Title: All rigid N=2 supersymmetric backgrounds and actions

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Abstract: <p>I will discuss how to classify (up to discrete identifications) all rigid 4D N=2 supersymmetric backgrounds in both Lorentzian and Euclidean signatures that preserve eight real supercharges. These include backgrounds such as warped S_3Ã—R, warped AdS_3Ã—R, and AdS_2Ã—S^2, as well as some more exotic geometries. I will also address how to construct all supersymmetric two-derivative actions involving hypermultiplets and vector multiplets in these backgrounds.</p>

All rigid $N = 2$ supersymmetric backgrounds and actions

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Based on [1505.03500] with Gianluca Inverso and Ivano Lodato

Motivation: Rigid SUSY on curved manifolds

Lots of work about exploiting SUSY on curved manifolds

- Wilson loop observables in $N = 4$ on S^4 [Pestun '07]
- Partition functions of $N = 2$ theories on S^3 to test various dualities [Kapustin, Willett, Yaakov '10]
- Computation of various indices for supersymmetric theories, etc. [Romelsberger '07] see also [Jafferis; Hama, Hosomichi, Lee; Imamura, Yokoyama; · · ·]

But how does one put a known supersymmetric field theory on a curved manifold in the first place?

[Festuccia and Seiberg] gave a systematic scheme...

Derive rigid SUSY from SUGRA.

Motivation: Rigid SUSY on curved manifolds

Characterizing rigid manifolds with some SUSY

- 4D $N = 1$ theories with one or more supercharges and applications
  - Classification of possible Euclidean theories [Dumitrescu, Festuccia, Seiberg '12]
  - Lorentzian theories (and holography) [Cassani, Klare, Martelli, Tomasiello, Zaffaroni '12]

Not a lot of work on 4D $N = 2$

- $N = 2$ theories have interesting features and more SUSY to exploit...
- Classification of backgrounds with one supercharge: $\exists$ CKV [Gupta, Murthy; Klare, Zaffaroni '13]

We will address the following questions:

What are all curved backgrounds consistent with *full* rigid $N = 2$ SUSY?
What are all rigid actions for vector multiplets and hypermultiplets?

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Lessons from rigid supergravity

Take a pause and recall the lesson of [Festuccia-Seiberg '11]:

A rigid SUSY matter action can be thought of as a coupled matter-SUGRA action with SUGRA fixed as background... and the auxiliary fields are important.

Finding a rigid SUSY means solving the SUGRA Killing spinor equation.

$$\delta\psi_m = 2\mathcal{D}_m\xi(x) + \text{auxiliary fields} = 0$$

Generically, $\xi(x) = A(x)\epsilon$ in terms of constants ϵ.

for S^n, AdS_n, H^n see [Lü, Pope, Rahmfeld '98]

Two observations:

- Number of solutions ϵ = number of rigid supercharges
  $\implies$ Requring more supercharges gives stronger conditions.
- Choice of $\mathcal{N}$ (and off-shell sugra) determines the form of the equation.
  $\implies$ Increasing $\mathcal{N}$ gives weaker conditions b/c more auxiliary fields.

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Lessons from rigid supergravity

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Example: Different off-shell $N=1$ SUGRAs lead to different backgrounds.

- Old minimal SUGRA: auxiliaries G_a and M

$$\mathcal{D}_m \xi_\alpha + iG^b(\eta_{ab} + \sigma_{ab})_\alpha{}^\beta \xi_\beta + iM\,(\sigma_m \bar{\xi})_\alpha = 0$$

  - Four (Euclidean) supercharges: $\mathbb{R} \times S^3$, $\mathbb{R} \times H^3$, S^4, or H^4

- New minimal SUGRA: $U(1)_R$ gauge field A_m and two-form auxiliary B_{mn}

$$\mathcal{D}_m^{(A)} \xi_\alpha + i\tilde{H}^b(\eta_{ab} + \sigma_{ab})_\alpha{}^\beta \xi_\beta = 0$$

  - Four (Euclidean) supercharges: $\mathbb{R} \times S^3$ or $\mathbb{R} \times H^3$

We should use the most general auxiliaries for $N=2$.

What is the most general off-shell $N=2$ SUGRA?

