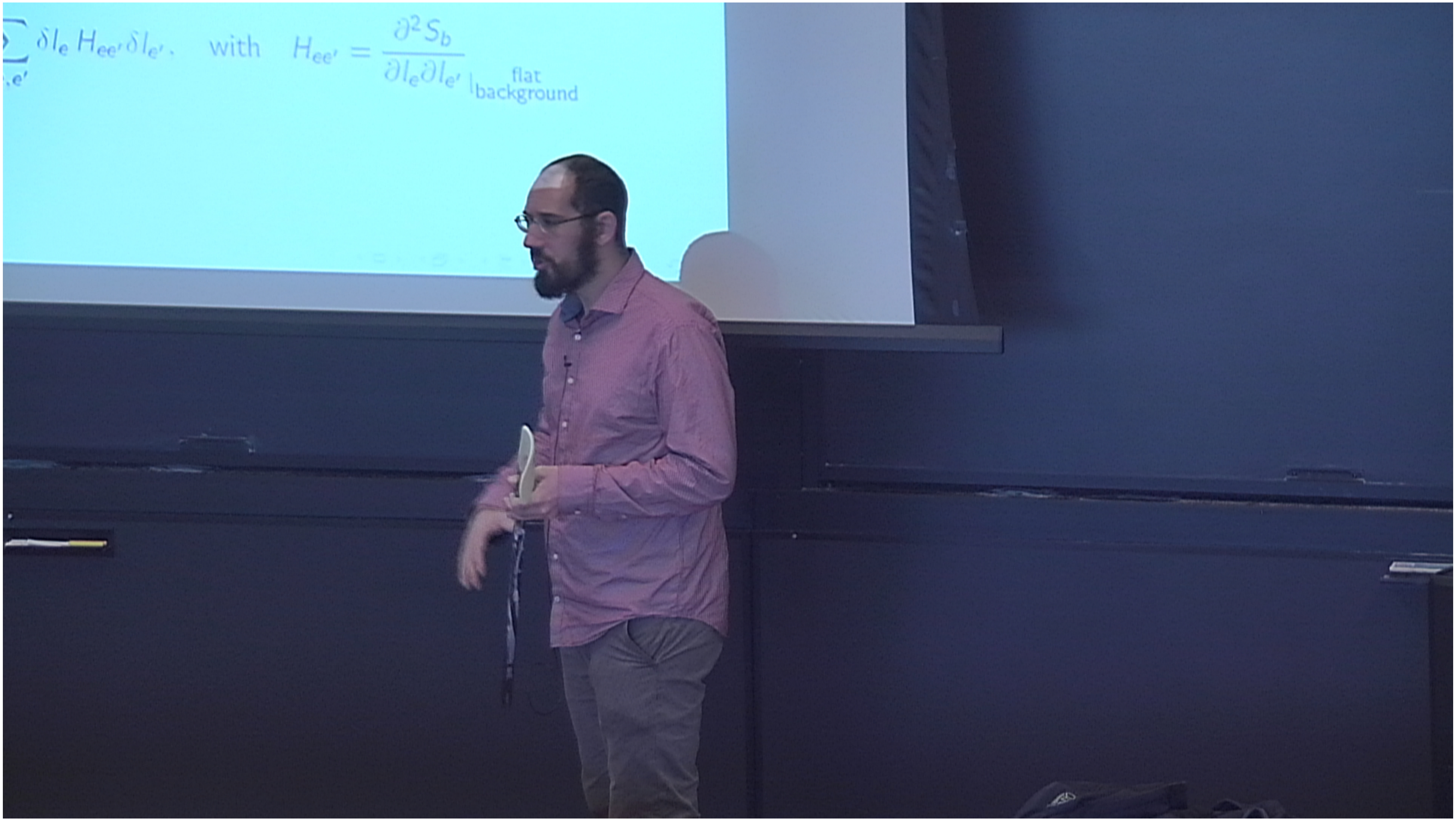
Title: TBA

Date: Sep 28, 2015 02:00 PM

URL: http://pirsa.org/15090078

Abstract:


