Title: Statistical Decoupling of Lagrangian Fluid Parcel in Newtonian Cosmology - Xin Wang

Date: Sep 15, 2015 11:00 AM

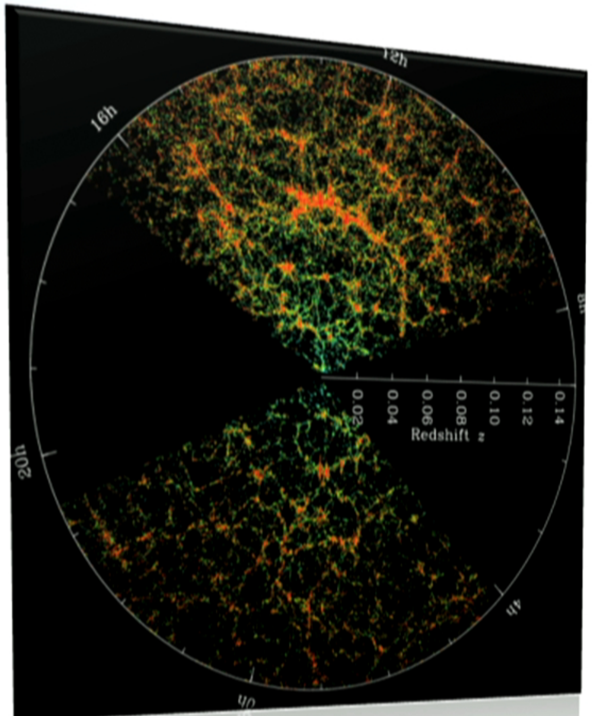
URL: http://pirsa.org/15090065

Abstract: <p>The Lagrangian dynamics of a fluid element within a self-gravitational matter field is intrinsically nonlocal due to the presence of the tidal force. Instead of searching for local approximations, we provide a statistical solution that could decouple the evolution of the fluid parcel from the surrounding environment. Given the probability distribution of the matter field, the method produces a set of ordinary differential equations to be solved locally. Mathematically, it corresponds to the characteristic curve of the transport equation of the density-weighted probability density function (Ï•PDF). Physically, it describes the mean evolution of the element with specific density and shape averaged over various environments. Furthermore, it is guaranteed that the one-point Ï•PDF would be preserved if one evolves these local, yet nonlinear, curves with the same set of initial data as the real system. This PDF based method, which is well developed in turbulence and other fields, provides a new perspective for understanding the non-linear structure formation in cosmology, e.g. the halo formation and the evolution of cosmic web. For Gaussian distributed dynamical variables, we demonstrate that the localized mean tidal tensor is proportional to the shear tensor, and the coefficient would recover the prediction of Zelâ€™dovich approximation (ZA) with the further assumption of the linearized continuity equation. For Weakly non-Gaussian field, the averaged tidal tensor could then be expressed as polynomial of other variables. Moreover, one could further generalize this concept of the mean evolution of the fluid element to incorporate some stochastic contributions, which we suggest would be valuable in describing a variety of processes in cosmology, such as the shell-crossing and realistic halo formation.</p>

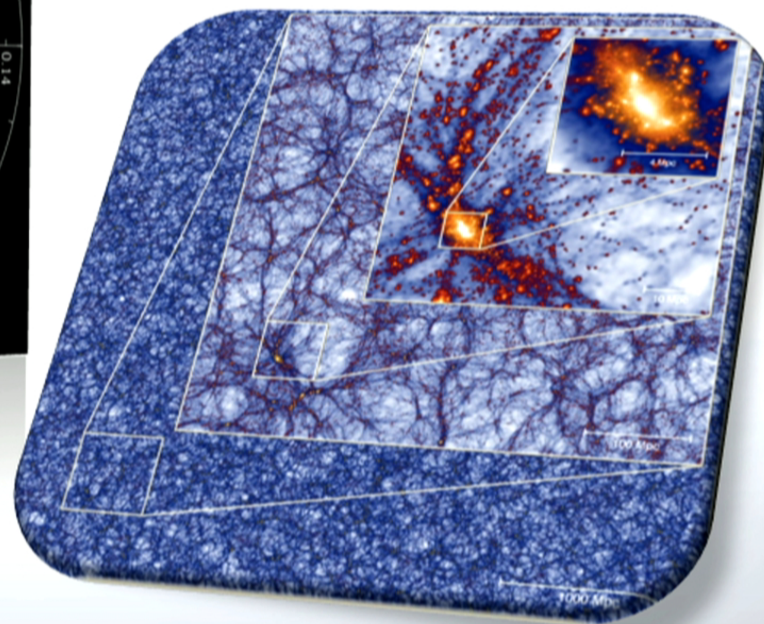
Statistical Decoupling of Lagrangian Fluid Parcel in Newtonian Cosmology

