \wedge d\tilde{x}^2 \wedge d\tilde{x}^4 \wedge d\tilde{x}^5 \wedge \ldots \wedge dx^n$.
- Reorder those terms - they are all

$$d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \ldots \wedge d\tilde{x}^n$$

up to a possible factor -1 because $dx^i \wedge dx^j = -dx^j \wedge dx^i$

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Jacobian determinant is negative if coordinate systems change handedness.

Now for a general n-dimensional diff'able mfld M, consider an n-form ω in a chart:

$$\omega = f(x)\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

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The following definition of the integral of n-forms in an n-dim. diffable mfld is chart-independent, i.e., is well-defined:

Definition:

Assume M is an oriented n-dim mfld

and $\omega \in \Lambda_n(M)$ reads in a chart α: $\omega = f(x)\, dx^1 \wedge \ldots \wedge dx^n$.

Then, if one chart suffices:

$$\int_M \omega := \overbrace{\int_{\alpha(M)} f(x)\, dx^1 dx^2 \ldots dx^n}^{\text{usual Riemann or Lebesgue integral}}$$

image of M in $\mathbb{R}^n$

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Else: Piece right hand side together from several charts

Note: how to piece together does not matter as long as charts are from the atlas that M is equiped with. That's why orientation is important.

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Definition: The boundary operator, ∂

- Assume $G \subset M$ is a region (i.e. an n-dim., open and connected subset) of the n-dim mfld M.

We denote the $(n-1)$ dim. boundary manifold of G by ∂G:

the boundary operator

$$\partial G := \text{boundary}(G)$$

- We say that ∂G is smooth if locally there exist charts so that:

image of boundary surface is

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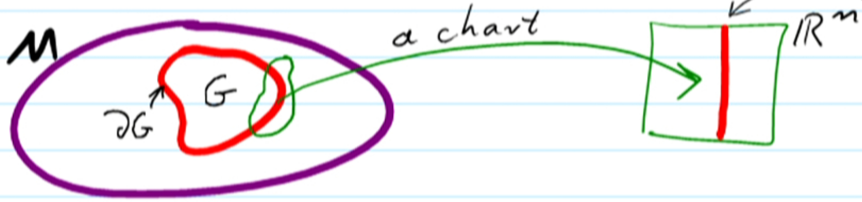

Proposition: If M is orientable, then so

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Proposition: If M is orientable, then so is G. Also, the orientation of G induces an orientation of ∂G.

We finally have all ingredients for one of Math's most important theorems:

Stokes' theorem: If closure $\bar{G}$ of G is a compact n-dim region, then:

$$\boxed{\int d\omega = \int \omega \quad \text{for all } \omega \in \Lambda(M)}$$

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Special case I:

Assume: $M = \mathbb{R}$, $G = (a, b)$

Therefore: $\partial G = \{a, b\}$

Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b df = f\Big|_a^b$$ (fund. thm of calculus)

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Special case II: ▫ $M = \mathbb{R}^2$, $G \subset \mathbb{R}^2$ a region with (closed) boundary curve ∂G.

recall: this is automatic because $\partial \circ \partial = 0$

▫ Consider a 1-form $\omega \in \Lambda_1(M)$:

$$\omega = \omega_1(x)\, dx^1 + \omega_2(x)\, dx^2$$

$$\Longrightarrow d\omega = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 \wedge dx^2$$

Now Stokes' theorem:

$$\int_G d\omega = \int_{\partial G} \omega \Rightarrow \iint_G \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 dx^2 = \int_{\partial G} \left(\omega_1 dx^1 + \omega_2 dx^2\right)$$

▫ This is known as "Green's theorem".