$\delta l_e \, H_{ee'} \delta l_{e'}$, with $H_{ee'} = \frac{\partial^2 S_b}{\partial l_e \partial l_{e'}}\Big|_{\substack{\text{flat}\\ \text{background}}}$

The partition function of 3D gravity on the solid torus

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January 28, 2015
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Context

- Flat metric

$$g = dt^2 + dr^2 + r^2\, d\phi^2$$

- Torus topology for each r,

$$\text{identification map } \gamma\text{: } (t, \phi) \sim (t + \beta, \phi + \theta)$$

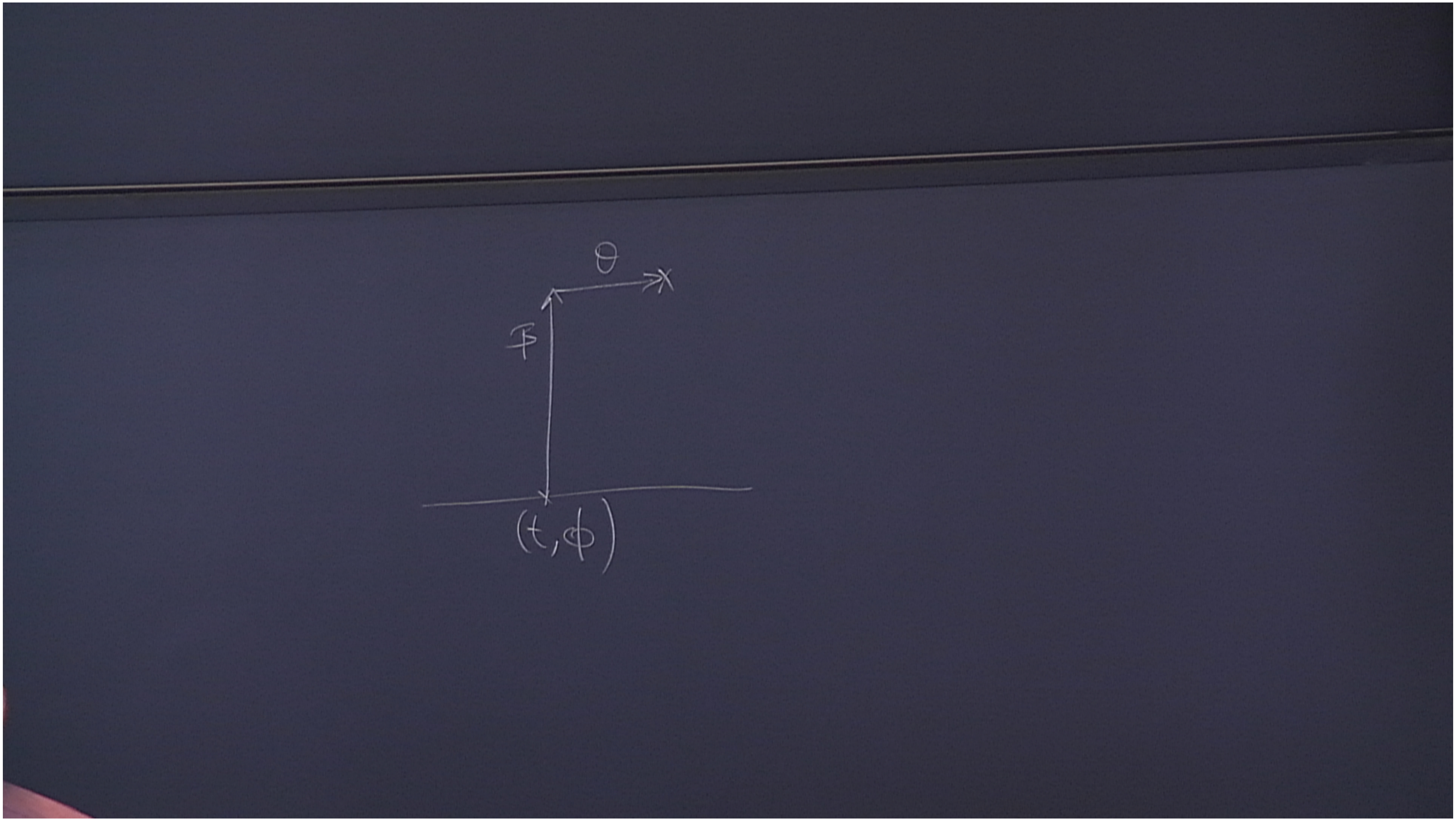
- On-shell action (GHY term)