Xin Wang (CITA, JHU)

Large-scale Structure as a Field



SDSS



Millennium Run (Springel et al. 2005)

Individual Cosmic Object

- In our human language, we keep talking about individual objects.
  - Halos, voids, filaments, clusters, galaxies, stars …
  - their various properties: masses, ages, time evolution, etc.
  - Convenient for model construction
    - e.g. Halo model
  - However, their evolution are not isolated.

Individual Cosmic Object

- **Example: dark matter halos model**
  - useful for large & small scales:
    - constitute LSS
    - hosting galaxies
  - Ingredients:
    - spatial distribution
    - halo density profile
    - halo evolution ***model***: collapse criteria
    - e.g. spherical collapse, ellipsoidal collapse (simplified)

Newtonian Gravity + Cosmic Fluid

- The reason: ***non-locality, non-linearity***

- Eulerian Fluid in Newtonian Gravity

$$\frac{\partial\delta(\boldsymbol{x},\tau)}{\partial\tau} + \nabla\cdot\{[1+\delta(\boldsymbol{x},\tau)]\boldsymbol{v}(\boldsymbol{x},\tau)\} = 0\,,$$

$$\frac{\partial\boldsymbol{v}(\boldsymbol{x},\tau)}{\partial\tau} + \mathscr{H}(\tau)\boldsymbol{v}(\boldsymbol{x},\tau) + [\boldsymbol{v}(\boldsymbol{x},\tau)\cdot\nabla]\boldsymbol{v}(\boldsymbol{x},\tau) = -\nabla\Phi(\boldsymbol{x},\tau)\,,$$

$$\boxed{\nabla^2\Phi(\boldsymbol{x},\tau) = \tfrac{3}{2}\Omega\mathscr{H}^2(\tau)\delta(\boldsymbol{x},\tau)}\,.$$

- Lagrangian fluid dynamics
  - Spherical collapse

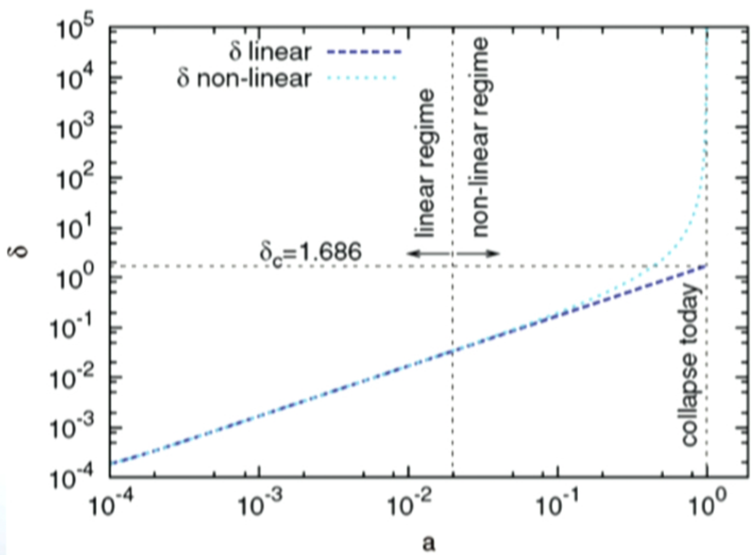
$$\frac{d}{d\tau}\delta_\rho = -(1+\delta_\rho)\theta,$$

$$\frac{d}{d\tau}\theta = -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \cancel{\sigma^{ij}}\sigma_{ij}\right] - 4\pi G_N\bar{\rho}a^2\delta_\rho,$$

