Title: TBA

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Abstract:

Coarse-graining of 3D spin foam models

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September 30, 2015

Work in collaboration with B. Dittrich

Outlook

- Continuum limit of spin foam models
- Tensor Renormalization Group algorithm
- Algorithm for 3D constrained spin foam models
- Discussion of the results
- Work in progress

Continuum limit

- **Discretization of space-time:**

  ⇒ Breaking of diffeomorphism symmetry for 4D gravity theories

  ⇒ Dependence of the path integral on the choice of triangulation

Continuum limit of the path integral

- **Achieving the continuum limit**: construction of a cylindrical consistent path integral w.r.t dynamical embedding maps [Bahr 14, Dittrich 12, Dittrich 14]
- Discrete notion of symmetry restored ⇔ discretization independence
- **Approximation scheme**: iterative coarse-graining procedure

  ⇒ Iterative improvement of the amplitudes

  ⇒ Fixed points of coarse-graining flow enjoy enhanced symmetries [Bahr, Dittrich 09]

Tensor network renormalization

- Iterative improvement of the amplitude via tensor network coarse-graining schemes
- 2D tensor networks are widely used in Condensed Matter Theory
  e.g. [Levin Nave 08 (TRG)][Gu Wen 09][Evenbly Vidal 14 (TNR)]
- Generalization to 3D: decorated tensor networks
  [Ditttich Mizera Steinhaus 14]

  $\Rightarrow$ Lattice gauge theory with abelian groups
- Modification of the algorithm to deal with gauge invariance with **non-abelian groups** and implement **simplicity constraints**

3D spin foam models

- Simplifications $\begin{cases} 3+1 \longrightarrow 2+1 \\ \text{Lie groups} \longrightarrow \text{finite groups} : \mathcal{S}_3,\ \text{q-groups} \end{cases}$
- Cubical regular lattice with **4**-valent intertwiners

  $\Rightarrow$ Necessary for implementation of **simplicity constraints**

$$\boxed{\text{Topological BF model} \longrightarrow \text{Spin foam models}}$$

- Study of the fate of the simplicity constraints throughout the coarse-graining procedure
- Constraints extend the phase space of standard lattice gauge theories

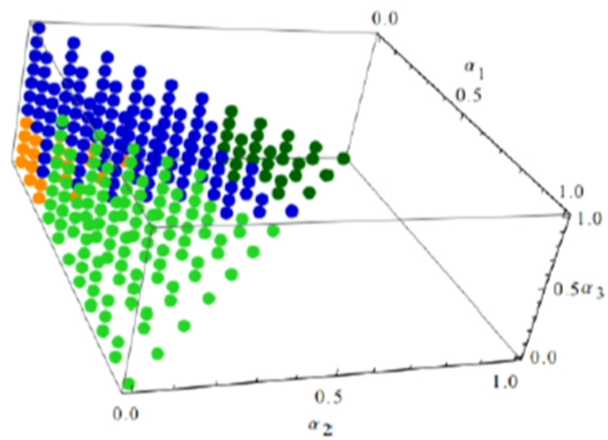
  $\rightarrow$ **new fixed points, new phases?**

  $\rightarrow$ new continuum representations
  [Dittrich Steinhaus 13]

Results with spin net models

[Dittrich, Martin-Benito, Schnetter 13][Dittrich, Martin-Benito, Steinhaus 14]

- Spin nets = 2D analogues of spin foam models
- Spin net models display a very rich phase structure



- End points of the CG flow are encoded in different colors
- Phase = set of parameters for which the system flows towards a given fixed point

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Embedding maps and vacuum

- New fixed points → new topological field theories
- Improved amplitudes define the dynamical embedding maps

  ⇒ New refinement limit

  ⇒ New vaccum

  ⇒ New representation of LQG
- Organization of the theory w.r.t. different notions of excitations

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Parametrization (1/2)

- Partition function:

$$Z = \int_G \prod_e dg_e \prod_f w_f(g_f) \quad \text{with} \quad w_f \equiv \overrightarrow{\prod_{e \subset f}} g_e$$

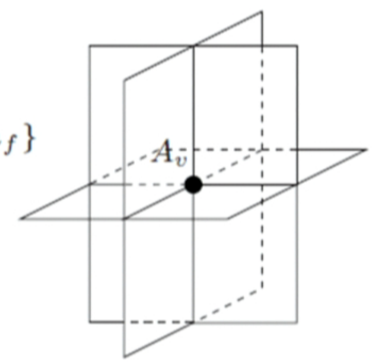
- Fourier transformation (group rep. → spin rep.): $w_f(g) = \sum_\rho \tilde{w}_f(\rho)\chi_\rho(h)$

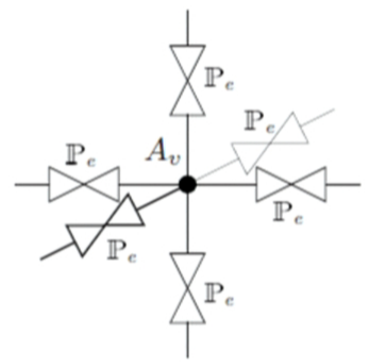
$$\boxed{Z = \sum_{\rho_f} \prod_f \tilde{w}_f(\rho_f) \prod_e (\mathbb{P}^e_{\text{Haar}})^{\{n_f\}_{f \supset e}}_{\{m_f\}_{f \supset e}}(\{\rho_f\}_{f \supset e})}$$

- Splitting of the Haar projector :

$$(\mathbb{P}^e_{\text{Haar}})^{\{n_f\}}_{\{m_f\}} = \sum_{\iota_e} {}^{\{n_f\}}|\iota_e\rangle\langle\iota_e|_{\{m_f\}}$$

- We contract the intertwiners ι_e associated with a vertex v to a **vertex amplitude** A_v