$$\int_{\text{fixed } r} \sqrt{h} K = 2\pi\beta$$

- 1-loop calculation, believed to be exact [Barnich et al. 2015]: $q = e^{i\theta}$,

$$\prod_{k \geq 2} \frac{1}{|1 - q^k|^2}$$

- Main question
  Can we reproduce that result via discrete gravity, Regge calculus?


θ
(t, φ)

Motivations

- Calculate something concretely using discrete gravity models
- Seems like a simple result
- Character of BMS
- Discrete asymptotic symmetries?
- AdS/CFT?

Interesting aspect: continuum limit!

- Triangulation independence in the bulk
- Is continuum limit required on the boundary?

Continuous calculation

- Expand Einstein-Hilbert action

$$F_{1-loop} = \sum (-1)^i \ln \det \Delta_i$$

- Heat kernel method

$$-\ln \det(\Delta - m^2) = \int_0^\infty \frac{ds}{s} \int d^3x \; K(s; x, x)$$

- Evaluate kernels for flat space and use method of images to find them on coset spaces

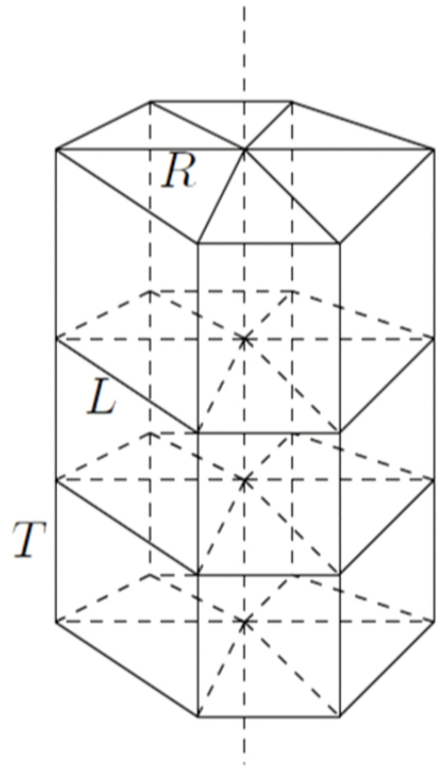
$$K_{\text{coset}}(s; x, x) = \sum_{n \in \mathbb{Z}} K(s; x, \gamma^n x)$$

Discretized torus

- Cylinder: N-gon boundary
- N angular pieces, M time pieces
  As fine as desired
- Flat space embedding
- Radius R
- Discretization scale: $L^2/R^2 = 2a$ with $a = 1 - \cos\frac{2\pi}{N}$

$$L \sim \frac{2\pi}{N} R$$

- Total height $\beta = MT$
- Angular twist $\theta = A/N$
- Triangulate each prism with 3 tetrahedra



Bulk Hessian

- Bulk action

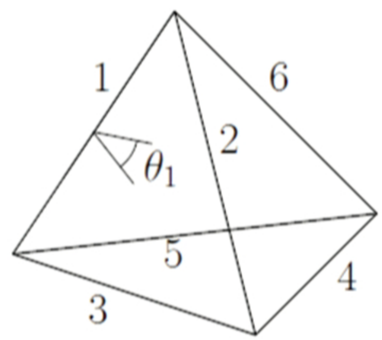
$$S_b = \sum_{\text{bulk edges}} l_e \Big(2\pi - \sum_{\text{tetrahedra}} \theta_{e\sigma} \Big)$$