Large-scale Structure & GR

- **Lagrangian fluid dynamics**
  - Spherical collapse

$$\ddot{\delta} + \frac{\dot{a}}{a}\dot{\delta} = \frac{4}{3}\frac{\dot{\delta}^2}{1+\delta} + (1+\delta)\left(\cancel{\frac{2}{3}\sigma^2} - \cancel{2\omega^2} + 4\pi G\bar{\rho}a^2\delta\right)$$



Pace et al. 2010

Large-scale Structure & GR

- **Lagrangian fluid dynamics**

  - In general,

$$
\begin{aligned}
\frac{d}{d\tau}\delta_\rho &= -(1+\delta_\rho)\theta, \\
\frac{d}{d\tau}\theta &= -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij}\right] - 4\pi G_N \bar{\rho} a^2 \delta_\rho, \\
\frac{d}{d\tau}\sigma_{ij} &= -\left[\mathcal{H}(\tau)\sigma_{ij} + \frac{2}{3}\theta\sigma_{ij} + \sigma_{ik}\sigma^k_j - \frac{1}{3}\sigma_{mn}\sigma^{mn}\delta^K_{ij}\right] + \varepsilon_{ij},
\end{aligned}
$$

  - Where velocity gradient $= \frac{\theta}{3}\delta^K_{ij} + \sigma_{ij} + \omega_{ij}$
  - and

$$
\Phi_{ij} = \frac{\nabla^2\Phi}{3}\delta^K_{ij} + \varepsilon_{ij} = \frac{4\pi G_N \bar{\rho} a^2 \delta_\rho}{3}\delta^K_{ij} + \varepsilon_{ij}
$$

Large-scale Structure & GR

- Time evolution of ε_{ij} ?
  - Newton Gravity:

$$\varepsilon_{ij}(\mathbf{x}) = G_N \bar{\rho} a^2 \int d^3x' \left[\frac{\delta^K_{ij}}{r^3} - 3\frac{r_i r_j}{r^5} \right] \delta_\rho(\mathbf{x}')$$

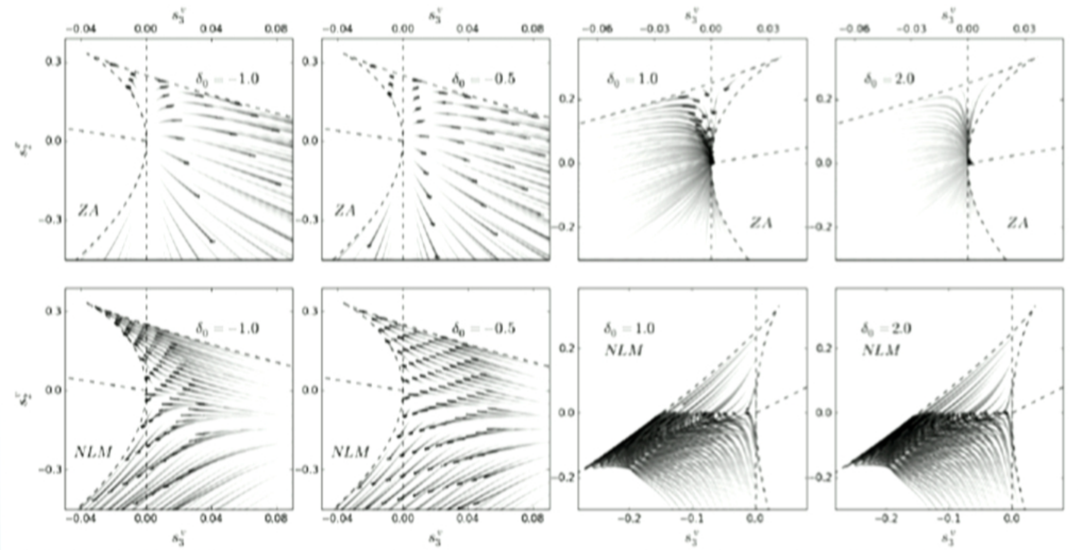
  - In GR:
    - E_ij: electric part of Weyl tensor (tidal tensor)
    - H_ij: magnetic part of Weyl tensor

$$\frac{dE_{ij}}{d\tau} + \frac{\dot{a}}{a} E_{ij} + \theta E_{ij} + \delta_{ij} \sigma^{kl} E_{kl} - 3\sigma^k{}_{(i} E_{j)k}$$
$$+ \epsilon^{kl}{}_{(i} E_{j)k} \omega_l - \boxed{\nabla_k \epsilon^{kl}{}_{(i} H_{j)l}} = -4\pi G \rho a^2 \sigma_{ij}$$