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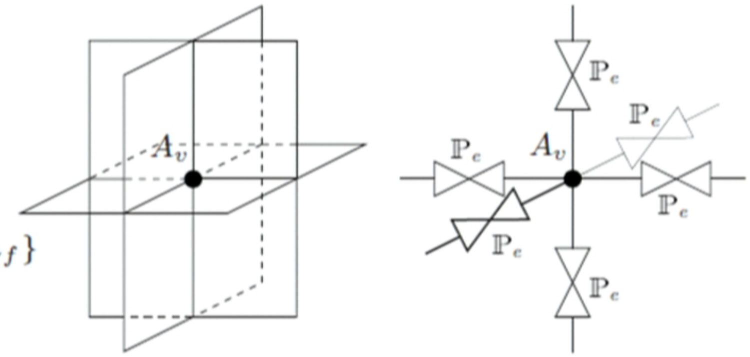


- Haar projectors attached to the edges

$$\mathbb{P}^e_{\text{Haar}} : \underbrace{\text{Inv}(V_{\rho_1} \otimes ... \otimes V_{\rho_4})}_{\text{Non-trivial invariant subspace}} \to \text{Inv}(V_{\rho_1} \otimes ... \otimes V_{\rho_4})$$

$\Rightarrow$ Implementation of simplicity constraints:

$$\mathbb{P}'_e \quad \text{projects onto} \quad V \subset \text{Inv}(V_{\rho_1} \otimes ... \otimes V_{\rho_4})$$

Tensor Renormalization Group

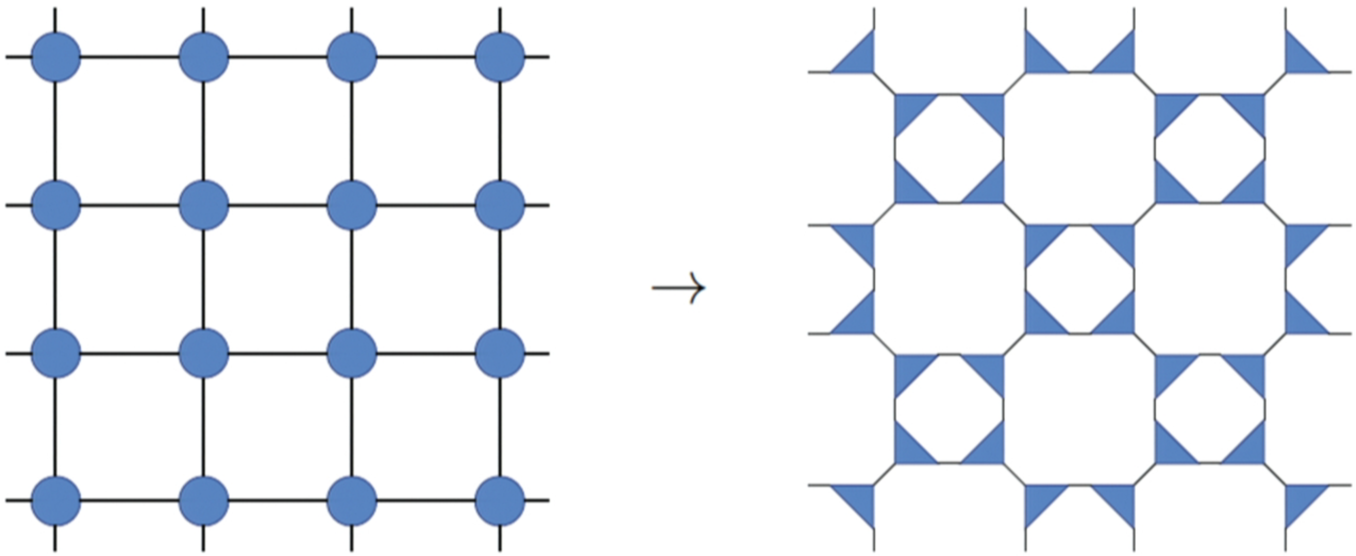
[Levin Nave 08]

- Renormalization of a 2D tensor network

  ⇒ Factorization of each tensor using **Singular Value Decompositions**

$$M_{AB} = \sum_K U_{AK} \lambda_K V^\dagger_{KB} = \sum_K \underbrace{(U_{AK}\sqrt{\lambda_K})}_{S^1_{AK}} \underbrace{(\sqrt{\lambda_K} V^\dagger_{KB})}_{S^2_{KB}}$$