General off-shell $N = 2$ SUGRA

Conformal SUGRA: 24b+24f

$e_m{}^a \quad \psi_{m\alpha}{}^i \quad V_m{}^i{}_j \quad V_m \quad \| \quad W^-_{ab} \quad \chi_{\alpha i} \quad D$

Use the longest possible compensator

General scalar multiplet: 128b + 128f

$\Omega \quad \lambda_{\alpha i} \quad \| \quad Y_{ab} \quad S^{ij} \quad G_a \quad G_a{}^i{}_j \quad \cdots$

General Poincaré SUGRA

$e_m{}^a \quad \psi_{m\alpha}{}^i \quad V_m{}^i{}_j \quad V_m \quad \| \quad \Xi_{ab} \,,\quad S^{ij} \,,\quad G_a \,,\quad G_a{}^i{}_j \,,\quad \cdots$

Convenient to combine $\Xi_{ab} := Y_{ab} - W^-_{ab}$

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General SUGRA to rigid SUSY

General SUGRA Killing spinor equation:

$$0 = \delta_Q \psi_{m\alpha}{}^i = 2\mathcal{D}_m \xi_{\alpha i} - i\bar{S}_{ij}(\sigma_m \bar{\xi}^j)_\alpha + i\bar{\mathcal{Z}}_{mn}(\sigma^n \bar{\xi}_i)_\alpha$$
$$+ 4iG^n(\sigma_{nm}\xi_i)_\alpha - 2G^{nj}{}_i(\sigma_n \bar{\sigma}_m \xi_j)_\alpha$$

Helpful to express this in superspace... [Howe '82]

General SUGRA algebra (schematic form)

$$\{D_\alpha{}^i, D_\beta{}^j\} = \text{Lorentz and } R\text{-symmetry curvatures},$$
$$\{D_\alpha{}^i, \bar{D}_{\dot\beta j}\} = -2i\,\delta^i_j D_{\alpha\dot\beta} + \text{Lorentz and } R\text{-symmetry curvatures}$$
$$\text{curvatures involve:} \quad S^{ij}, \quad \mathcal{Z}_{ab}, \quad G_a, \quad G_a{}^{ij} = \epsilon^{jk} G_a{}^i{}_k$$

A rigid SUSY must leave the curvatures invariant.

$$\delta_\xi S^{ij} = \xi^\alpha_k D^k_\alpha S^{ij} = 0 \quad \Longrightarrow \quad D^k_\alpha S^{ij} = 0 \quad \Longrightarrow \quad \{D^k_\alpha, D^l_\beta\} S^{ij} = 0$$

Integrability conditions imply that all curvatures are (covariantly) constant.

[Kuzenko '12; + Novak, Tartaglino-Mazzucchelli '14]

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From constant curvatures to coset spaces

Riemann tensor is explicitly determined

$$R_{ab}{}^{cd} = S^{ij}\bar{S}_{ij}\delta_a{}^{[c}\delta_b{}^{d]} - \frac{1}{2}(\mathcal{Z}_{ab}\bar{\mathcal{Z}}^{cd} + \bar{\mathcal{Z}}_{ab}\mathcal{Z}^{cd})$$
$$+ 8\,G^2\delta_a{}^{[c}\delta_b{}^{d]} - 16\,G_{[a}G^{[c}\delta_{b]}{}^{d]} + 4G^f_{ij}G^{ij}_f\delta_a{}^{[c}\delta_b{}^{d]} - 8G^{ij}_{[a}G^{[c}_{ij}\delta_{b]}{}^{d]}$$

Although all curvature tensors specified, we really want to know:

- What is the (global) structure of these spaces?
- How do we know the full set of Killing spinors actually exists?

We can easily resolve all these issues if we realize one important fact:

constant curvature tensors $\Longrightarrow$ (super) coset space

More accurately: for any superspace algebra with constant curvatures, we can construct a (global) super coset space with the same curvatures.

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Review: Coset spaces

Consider a Lie group G with a subgroup H.
The coset space G/H is the space of equivalences $g \cong gh$ for $g \in G$ and $h \in H$.
Take Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and assume $\mathfrak{g} = \mathfrak{K} \oplus \mathfrak{h}$ where

$$[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h} \ , \qquad [\mathfrak{h}, \mathfrak{K}] = \mathfrak{K} \ , \qquad [\mathfrak{K}, \mathfrak{K}] = \mathfrak{K} + \mathfrak{h}$$

Schematically, G/H is generated by $\mathfrak{K}$.