Hawking1966, Ellis & Bruni 1989

Large-scale Structure & GR

- **Silent Universe model**
  - $H_{ab} = 0$ $\quad \frac{d}{d\tau}\varepsilon_{ij} + \mathcal{H}(\tau)\varepsilon_{ij} + \theta\varepsilon_{ij} + I_{ij}\sigma^{kl}\varepsilon_{kl}$
  $$-3\sigma^{k}{}_{(i}\varepsilon_{j)k} = -4\pi G\rho a^2\sigma_{ij}$$
  - More filaments (contradict with ZA prediction)



Wang et al. 2014

Large-scale Structure & GR

- However,
  - (1/c) expansion of GR equation
  - Nonlocal ε_{ij} evolution equation

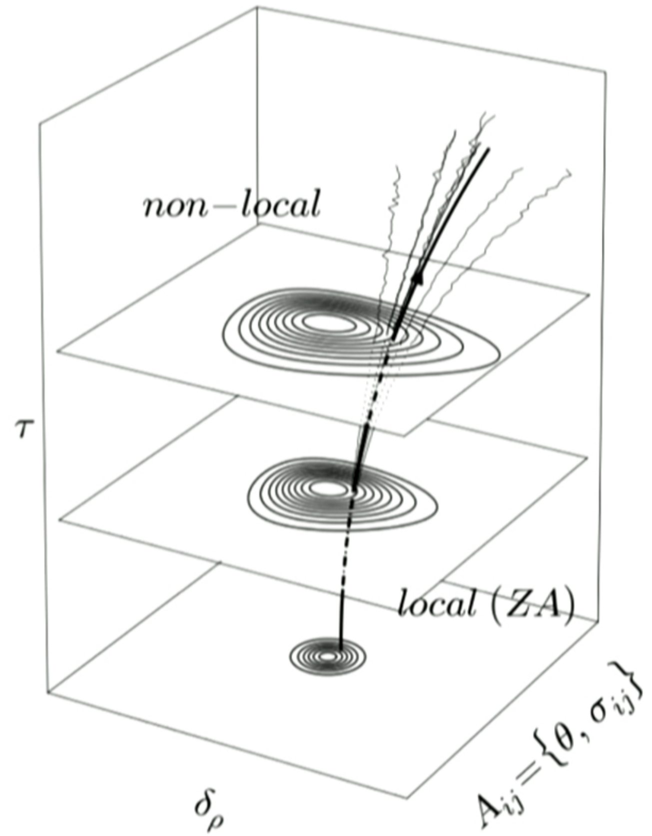
$$\frac{\partial}{\partial\tau} E^{(\mathrm{N})}_{\alpha\beta} + \frac{a'}{a} E^{(\mathrm{N})}_{\alpha\beta} - \frac{4\pi G}{3} g_{\alpha\beta}(\rho v^{\gamma})_{,\gamma} = Ga^2 \left[\int \frac{d^3x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \rho v^{\gamma} \right]_{,\gamma\alpha\beta}$$

Kofman & Pogosyan, 1995

  - Problem remains (open question) …
    - Local approximation (Hui & Bertschinger 1996)