- Exact discretized gauge symmetries [Freidel, Louapre, Baratin, Bahr, Dittrich, Steinhaus, VB-Smerlak, ...]
  vertex displacements in time and space directions
- Expect them to correspond to nullvectors of the Hessian [Bahr-Dittrich]
- Linearization

$$S_b = 0 + \sum_{e,e'} \delta l_e \, H_{ee'} \delta l_{e'} , \quad \text{with} \quad H_{ee'} = \frac{\partial^2 S_b}{\partial l_e \partial l_{e'}}\Big|_{\substack{\text{flat} \\ \text{background}}}$$

Bulk Hessian II

$$H_{ee'} = \frac{\partial \theta_e}{\partial l_{e'}}$$



Elementary geometry

$$\begin{cases} \cos\theta_1 = \dfrac{2l_1^2 l_4^2 + l_1^2(l_1^2 - l_2^2 - l_3^2 - l_5^2 - l_6^2) + (l_2^2 - l_3^2)(l_6^2 - l_5^2)}{16 A(l_1, l_2, l_3) A(l_1, l_5, l_6)} \\ \sin\theta_1 = \dfrac{3\, l_1\, V(l_1, l_2, l_3, l_4, l_5, l_6)}{2\, A(l_1, l_2, l_3) A(l_1, l_5, l_6)} \end{cases}$$

Structure [Dittrich-Freidel-Speziale]

$$H_{ee'} = \frac{l_e\, l_{e'}}{V}\; h_{ee'}(a)$$

Hessian matrix

- Set $\Delta_k = 2 - \omega_k - \bar{\omega}_k$: angular Laplacian eigenvalues

$$h_{k,p} = \begin{pmatrix} 0 & -2x\sqrt{N}\delta_{k,0} & 0 \\ -2x\sqrt{N}\delta_{k,0} & \Delta_k & -\Delta_k + 2x(1-\nu_p) \\ 0 & -\Delta_k + 2x(1-\bar{\nu}_p) & \Delta_k \end{pmatrix}$$

- t_p only couples to $k = 0$

Nullvectors: 3 gauge degrees of freedom

- $(k = 0, p)$ $\rightarrow$ timelike displacement
- $(k = \pm 1, p)$ $\rightarrow$ spatial displacements
  Due to $\Delta_{\pm 1} = 2a$
- Only $(k \geq 2, p)$ non-gauge

1-loop determinant

$$\prod_{k=2}^{(N-1)/2} \prod_{p} \det h_{k,p} \sim \prod_{k=2}^{(N-1)/2} |1 - q^k|^2 \quad \Rightarrow \prod_{k=2}^{(N-1)/2} \frac{1}{|1 - q^k|^2}$$

- Result of desired form found in a simple way
- Independent of number of time steps M!
- Coarse-graining trivial in time direction
  Invariant under integrating a time slice
- At finite N, truncation of the Fourier modes
- Continuum limit required in angular direction
- Determined by the edges incident to the boundary

Gauge theory with boundary

- Around a solution A_*, parametrize

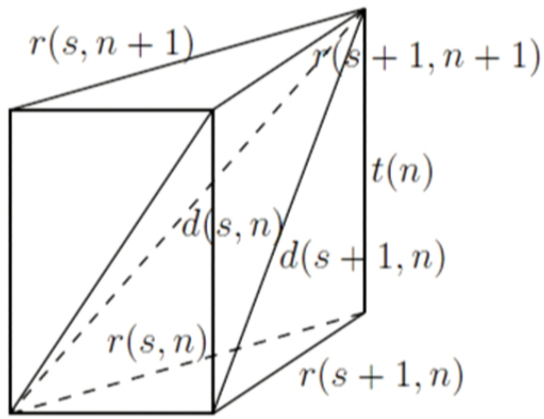
$$A = g^{-1} A_* g + g^{-1} dg$$

- Gauge parameters become dynamical on the boundary

$$S(A) = S_{\text{bulk}}(A_*) + S_{\text{bdry}}(A_*, g)$$

- Chern-Simons $\rightarrow$ Wess-Zumino-Witten(g)
- Boundary conditions for AdS $\rightarrow$ Liouville action
  with the Brown-Henneaux central charge
- Metric formalism: Liouville field related to fluctuations $r(s, n)$