More constructively...

- Denote $\mathfrak{K} = \{\hat{P}_a\}$ and $\mathfrak{h} = \{\hat{M}_{ab}\}$.
- Introduce representative coset element: $L(x) = \exp(x^a \hat{P}_a)$.
- Action of G on the coset space G/H can always be written

$$g L(x) = L(x') h(g, x) \cong L(x') \qquad \forall g \in G$$

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Review: Local coset space geometry

Local geometry is encoded in the coset representative $L(x) = \exp(x^a \hat{P}_a)$.

1. Construct Cartan-Maurer form

$$L^{-1}\mathrm{d}L = \mathrm{d}x^m \Big(e_m{}^a(x)\hat{P}_a + \frac{1}{2}\omega_m{}^{ab}(x)\hat{M}_{ab} \Big)$$

2. Covariant derivs $\mathcal{D}_a = e_a{}^m(\partial_m - \frac{1}{2}\omega_m{}^{ab}M_{ab})$ inherit algebra.

$$[\hat{P}_a, \hat{P}_b] = -f_{ab}{}^c\hat{P}_c - \frac{1}{2}f_{ab}{}^{cd}\hat{M}_{cd} \ ,$$
$$[\mathcal{D}_a, \mathcal{D}_b] = -T_{ab}{}^c\mathcal{D}_c - \frac{1}{2}R_{ab}{}^{cd}M_{cd} \ , \qquad T_{ab}{}^c = f_{ab}{}^c \ , \quad R_{ab}{}^{cd} = f_{ab}{}^{cd} \ .$$

3. Local isometries are the Killing vectors $\xi^a(x)$ that obey $\mathcal{D}_{(a}\xi_{b)} = 0$.
But they are *also* encoded algebraically

$$L^{-1}(\epsilon^a\hat{P}_a + \tfrac{1}{2}\lambda^{ab}\hat{M}_{ab})L = \xi^a(x)\hat{P}_a + \tfrac{1}{2}\xi^{ab}(x)\hat{M}_{ab}$$

Schematically, $\xi^a(x) = A(x)^a{}_b\,\epsilon^b + B(x)^a{}_{bc}\,\lambda^{bc}$

Supercoset spaces

Same approach holds for supercoset spaces.

- Choose a supercoset representative: $L = \exp(x^a \hat{P}_a + \theta_i \hat{Q}^i + \bar{\theta}^i \hat{\bar{Q}}_i)$.
- Killing spinors are trivial to calculate (*only* $\theta = 0$ *part is needed*)

$$L^{-1}(\epsilon^i \hat{Q}_i + \bar{\epsilon}_i \hat{\bar{Q}}^i) L = \xi^i(x) \hat{Q}_i + \bar{\xi}_i(x) \hat{\bar{Q}}^i$$

Schematically, $\begin{pmatrix} \xi^i \\ \bar{\xi}_i \end{pmatrix} = A(x) \begin{pmatrix} \epsilon^i \\ \bar{\epsilon}_i \end{pmatrix}$

see e.g. [Alonso-Alberca, Lozano-Tellechea, Ortin '02]

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Classifying the allowed spaces

Background constant fields:

$$S^{ij}\ , \qquad \mathcal{Z}_{ab}\ , \qquad G_a\ , \qquad G_a{}^i{}_j$$

- $\mathcal{Z}_{ab}$ is a complex field strength, $\mathrm{d}\mathcal{Z} = 0$.
  If the SUGRA algebra has (complex) central charge, $\mathcal{Z}_{ab}$ is its field strength.
- G_a may be thought of as dual of three-form field strength H_{abc}.

Three sets of solutions to integrability conditions for background fields:

1. S_{ij} alone is nonzero
2. $G_a{}^{ij}$ alone is nonzero and decomposes as $G_a{}^{ij} = g_a t^{ij}$
3. G_a and/or $\mathcal{Z}_{ab}$ are nonzero and obey $G^a \mathcal{Z}_{ab} = 0$

Background fields determine R-symmetries in two ways:

- Their VEVs break some of the R-symmetry.
- They generate R-symmetry within the SUSY algebra.