  ⇒ Contraction of four isometries → new tensor



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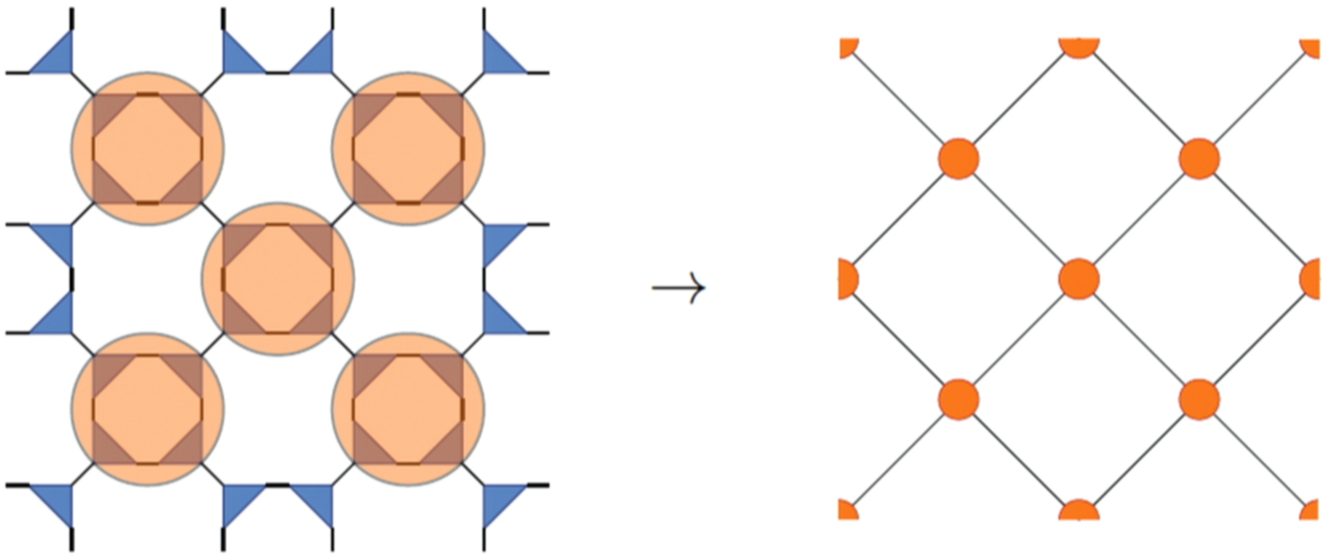
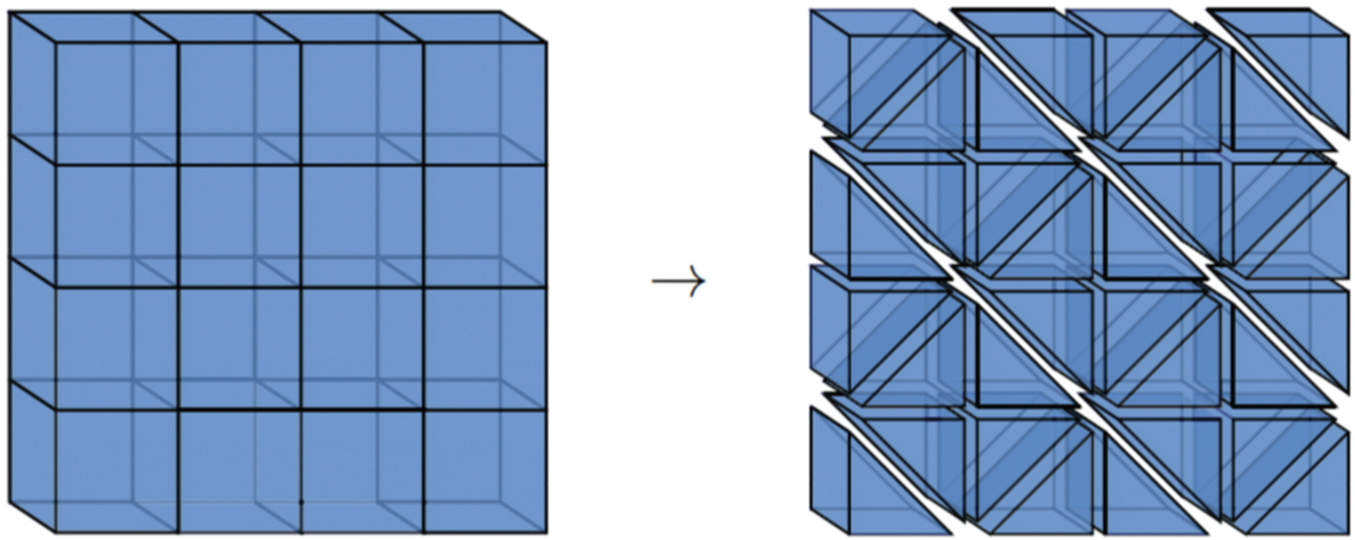


3D generalization of TRG

- TRG algorithm in each plane of the cubical lattice

  ⇒ Splitting of the cubes into prisms via SVD

  ⇒ Gluing of four prisms → new cube



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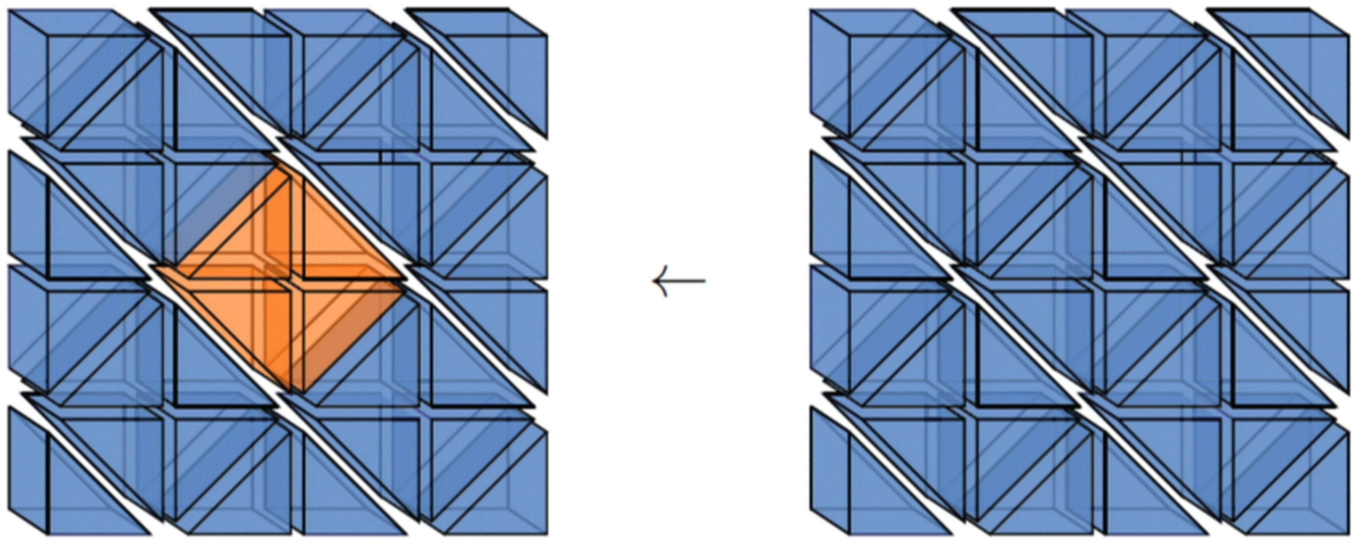