Fourier transform



s: angular position, n: time step
Space-time Fourier transform

$$r(s,n) = \sum_{k=-N/2}^{N/2} \sum_{p=-M/2}^{M/2} r_{k,p}\, \omega_k^s \nu_p^n$$

with $\omega_k = \exp 2i\pi k \frac{L}{R}$

$$\int \prod dr(s,n) dd(s,n) dt(n)\; e^{-\sum_{n,s} (\,r\;\; d\;\; t\,) h \begin{pmatrix} r \\ d \\ t \end{pmatrix}}$$

$$= \prod_{k,p} \int dr\, d\bar{r}\, dd\, d\bar{d}\, dt\, d\bar{t}\; e^{-(\,r\;\; d\;\; t\,) h_{k,p} \begin{pmatrix} \bar{r} \\ \bar{d} \\ \bar{t} \end{pmatrix}} = \prod_{k,p} \frac{1}{\det h_{k,p}}$$

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Boundary

- Hartle-Sorkin term

$$S_{\text{bdry}} = \sum_{e \in \text{bdry}} l_e (\pi - \theta_e)$$

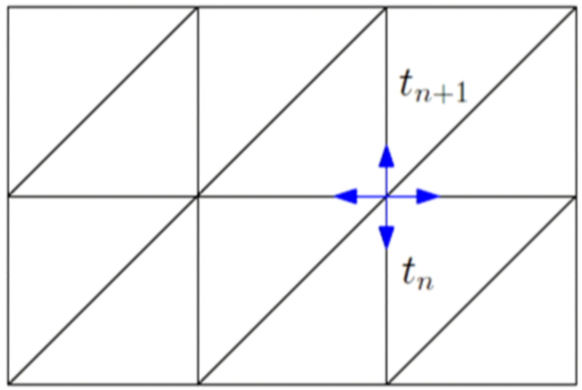
- Regular configuration

$$\theta_{t_n} = \frac{2\pi}{N}$$

Other θ_e vanish

- Evaluation

$$S_{\text{bdry}} = 2\pi\beta = \int \sqrt{h} K$$



- Invariant to 1st order under vertex displacements in time direction
- Invariant to 1st order under vertex displacements in angular direction provided $t_n = t_{n+1}$

Boundary fluctuations

- Expand total action

$$\sum_{e,e'} \delta l_e \frac{\partial^2 S}{\partial l_e \partial l_{e'}} \delta l_{e'}$$

- Hessian

$$\begin{pmatrix} H_{\text{bulk-bulk}} & H_{\text{bulk-bdry}} \\ H^{\dagger}_{\text{bulk-bdry}} & H_{\text{bdry-bdry}} \end{pmatrix}$$

Nullvectors

- Vertex displacements in the bulk
- Vertex displacements on the boundary in time direction only

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Effective boundary theory

- Fourier transform decouples the modes
- Integrate bulk fluctuations

$$H_{\text{eff}} = H_{\text{bdry-bdry}} - H^{\dagger}_{\text{bulk-bdry}} H^{-1}_{\text{bulk-bulk}} H_{\text{bulk-bdry}}$$

- In the basis of (horizontal, diagonal, time) fluctuations
- Single nullvector (time displacements)
- Continuum limit: $L \to 0$
- Still no invariance under angular vertex displacements

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Boundary invariance

Fourier mode (k, v):

$$\begin{pmatrix} \frac{8\pi^2(Rv+k\beta)^2}{\beta^2} - 2 & -8\pi^2k^2 - \frac{8\pi^2 Rvk}{\beta} + 2 & \frac{2(Rv+k\beta)\left(4\pi^2\beta k^2+4\pi^2 Rvk-\beta\right)}{Rv\beta} \\ -8\pi^2k^2 - \frac{8\pi^2 Rvk}{\beta} + 2 & 8k^2\pi^2 - 2 & -\frac{2\left(4k^2\pi^2-1\right)(Rv+k\beta)}{Rv} \\ \frac{2(Rv+k\beta)\left(4\pi^2\beta k^2+4\pi^2 Rvk-\beta\right)}{Rv\beta} & -\frac{2\left(4k^2\pi^2-1\right)(Rv+k\beta)}{Rv} & \frac{2\left(4k^2\pi^2-1\right)(Rv+k\beta)^2}{R^2v^2} \end{pmatrix}$$

- Notice that matrix entries are even polynomial of degree 2 in Fourier modes
- Keep only terms of degree 2 as Fourier modes are arbitrarily large
- 2 nullvectors (time and angular diffeos)

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Relation to Liouville?