Classifying the allowed spaces

Background constant fields:

$$S^{ij}\ , \qquad \mathcal{Z}_{ab}\ , \qquad G_a\ , \qquad G_a{}^i{}_j$$

- $\mathcal{Z}_{ab}$ is a complex field strength, $\mathrm{d}\mathcal{Z} = 0$.
  If the SUGRA algebra has (complex) central charge, $\mathcal{Z}_{ab}$ is its field strength.
- G_a may be thought of as dual of three-form field strength H_{abc}.

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Menagerie of $N = 2$ backgrounds: The simplest cases

Start with the simple Lorentzian cases...

Geometry	Active backgrounds	Supergroup
AdS_4	S^{ij}	$OSp(2\vert 4)$
$\mathbb{R} \times S^3$	$G_a{}^i{}_j$ timelike	$SU(2\vert 1) \times SU(2\vert 1)$
$\text{AdS}_3 \times \mathbb{R}$	$G_a{}^i{}_j$ spacelike	$SU(1,1\vert 1) \times SU(1,1\vert 1)$
plane wave	$G_a{}^i{}_j$ null	
$\mathbb{R} \times S^3$	G_a timelike	$SU(2\vert 2) \times SU(2)$
$\text{AdS}_3 \times \mathbb{R}$	G_a spacelike	$SU(1,1\vert 2) \times SU(1,1)$
plane wave	G_a null	
$\text{AdS}_2 \times S^2$	Ξ_{ab} elliptic + hyperbolic	$D(2,1;\alpha)$
$\mathbb{R}^{1,1} \times S^2$	Ξ_{ab} elliptic ($\Xi^2 > 0$)	$D(2,1;\infty) = SU(2\vert 2)$
$\text{AdS}_2 \times \mathbb{R}^2$	Ξ_{ab} hyperbolic ($\Xi^2 < 0$)	$D(2,1;0) = SU(1,1\vert 2)$
plane wave	Ξ_{ab} parabolic ($\Xi^2 = 0$)	

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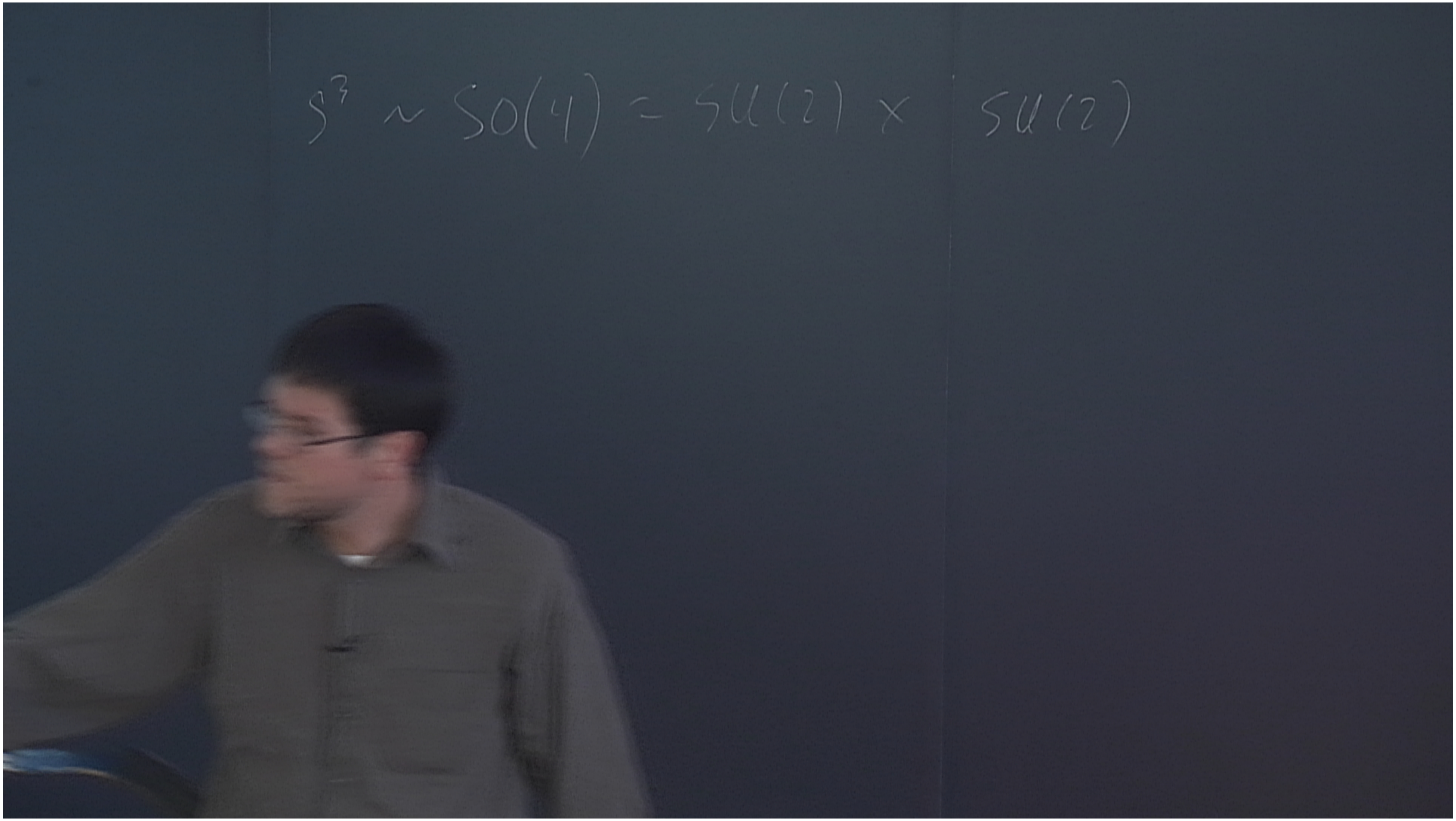
S^3 ~ SO(4) = SU(2) x SU(2)