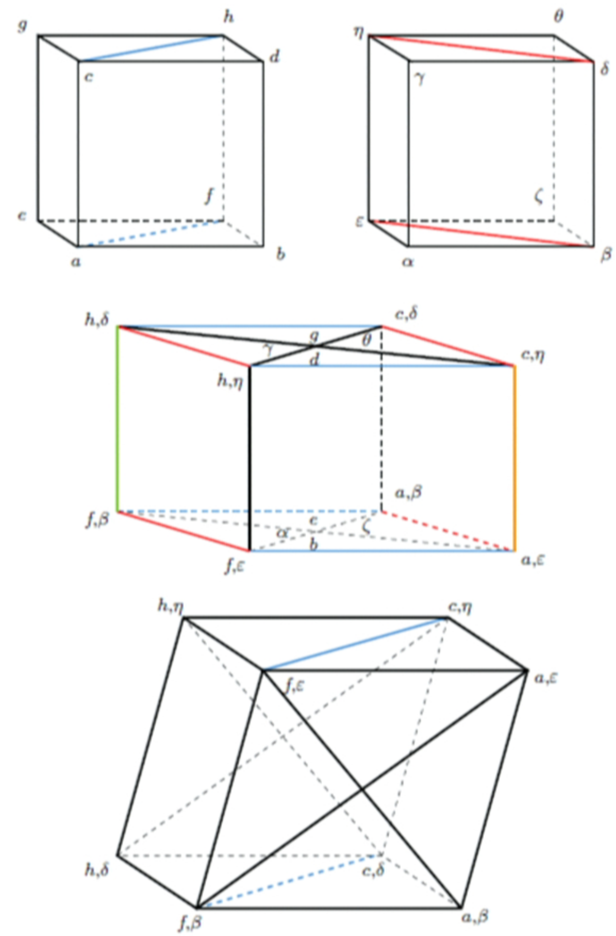
Overview

One iteration:

- Translation invariance ⇒ we focus on a single cube/tensor
- Splitting of the cube along two diagonals

  ⇒ 4 prisms
- The prisms are glued back together

  ⇒ New bigger shape
- Rotation



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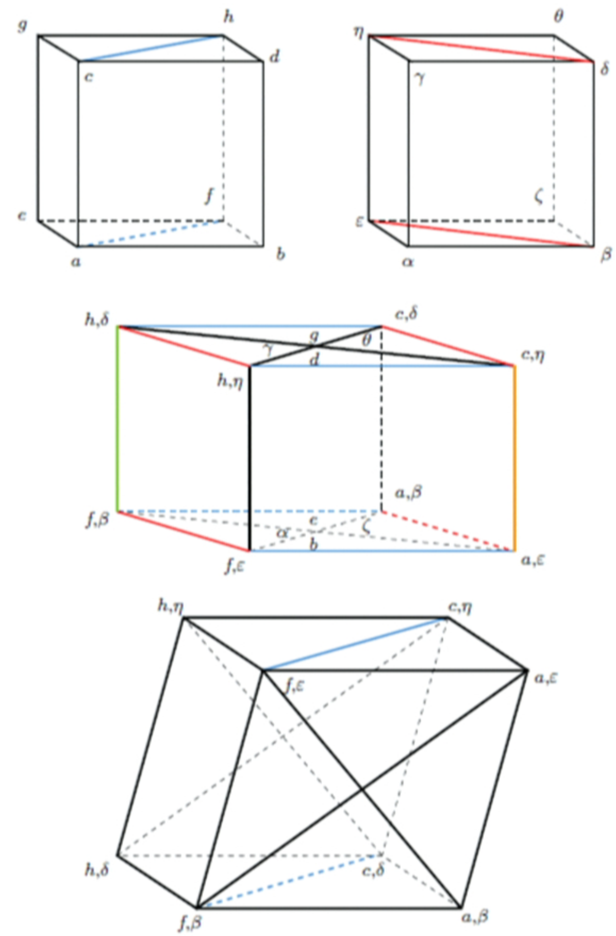


Spin network on the surface

- Change of perspective:

  Bulk lattice gauge theory with tensor based vertices $\rightarrow$ amplitudes associated to blocks

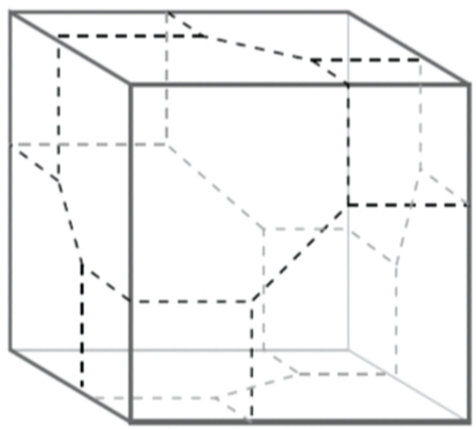
- Dual graph to the surface $\rightarrow$ **Spin networks** with three-valent vertices

- Completely gauge invariant spin network

  $\Rightarrow$ variables on each edge

  $\Rightarrow$ non-local coupling rules

  $\Rightarrow$ Loss of Gauss constraint during embedding



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Gauge fixing

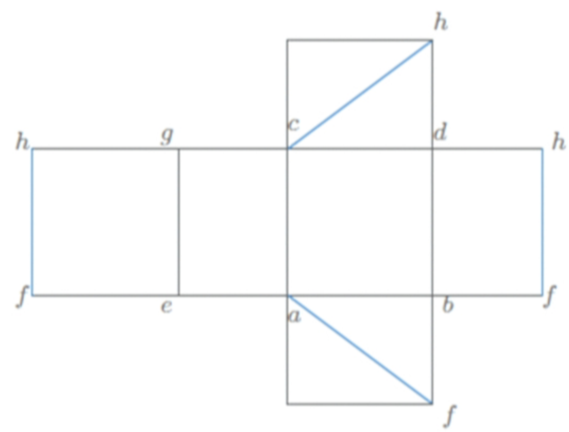
- Invariance under the gauge transformation

$$g_e \to k_{s(e)}^{-1} g_e k_{t(e)}, \quad \forall e \in \text{cube}$$

- Spin networks on the surface $\to$ gauge-fixing via spanning tree

  $\Rightarrow$ $L = E - V + 1$ leaves in one-to-one correspondence with the cycles