- 1 non-vanishing eigenvalue

$$\frac{(R^2 v^2 + kRv\beta + k^2\beta^2)^2}{R^2\beta^2(v^2 + k^2)}$$

- Investigate more closely relationship to Liouville
- Maybe keep the fluctuations $r(n, s)$

Summary

Discretization + continuum limit in a gravitational model

- Linearized Regge calculus reproduces BMS character
- 1-loop determinant invariant under time coarse graining
- But continuum limit required in angular direction
- Fixed fluctuations on boundary at finite radius R
- Need continuum limit in both directions
- Use triangles with extrinsic curvature on the boundary? Similar to [Bahr-Dittrich]
- Measure? [Dittrich-Steinhaus] R, N-dependent terms

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Perspectives

- $R \to \infty$

$$\frac{1}{16\pi G}\int_M \sqrt{g}R + \frac{\alpha}{8\pi G}\int_{\partial M}\sqrt{h}K$$

  - $\alpha = 1$ for finite R
  - $\alpha = 1/2$ for $R \to \infty$
- AdS case [Dittrich-Mizera-Riello]
- Ring configuration: propagates between two cylinders, interesting for tensor networks
- Relation to Liouville?
- Charges on the boundary, in the discrete?
  Which Hamiltonian formulation? [Bahr-Dittrich, Dittrich-Höhn]

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Non-perturbative

- Quantum non-perturbative version!
- Ponzano-Regge instead of Regge calculus [under investigation w/ Livine, Riello]
- Gauge-fixing [Freidel-Louapre]
- Boundary states: spin networks
- Choice of boundary conditions

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Boundary conditions in BF theory

- BF action

$$S_{\mathsf{BF}}(B,A) = \int_M \operatorname{tr} B \wedge (dA + A \wedge A)$$

B: metric/Lagrange multiplier

- Variations

$$\delta S_{\mathsf{BF}}(B,A) = \underbrace{\int_M \operatorname{tr} \delta B \wedge F(A) + \int_M \operatorname{tr} d_A B \wedge \delta A}_{\text{e.o.m.}} + \int_{\partial M} \operatorname{tr} B \wedge \delta A$$

Boundary Conditions

- Connection fixed: no boundary term
- Metric B fixed: $-\int_{\partial M} \operatorname{tr} B \wedge A$

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Change of boundary triangulation

- 3-1 and 2-2 Pachner moves
- $|\psi_i\rangle$ and $|\psi_f\rangle$ live in different Hilbert spaces

$$\psi_f(j_e) = \sum \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \psi_i(j_{e'})$$

  Proof using recursion relations (no projection operator)
  [VB-Dupuis-Girelli-Livine]
- Embedding maps to interpret states on coarse triangulations as states in finer ones
  [Dittrich-Geiller]
- All the coarse-graining issues can be answered in 3D!

BF with boundary [VB-Smerlak]

- Discrete connection and holonomies $A = (g_e)$
- Discrete curvature $H(A) = \left(H_f(A) = \prod_{e \in \partial f} g_e\right)$
- Gauge transformations I: $\gamma_A(h) = \left(h_{t(e)}^{-1} g_e h_{s(e)}\right)$
- Gauge transformations II: Vertex displacements $\to$ Action of 3-cells

$$\int dA \, \delta(H(A)) \underset{\text{g.f.}}{=} \sum_{[\phi] \in \mathcal{M}} \text{R-torsion}([\phi])$$

- Extends to presence of boundary using relative cohomology