$AdS_2 \times S^2$ and $D(2,1;\alpha)$

Some historical observations

- A non-trivial spherically symmetric solution of $N = 2$ gauged supergravity is $AdS_2 \times S^2$ of equal radii. The eight supercharges give $SU(1,1|2)$.

  This describes the near horizon geometry of an extremal BPS Reissner-Nordstrom black hole. (Related to *attractor mechanism*.)

  [Ferrara, Kallosh, Strominger '95; Ferrara, Kallosh '96]

- Can be generalized to different radii supergeometries with supergroup $D(2,1;\alpha)$. (Not SUGRA solutions!) [Bandos, Ivanov, Lukierski, Sorokin '02]

$D(2,1;\alpha)$ has bosonic part $SU(1,1) \times SU(2) \times SU(2)_R$ with $Q_{\bar a\,\bar\alpha\,i} \in (\mathbf{2},\mathbf{2},\mathbf{2})$

$$\{Q_{\bar a\,\bar\alpha\,i},\ Q_{\bar b\,\bar\beta\,j}\} = -\lambda_- \epsilon_{\bar\alpha\bar\beta}\epsilon_{ij} \underbrace{T_{\bar a\bar b}}_{AdS_2} - \lambda_+ \epsilon_{\bar a\bar b}\epsilon_{ij} \underbrace{T_{\bar\alpha\bar\beta}}_{S^2} + (\lambda_+ + \lambda_-)\epsilon_{\bar a\bar b}\epsilon_{\bar\alpha\bar\beta} \underbrace{I_{ij}}_{SU(2)_R}$$

The Euclidean version has been studied recently with various applications.

[Bawane, Bonelli, Ronzani, Tanzini '14; Sinamuli '14; Rodriguez-Gomez and Schmude '15]

Deforming these spaces: Squashing $\mathbb{R} \times S^3$

Squashing the S^3 is only possible for one of the supergroups

G_a	${G_a}^{ij}$
$SU(2\|2) \times SU(2)$	$SU(2\|1) \times SU(2\|1)$
$\downarrow \mathcal{Z}_{ab}$	$\downarrow$
$SU(2\|2) \times U(1)$	Not possible

Geometrically, we turn on $\mathcal{Z}_{ab}$ along S^3 and squash along $S^1 \hookrightarrow S^3 \to S^2$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{v}{16|G^2|}\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + v(\mathrm{d}\omega + \cos\theta\,\mathrm{d}\phi)^2\right]$$

$$v \equiv \left(1 + \frac{|\mathcal{Z}|^2}{32|G^2|}\right)^{-1}, \qquad 0 \leq v < 1$$

Can repeat for spacelike G_a to give deformations of $AdS_3 \times \mathbb{R}$.