- **Initial tensor**:
  9 leaves/variables



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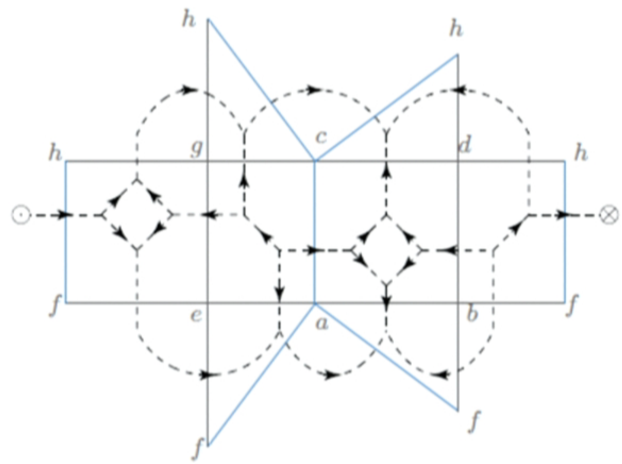

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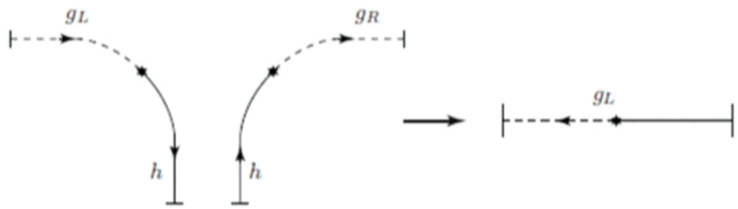
Gluing

- Integration over half edges



$$\Psi_{\text{glue}}(g_L) = \int_G dh \Psi_L(hg_L)\Psi_R(h)$$

$$= \sum_{\{\rho,m,n\}} \int_G dh \widetilde{\Psi}_L(\rho_L, m_L, n_L)\widetilde{\Psi}_R(\rho_R, m_R, n_R) D^{\rho_L}_{m_L n_L}(hg_L)\overline{D^{\rho_R}_{m_R n_R}(h)}\sqrt{d_{\rho_L} d_{\rho_R}}$$

$$= \sum_{\substack{\{\rho,m,n\}\\ p}} \int_G dh \widetilde{\Psi}_L(\rho_L, m_L, n_L)\widetilde{\Psi}_R(\rho_R, m_R, n_R) D^{\rho_L}_{m_L p}(h) D^{\rho_L}_{p n_L}(g_L)\overline{D^{\rho_R}_{m_R n_R}(h)}\sqrt{d_{\rho_L} d_{\rho_R}}$$

$$= \sum_{\substack{\rho,\{m,n\}\\ p}} \widetilde{\Psi}_L(\rho, m_L, n_L)\widetilde{\Psi}_R(\rho, m_R, n_R)\delta_{m_L, m_R}\delta_{p, n_R} D^{\rho}_{p n_L}(g_L)$$

$$= \sum_{\rho, n_R, n_L} \underbrace{\left(\sum_{m'} \frac{1}{\sqrt{d_\rho}} \widetilde{\Psi}_L(\rho, m', n_L)\widetilde{\Psi}_R(\rho, m', n_R)\right)}_{\widetilde{\Psi}_{\text{glue}}(\rho, n_R, n_L)} \sqrt{d_\rho} D^{\rho}_{n_R n_L}(g_L)$$

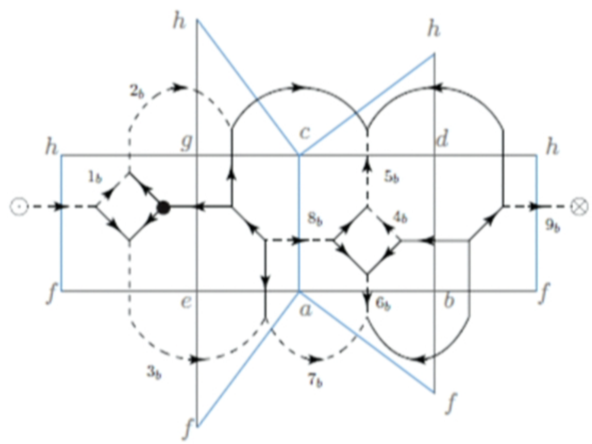
- Tensor formalism $\Rightarrow$ contraction of indices

Splitting

- Splitting of the amplitude via singular value decomposition

$$\boxed{M_{AB} = \sum_K U_{AK} \lambda_K V^\dagger_{KB} \sim S_{A1} S_{1B}}$$

- Cutting of the 3 leaves shared by both sides



- Distribution of the magnetic indices provided by the SVD

$\Rightarrow$ "super-index" $K \longrightarrow (m1, n1, m2, n2, m3, n3)$

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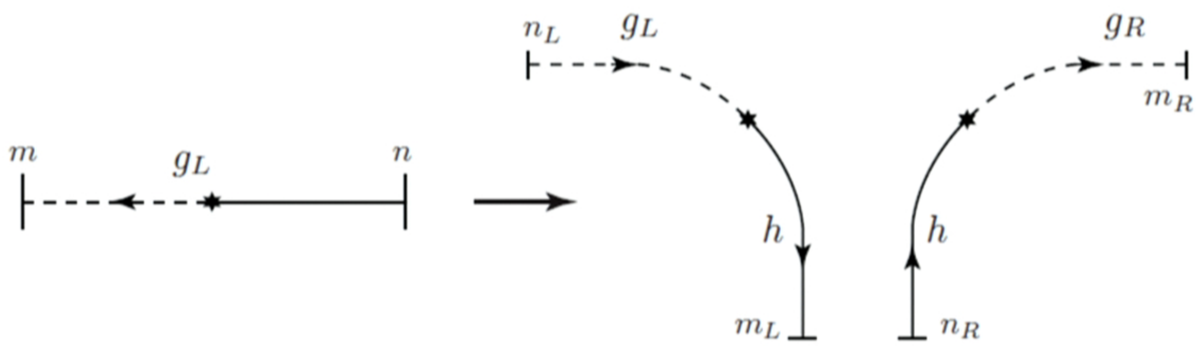

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Splitting

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- Key step of the approximation