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Special case III: (exercise)

Similarly, one can show that what is often called the Stokes theorem for $M = \mathbb{R}^3$, namely

"cross product": $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ "

$$\int_G \vec{\nabla} \times \vec{w} \, dG = \int_{\partial G} \vec{w} \cdot \vec{ds}$$

vector field

a 2dim submanifold of M

1dim boundary of A.

is indeed this special case:

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$$\text{"}\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)\text{"} \qquad \int_G \vec{\nabla} \times \vec{w} \, dG = \int_{\partial G} \vec{w} \cdot \vec{ds}$$

G: a 2dim submanifold of M

$\vec{w}$: vector field

∂G: 1dim boundary of A.

is indeed this special case:

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$$\boxed{\mathrm{div}_\Omega \xi := L_\xi(\Omega)}$$

Lie derivative

Wait! Isn't the divergence, at least on $\mathbb{R}^n$, simply

$$\mathrm{div}\,\xi = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\xi^i = \xi^i_{,i}$$

Let's check in a chart:

- Assume $\Omega = a(x)\, dx^1 \wedge \ldots \wedge dx^n$ (volume form)

$a(x) \in \Lambda_0(M)$

and $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ (vector field)

- Then:

by Leibnitz rule

$= \xi^i \frac{\partial}{\partial x^i} a(x)$

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- Then:

$$\mathrm{div}_\Omega \xi = L_\xi \Omega \overset{\text{by Leibnitz rule}}{=} \underbrace{\xi(a)}_{= \xi^i \frac{\partial}{\partial x^i} a(x)} dx^1 \wedge \dots \wedge dx^n + \dots$$

$$+ a \sum_{i=1}^{n} dx^1 \wedge \dots \wedge L_\xi(dx^i) \wedge \dots \wedge dx^n$$

$$\left(\text{recall: } L_\xi(dx^i) = d(\xi(x^i)) = d\left(\xi^j \frac{\partial}{\partial x^j} x^i\right) = d(\xi^j \delta_j^i) = d(\xi^i) = \underbrace{\frac{\partial \xi^i}{\partial x^r} dx^r}_{\text{only } dx^i \text{ term}} \right)$$

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$$\Rightarrow \quad \mathrm{div}_\Omega \xi = \left(\xi^i a_{,i} + a \xi^i_{,i} \right) dx^1 \wedge \ldots \wedge dx^n$$

$$\Rightarrow \quad \boxed{\mathrm{div}_\Omega \xi = \frac{1}{a} (a \xi^i)_{,i}\, \Omega}$$

Notice: If $a \equiv 1$ then $\mathrm{div}_\Omega \xi = \frac{\partial \xi^i}{\partial x^i}$ as expected for the divergence in the simplest case

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Let's check on a chart:

□ Assume $\Omega = a(x)\, dx^1 \wedge \ldots \wedge dx^n$ (volume form)

$a(x) \in \Lambda_0(M)$

and $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$ (vector field)

□ Then:

$$\mathrm{div}_\Omega \xi = L_\xi \Omega \overset{\text{by Leibniz rule}}{=} \underbrace{\xi(a)}_{= \xi^i \frac{\partial}{\partial x^i} a(x)} dx^1 \wedge \ldots \wedge dx^n + \ldots$$

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$$\Rightarrow \quad \mathrm{div}_\Omega \xi = \left(\xi^i a_{,i} + a\, \xi^i_{\ ,i}\right) dx^1 \wedge \ldots \wedge dx^n$$

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Then:

$$\mathrm{div}_\Omega \xi = L_\xi \Omega \overset{\text{by Leibnitz rule}}{=} \underbrace{\xi(a)}_{= \xi^i \frac{\partial}{\partial x^i} a(x)} dx^1 \wedge \ldots \wedge dx^n + \ldots$$

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only dx^i term survives in wedge product

$$\Rightarrow \quad \mathrm{div}_\Omega \xi = \left(\xi^i a_{,i} + a \xi^i_{,i}\right) dx^1 \wedge \ldots \wedge dx^n$$

$$\Rightarrow \quad \boxed{\mathrm{div}_\Omega \xi = \frac{1}{a}(a \xi^i)_{,i} \, \Omega}$$

Notice: If $a \equiv 1$ then $\mathrm{div}_\Omega \xi = \frac{\partial \xi^i}{\partial x^i}$ as expected for the divergence in the simplest case.

Thus: Indeed, if $a(x) = 1 \ \forall x$ then $\mathrm{div}_\Omega \xi = \xi^i_{,i} \, dx^1 \wedge \ldots \wedge dx^n$.