Menagerie of $N = 2$ backgrounds: Mixed cases

Mixed cases arise with G_a and $\mathcal{Z}_{ab}$ turned on

Active backgrounds	Geometry
G_a timelike	$\mathbb{R} \times S^3$
$\mathcal{Z}_{ab}$ elliptic ($\lvert\mathcal{Z}\rvert^2 > 0$)	$\mathbb{R} \times S^3$ squashed
G_a null	plane wave
$\mathcal{Z}_{ab}$ elliptic	'lightlike' $S^3 \times \mathbb{R}$
$\mathcal{Z}_{ab}$ parabolic ($\lvert\mathcal{Z}\rvert^2 = 0$)	plane wave
G_a spacelike	$AdS_3 \times \mathbb{R}$
$\mathcal{Z}_{ab}$ elliptic	
$0 < \lvert\mathcal{Z}\rvert^2 < 32\, G^2$	timelike stretched $AdS_3 \times \mathbb{R}$
$\lvert\mathcal{Z}\rvert^2 = 32\, G^2$	$Heis_3 \times \mathbb{R}$
$\lvert\mathcal{Z}\rvert^2 > 32\, G^2$	warped 'Lorentzian' $S^3 \times \mathbb{R}$
$\mathcal{Z}_{ab}$ parabolic	null warped $AdS_3 \times \mathbb{R}$
$\mathcal{Z}_{ab}$ hyperbolic ($\lvert\mathcal{Z}\rvert^2 < 0$)	spacelike squashed $AdS_3 \times \mathbb{R}$

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Euclidean backgrounds

The entire analysis can be repeated for Euclidean signature.

- We chose 4D $N = 2$ spinors to be symplectic Majorana-Weyl.
  Left-handed and right-handed supercharges completely independent of each other.
  $\Longrightarrow$ independently choose S_{ij} and $\widetilde{S}_{ij}$ as well as $\mathcal{Z}_{ab}$ and $\widetilde{\mathcal{Z}}_{ab}$

Active backgrounds	Geometry
S^{ij} and $\widetilde{S}^{ij}$	S^4 and H^4
${G_a}^i{}_j$	$H^3 \times \mathbb{R}$
G_a	$S^3 \times \mathbb{R}$
$\mathcal{Z} \cdot \widetilde{\mathcal{Z}} < 32\,\lvert G \rvert^2$	Warped $S^3 \times \mathbb{R}$
$\mathcal{Z} \cdot \widetilde{\mathcal{Z}} = 32\,\lvert G \rvert^2$	$\text{Heis}_3 \times \mathbb{R}$
$\mathcal{Z} \cdot \widetilde{\mathcal{Z}} > 32\,\lvert G \rvert^2$	Warped Euclidean $AdS_3 \times \mathbb{R}$
$\mathcal{Z}_{ab}$ and $\widetilde{\mathcal{Z}}_{ab}$	$H^2 \times S^2$, $\mathbb{R}^2 \times S^2$ and $H^2 \times \mathbb{R}^2$
S^{ij}, $\mathcal{Z}_{ab}$	Flat space (deformed susy)

- Last case is flat space but includes full SUSY limit of Ω background.
  see e.g. [Klare, Zaffaroni '13]

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(Abelian) vector multiplets

The Minkowski story

- Vector multiplet superfield: $\mathcal{X}^I \sim X^I + \theta^{\alpha i}\lambda^I_{\alpha i} + (\theta_j \sigma^{ab}\theta^j)F^I_{ab} + (\theta_i\theta_j)Y^{ijI}$
- Holomorphic prepotential: $F(\mathcal{X})$