  $\rightarrow$ Truncation by keeping only the first set of singular values
- The maps U and V define a **dynamical embedding mapping** from coarser to finer boundary graphs
- Cylindrical consistency w.r.t. to these embedding maps

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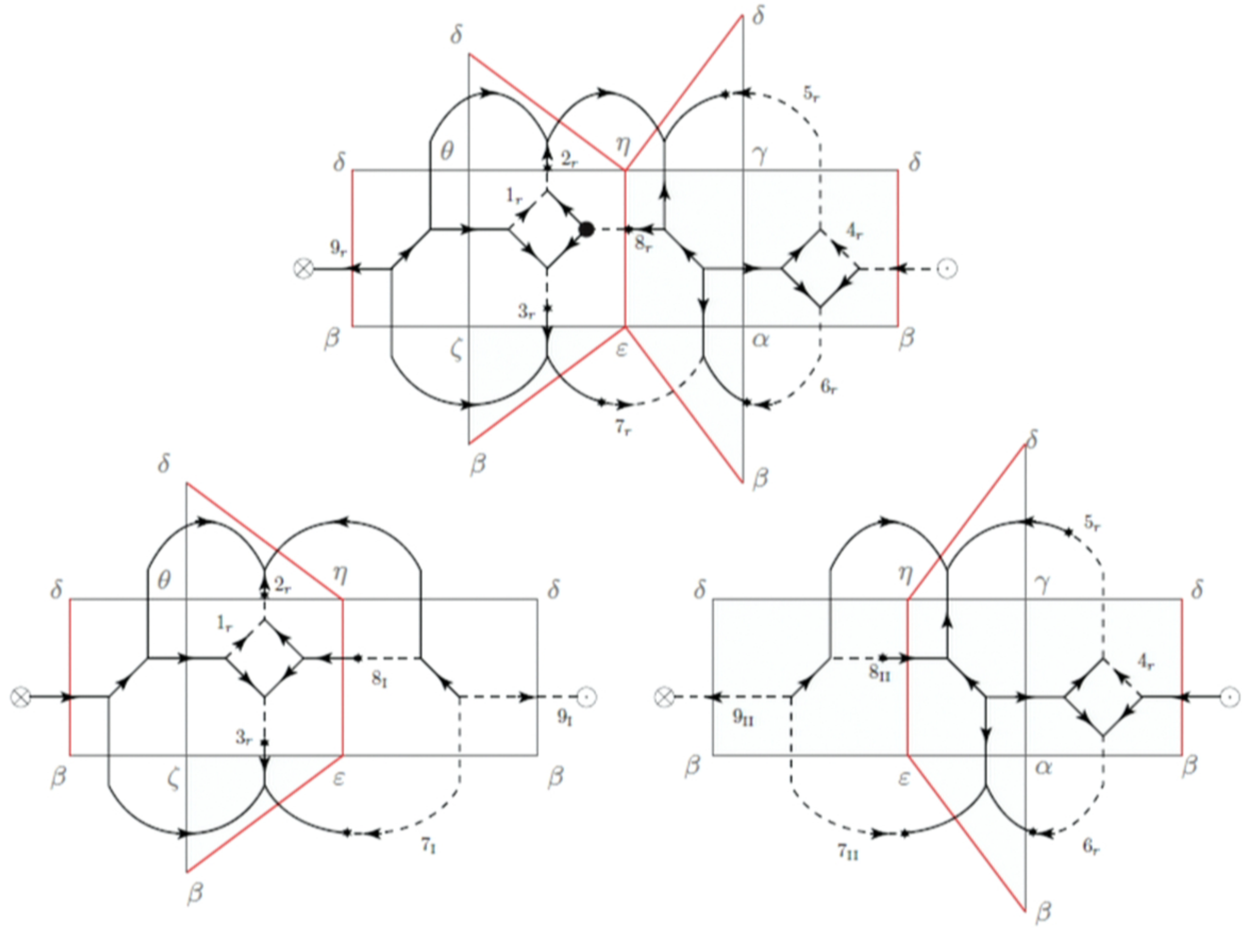
First splitting