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Now, we can derive Gauss' theorem from Stokes':

$$\operatorname{div}_\Omega \xi := L_\xi \Omega \in \Lambda_n(\mathcal{M})$$

$$\operatorname{div}_\Omega \xi = (d \circ i_\xi + i_\xi \circ d)\,\Omega$$

Recall: $d\Omega = 0$ because anti-symmetry doesn't allow $(n+1)$ forms.

$$\Rightarrow \boxed{\operatorname{div}_\Omega \xi = d \circ i_\xi(\Omega)}$$

We can now apply Stokes' theorem $\int_G dv = \int_{\partial G} v$:

$$\int_G d\, i_\xi(\Omega) = \int_{\partial G} i_\xi(\Omega)$$

i.e.: n-form (n-1) form

"Gauss' th

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i.e.:

$$\boxed{\int_G \overbrace{\mathrm{div}_\Omega \xi}^{n\text{-form}} = \int_{\partial G} \overbrace{i_\xi(\Omega)}^{(n-1)\text{ form}} \qquad \text{"Gauß' theorem"}}$$

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So far, we have:

- differentiations, ξ, d, i_ξ, L_ξ, $\mathrm{div}_\xi \Omega$
- integration $\int_G v$

But, still lacking:

- A notion of distance between points!

The problem:

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→ **So far, we have:**

- differentiations, ξ, d, i_ξ, L_ξ, $\mathrm{div}_\xi \Omega$
- integration $\int_G \nu$

But, still lacking:

- A notion of **distance** between points!

The problem:

- Numerical distance of cds $(x^1, x^2, \dots, x^n)$

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Definition:

A (pseudo)-Riemannian metric on M is a covariant 2-tensor field g, i.e., a map $g : M \to T_p(M)_2$ which obeys

- $g : (\xi, \eta) \to \mathbb{R}$ ← any two vector fields

  with $g(\xi,\eta) = g(\eta,\xi)$ (symmetry)

- $g_p(\xi,\eta) = 0$ for all $\xi \in T_p(M)$ means g is non degen...

  ↑ any point $p \in M$.

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  any point $p \in M$.

  only if $\eta = 0$ at p.

means g is non degenerate i.e. has no kernel i.e. is invertible as a ...

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g is called a Riemannian metric if $g(\xi,\xi) > 0$ for all ξ at all $p \in M$.

Definition: A differentiable mfdl M together with a (pseudo) Riemannian metric g is called a (pseudo) Riemannian manifold.

The metric in a basis:

Assume the $\{\theta^i\}^n$ $\theta^i(x)$ $(e.g.\ \theta^i = dx^i)$ are

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The metric in a basis:

$\theta^i(x)$

- Assume the $\{\theta^i\}_{i=1}^n$ (e.g. $\theta^i = dx^i$) are a basis of $\Lambda_1(p) = T_p(M)$, at each $p \in M$.

- Then: $g = g_{ij}\, \theta^i \otimes \theta^j$

$g_{ij}(x)$

$e_i(x)$

- Assume $\{e_i\}_{i=1}^n$ (e.g. $e_i = \frac{\partial}{\partial x^i}$) is the dual basis of the tangent vector space, $T_p(M)'$, at all $p \in M$:

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▫ Then: $g = g_{ij}\, \theta^i \otimes \theta^j$

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▫ Assume $\{e_i\}_{i=1}^{n}$ $\left(\text{e.g. } e_i = \frac{\partial}{\partial x^i}\right)$ is the dual basis of the tangent vector space, $T_p(M)'$, at all $p \in M$:

$$\theta^i(e_r) = \delta^i_r$$

▫ Then: $g(e_r, e_s) = g_{ik}\, \theta^i(e_r)\, \theta^k(e_s)$

$$= g_{rs}$$

For Space-Time the equivalence

GR always assumes Lorentzian mfld.

▫ M is called Lorentzian, if for all $p \in M$ th

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For Space-Time the equivalence principle requires:

GR always assumes Lorentzian mfld.

- M is called Lorentzian, if for all $p \in M$ there are bases so that $g_{ij}(p) = \eta_{ij} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

I.e.: $\forall p \in M$, $\exists$ dual bases θ^i, e_j, so that $g(e_r, e_s) = \eta_{rs}$ (We'll later use the freedom to choose the θ^i basis this way to get $g_{ij}(x) = \eta_{ij}$!)

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$e_i(x)$

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