  Target space is special Kähler with potential $K = iX^I\bar{F}_I - i\bar{X}^I F_I$
- Action principle:

$$-i\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,F(\mathcal{X}) + \text{h.c.}\,, \qquad\qquad g_{IJ} := \partial_I\bar{\partial}_J K = 2\,\mathrm{Im}\,F_{IJ}$$

$$= \int \mathrm{d}^4x\Big[-g_{IJ}\mathcal{D}_a X^I\mathcal{D}^a\bar{X}^J - \frac{1}{4}\mathrm{Im}\,F_{IJ}\,F^I_{ab}F^{abJ} - \frac{1}{4}\mathrm{Re}\,F_{IJ}\,F^I_{ab}\widetilde{F}^{abJ} + \cdots\Big]$$

- Duality transformations lie in $\mathrm{ISp}(2n,\mathbb{R})$

  field strengths and duals: $\begin{pmatrix} F_{ab}{}^I \\ G_{abI}\end{pmatrix} \longrightarrow \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J\end{pmatrix}\begin{pmatrix} F_{ab}{}^J \\ G_{abJ}\end{pmatrix}$

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(Abelian) vector multiplets

General rigid curved background (8 supercharges)

- Superfield / superspace description fundamentally unchanged.
- If $U(1)_R$ is present, $F(X)$ must be superconformal.
- Background fields introduce new couplings in the action.
  - Ξ_{ab} gives new moment couplings like a background vector multiplet, e.g.

$$\frac{1}{4} F_{ab}{}^{I} \Big(\epsilon^{abcd} \Xi_{cd} F_I - \text{Re}\, F_{IJ} X^J - \Xi^{ab} g_{IJ} X^J + \text{h.c.} \Big)$$

  - G_a gives composite $B \wedge F$ term via its dual two-form

$$2 \epsilon^{abcd} B_{ab}\, g_{IJ}\, D_c X^I D_d \bar{X}^J$$

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- There is a simple interpretation of these results in terms of a frozen background vector multiplet generating Ξ_{ab}.

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(Abelian) vector multiplets

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General rigid curved background (8 supercharges)

SUSY transformations are deformed

$$\delta\lambda_{\alpha i}{}^I = (F_{ab}{}^I + \mathcal{Z}_{ab}X^I + \bar{\mathcal{Z}}_{ab}\bar{X}^I)(\sigma^{ab}\xi_i)_\alpha + (Y_{ij}{}^I + 2S_{ij}X^I)\xi_\alpha{}^j \\ - 2i\,\mathcal{D}_a X^I\,(\sigma^a\bar{\xi}_i)_\alpha + 4i\,G_{a\,ij}X^I\,(\sigma^a\bar{\xi}^j)_\alpha$$

This modifies the conditions for SUSY vacua in certain backgrounds

- $G_{a\,ij}X^I = 0$ — If $U(1)_R$ present, X^I must vanish.
- $Y_{ij}{}^I = -2S_{ij}X^I = -2\bar{S}_{ij}\bar{X}^I$ — Fixes phase of X^I
- $F_{ab}{}^I = -\mathcal{Z}_{ab}X^I - \bar{\mathcal{Z}}_{ab}\bar{X}^I$ — Generalized attractor equation

Last result generalizes the standard BPS attractor equation

[Ferrara, Kallosh, Strominger '95; Ferrara, Kallosh '96]

Straightforward to generalize to non-abelian vector multiplets.

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The hypermultiplets

<u>The Minkowski story</u>

Hypermultiplet scalars: complex scalars A, B in conjugate reps

Interacting case generically described by sigma model with fields ϕ^M

- Metric g_{MN} describes hyperkähler target space manifold
- Three covariantly constant integrable complex structures $(\mathcal{J}_I)^M{}_N$ obeying

$$\mathcal{J}_I \mathcal{J}_J = -\delta_{IJ} + \varepsilon_{IJK} \mathcal{J}_K$$

∃ Description in extended harmonic / projective superspace

[Galperin, Ivanov, Ogievetsky, Sokatchev '88; Lindström, Roček '08]

- Fields ϕ^M grouped into superfields $\mathcal{Q}$ depending on auxiliary S^2 with coordinates u_i^+ and u_i^-
- Harmonic case described by "Hamiltonian" $\mathcal{H}(\mathcal{Q}, u_i^\pm)$
- Projective (twistor) case described by "canonical transformation" $\mathcal{F}(\mathcal{Q}, u_i^+)$. Projective version can be derived from harmonic. [DB '12]
- One can produce component actions using either. [DB 1410.3604, 1508.07718]