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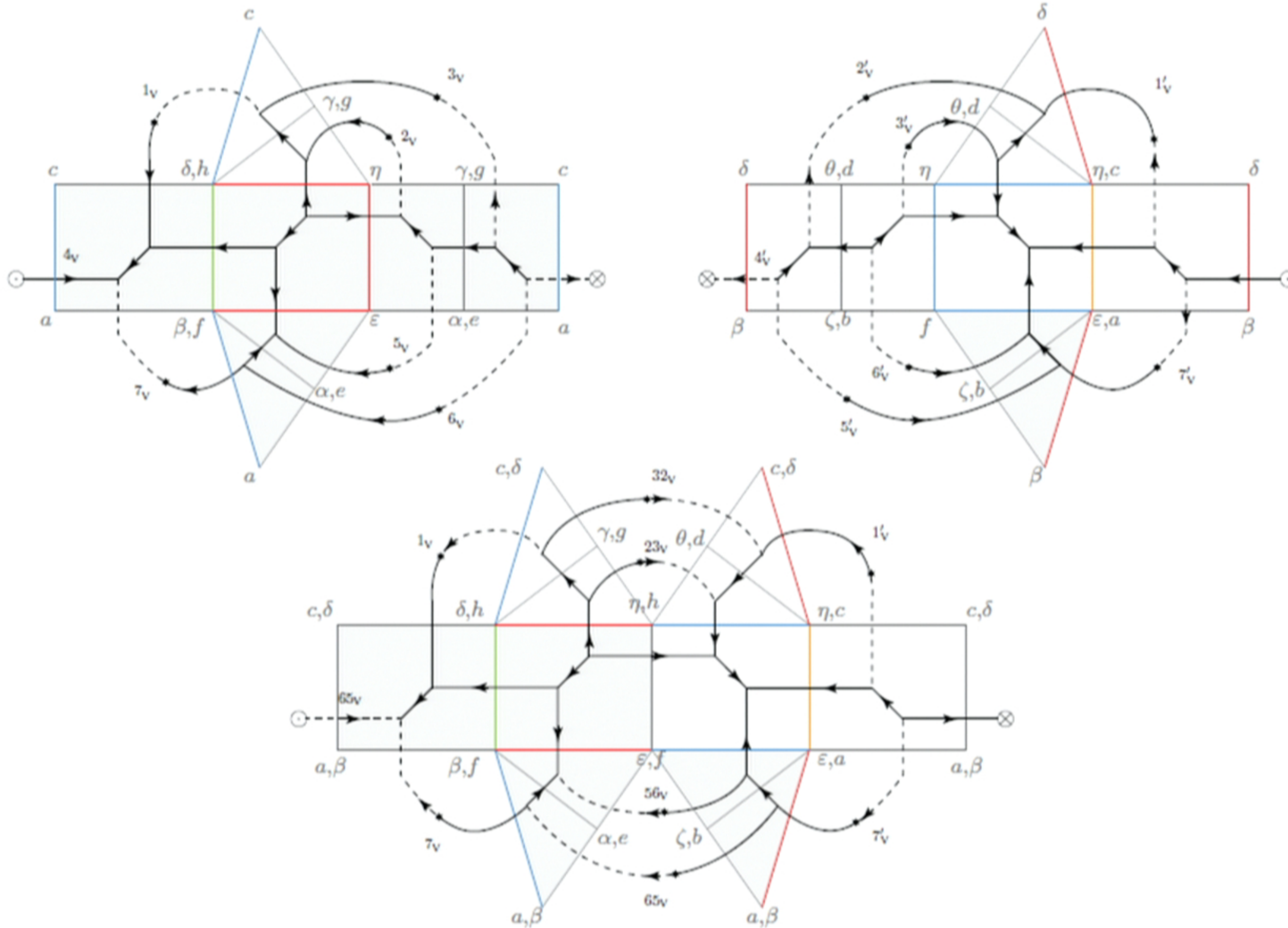
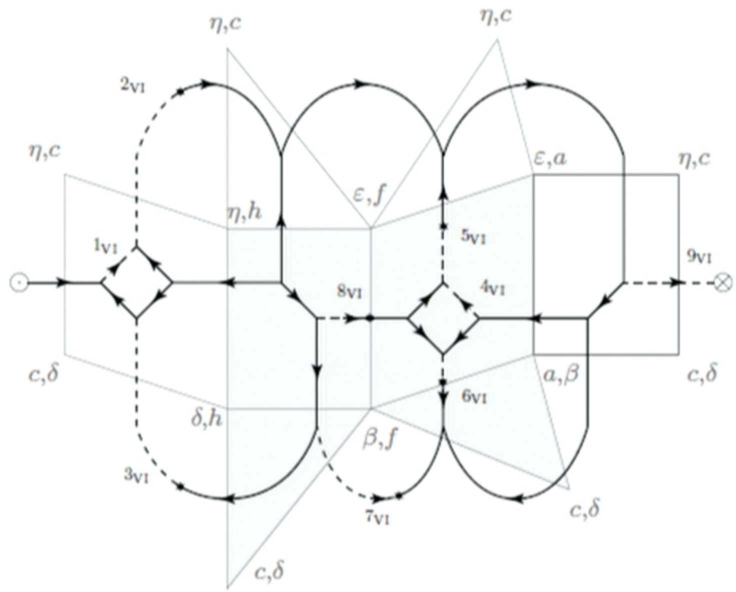
Third gluing



Rotation

- We perform a rotation to coase grain in a orthogonal plane



- Tree transformation

  $\Rightarrow$ End of the first iteration $\leftrightarrow$ back to square one

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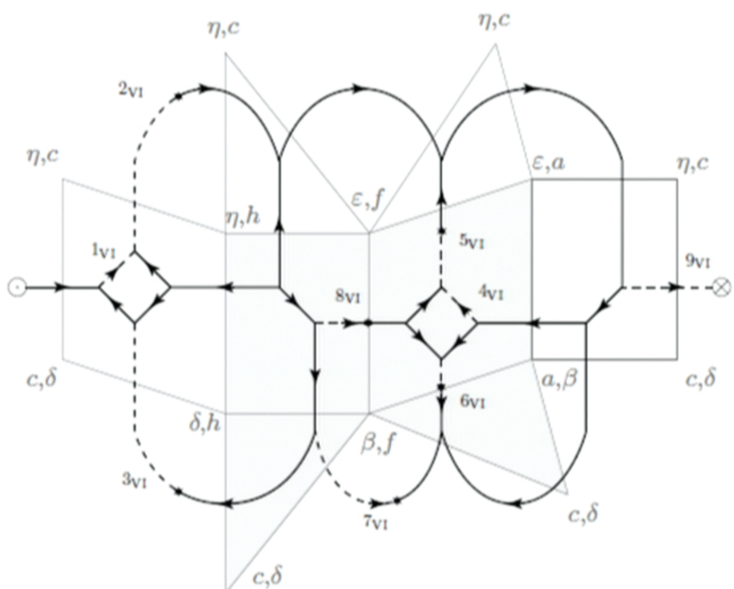

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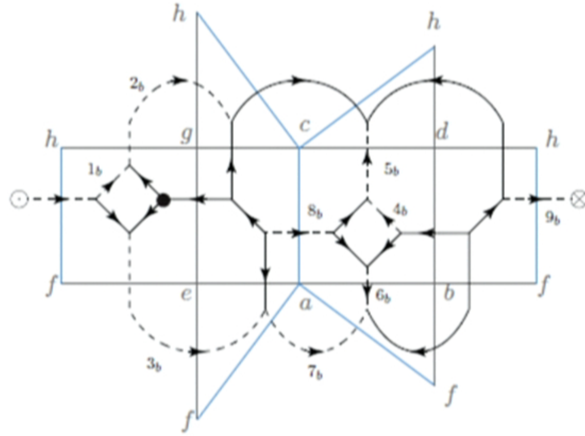
Parametrization (2/2)

- Implementation of constraints in the spin network picture



- Computation of the spin network amplitude:

$$\mathrm{SNW}(\{g\},\{\rho\}) = \Big(\prod_{e\in\text{leaves}} \overline{D^{\rho_e}_{m_e n_e}(g_e)}\Big)\Big(\prod_{e\in\text{edges}} \sqrt{d_{\rho_e}}\Big) \prod_{v\in\text{vertices}} \big(3jm\big)_v$$

- BF amplitude : $\mathcal{A}(\{\rho\}) = \mathrm{SNW}(\mathbb{1},\{\rho\})$

  $\Rightarrow$ In the group rep. : $\mathcal{A}(\{g\}) = \sum_{\{\rho\}} \overline{\mathrm{SNW}(\{g\},\{\rho\})}\mathcal{A}(\{\rho\})$

- **Parametrization**: coefficients on the dimensional factors of the diagonal edges $d^{\text{diag.}}_\rho = (1,1,2) \to (1,\alpha,2\beta)$

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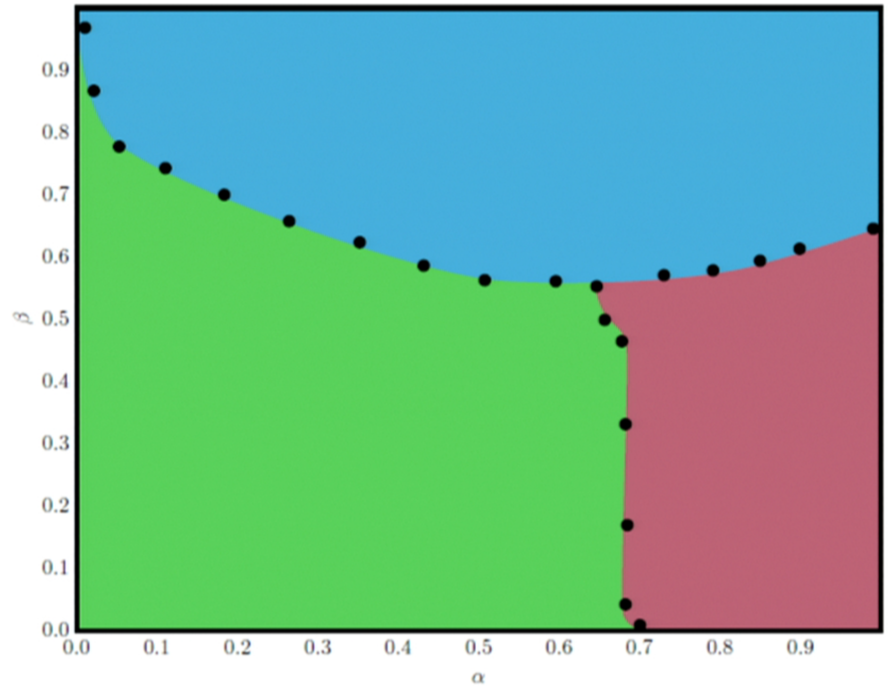
Results for $\mathcal{S}_3$

- Lattice gauge theory fixed points:
  - Ordered $\mathcal{S}_3$ phase
  - High temperature limit
  - Disordered phase with respect to the normal subgroup $\mathbb{Z}_3$



- Consistent with previous results

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Discussion

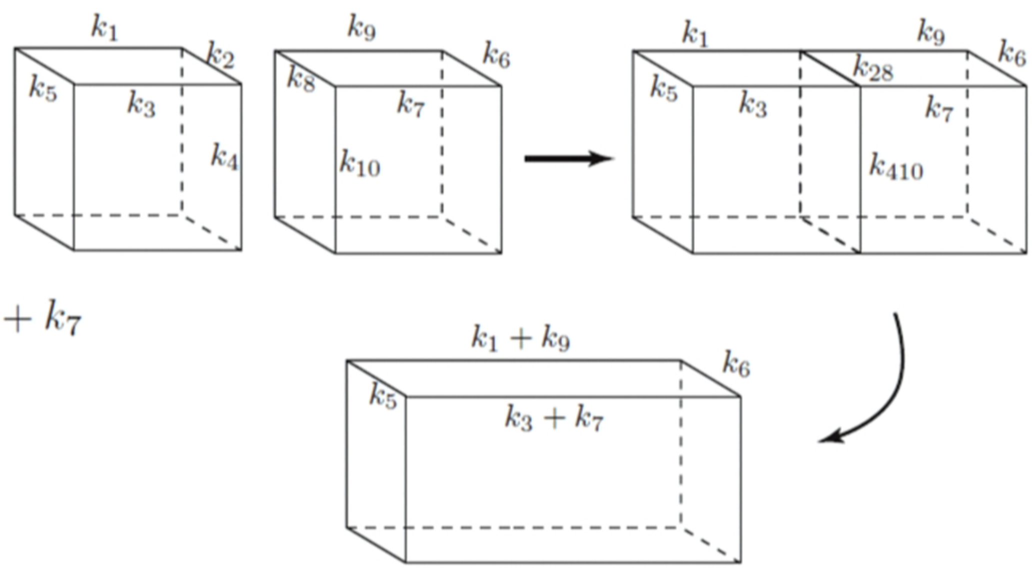
- Absence of additional phases near the phase transitions
- Symmetric group too simple : only three representations
- Lack of control over the distribution of the indices during the embedding

  $\Rightarrow$ Geometrical embedding
- No explicit removal of the short-range correlations (cf TNR)

Geometrical embedding

- The goal is to have more control over the embedding maps
- Redefinition of the variables as **coarser and finer variables**

  $\Rightarrow$ We keep the coarser variables and embed the finer ones
- Example in the case of an abelian lattice gauge theory:



$$\begin{cases} k_{\text{finer}} = k_{28}, k_{410} \\ k_{\text{coarser}} = k_1 + k_9, k_3 + k_7 \end{cases}$$

Conclusion

Summary

- Algorithm for 3D lattice gauge theories with non-abelian groups
- Implementation of simplicity constraints

Outlook

- Symmetry protecting algorithm
- Algorithm with geometrical embedding of the variables
- Implementation of cosmological constant via quantum groups $SU(2)_k$
- 4D: Hypercubes (17 variables) or 4-simplices (6 variables)