The hypermultiplets

The Minkowski story
Hypermultiplet scalars: complex scalars A, B in conjugate reps

Interacting case generically described by sigma model with fields ϕ^M

- Metric g_{MN} describes hyperkähler target space manifold
- Three covariantly constant integrable complex structures $(\mathcal{J}_I)^M{}_N$ obeying

$$\mathcal{J}_I \mathcal{J}_J = -\delta_{IJ} + \varepsilon_{IJK} \mathcal{J}_K$$

$\exists$ Description in extended harmonic / projective superspace
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The hypermultiplets

General rigid curved background (8 supercharges)

Same basic procedure holds... *except* target space structure has additional requirements inherited from R-symmetry in SUSY algebra.

SU(2) R-symmetry	Target space
none	arbitrary HK
$\mathrm{SO}(2)_R$	HK with special K.V. that rotates $\mathcal{J}_I$
$\mathrm{SU}(2)_R$	hyperkähler cone (conformal K.V.)

- These restrictions appear in harmonic / projective superspace because prepotential $\mathcal{H}$ or $\mathcal{F}$ can only depend on S^2 coordinates in certain ways.
  The $SO(2)_R$ case was previously noted in AdS_4 and AdS_5.
  [DB, Kuzenko '11; Bagger, Xiong '11]
- New couplings present: e.g. $G_a{}^{ij}$ contributes $B \wedge F$ term

  [illegible]

- Full SUSY configurations must obey:
  - If $SU(2)_R$ present, scalars at origin of HK cone. (C.K.V. vanishes)
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$$\epsilon^{mnpq} {B_{mn}}^{ij} \, \partial_p \phi^M \partial_q \phi^N \Omega_{MN\,ij}$$

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Example: $N = 2^*$ action

Choose diagonal metric $g_{IJ} = \delta_{IJ}$ and adjoint hypermultiplet with scalars (A^I, B_I).
In Minkowski background, mass term m softly breaks $N = 4$ to $N = 2^*$.

- In a general rigid background, the Lagrangian is

$$\mathcal{L} = -\mathcal{D}_m \bar{A}_I \mathcal{D}^m A^I - \mathcal{D}_m \bar{B}^I \mathcal{D}^m B_I - \mathcal{D}_m \bar{X}^I \mathcal{D}^m X^I - \frac{1}{8} F_{ab}{}^I F^{abI}$$
$$- \frac{1}{2} F_{ab}{}^I (\mathcal{Z}^{ab+} X^I + \bar{\mathcal{Z}}^{ab-} \bar{X}^I) + \mathcal{L}_{BF} + \mathcal{L}_{\text{pot}} + \text{fermions}$$

- The BF term involves couplings to the potentials for G_a and $G_a{}^{ij}$:

- New contributions to scalar potential:

Conclusions / Open questions

We have found all (global) rigid $N = 2$ spaces and constructed general rigid actions for vector and hypermultiplets. Some gaps / unanswered questions.

- We assumed global manifolds, but what about discrete quotients?
  e.g. The $\mathbb{R} \times S^3$: one can quotient along $U(1)$ fiber, giving a lens space $S^3/\mathbb{Z}_p$.

- Is there a dynamical origin of all rigid supersymmetric backgrounds?
  Not for $4D$ supergravity + normal matter! [Hristov, Looyestijn, Vandoren '09]
  But perhaps by compactifying higher dimensional theories.
  e.g. $D(2, 1; \alpha)$ from 6D theory with vacuum $AdS_2 \times S^2 \times S^2$
  [Zarembo '10; Wulff '14]

- Many spaces include trivial $\mathbb{R}$ factors, so reduction to Euclidean or Lorentzian 3D $N = 4$ is clearly possible. What are the other 3D $N = 4$ spaces?

- We exploited coset structure to radically simplify analysis.
  What about four supercharges for $N = 2$ where this does not apply?

- General compensator doesn't seem to give new backgrounds for our case.
  What about for four supercharges for $N = 2$ or two supercharges for $N = 1$?
  see [Triendl 1509.02926]

Thanks for your attention!