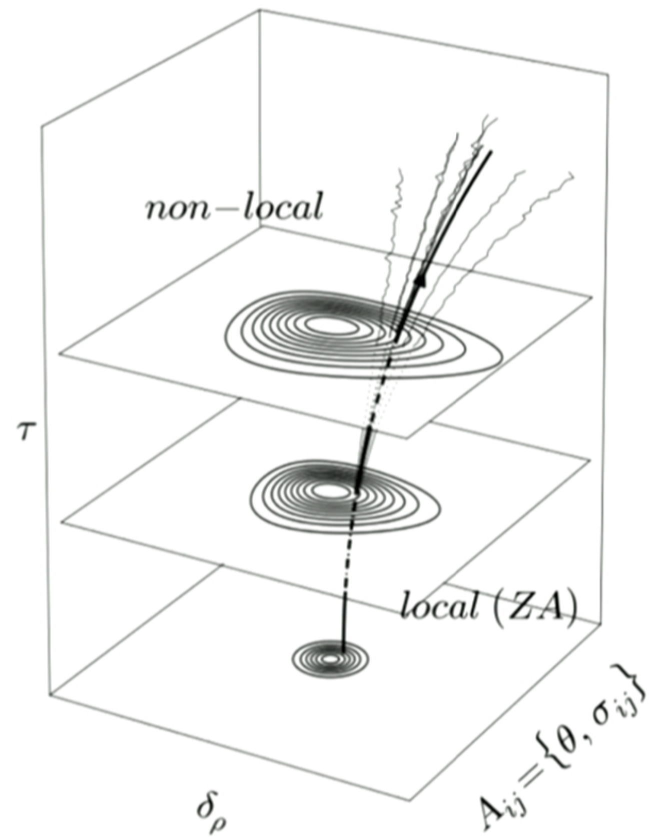
Large-scale Structure & Non-locality

- Question
  - Is that possible to construct a nonlinear but local theory ?
  - Many approximations…
  - What's the best approach? (theoretically)
  - The "local approximation" == some "mean effective evolution"
  - Mean?? -> ***statistical description ?***



Large-scale Structure & Non-locality

- ***What to do*** with data ? (experimental *or* simulated)
  - Measure it !!
  - Problem: How to describe such procedure analytically?



Statistical Decoupling of Lagrangian Fluid Parcel in Newtonian Cosmology

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Dynamical to PDF Evolution

- **From Dynamics to statistics**
  - Define dynamical variables

$$\psi = \{\delta_\rho, A_{ij}\} = \{\delta_\rho, \theta, \sigma_{ij}\}$$

  - Lagrangian dynamical evolution

$$\frac{d}{d\tau}\psi(\tau) = \boldsymbol{\chi}[\psi, \varepsilon_{ij}]$$

  - Which in details

$$\frac{d}{d\tau}\delta_\rho = -(1+\delta_\rho)\theta,$$
$$\frac{d}{d\tau}\theta = -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij}\right] - 4\pi G_N \bar{\rho} a^2 \delta_\rho,$$
$$\frac{d}{d\tau}\sigma_{ij} = -\left[\mathcal{H}(\tau)\sigma_{ij} + \frac{2}{3}\theta\sigma_{ij} + \sigma_{ik}\sigma^k_j - \frac{1}{3}\sigma_{mn}\sigma^{mn}\delta^K_{ij}\right] - \varepsilon_{ij}$$

Dynamical to PDF Evolution

- Lagrangian approach
  - Define fine-grained PDF

$$\mathcal{P}_L^f(\mathbf{\Psi};\tau) = \delta_D\left[\psi(\tau) - \mathbf{\Psi}\right] = \delta_D\left[\delta_\rho(\tau) - \Delta_\rho\right]\delta_D\left[\boldsymbol{A}(\tau) - \boldsymbol{\mathcal{A}}\right],$$

  - whose ensemble average relates to

$$\langle P_L^f(\mathbf{\Psi};\tau)\rangle_L = \int d\mathbf{\Psi}' P_L(\mathbf{\Psi}';\tau)\delta_D(\mathbf{\Psi}' - \mathbf{\Psi}) = P_L(\mathbf{\Psi};\tau).$$

  - Simply taking time derivatives

$$\begin{aligned}\frac{\partial}{\partial\tau}\mathcal{P}_L(\mathbf{\Psi};\tau) &= \left\langle \frac{\partial}{\partial\tau}\mathcal{P}_L^f(\mathbf{\Psi};\tau)\right\rangle_L = \left\langle \frac{d\psi_\alpha}{d\tau}\left[\frac{\partial}{\partial\psi_\alpha}\delta_D(\psi(\tau) - \mathbf{\Psi})\right]\right\rangle_L \\ &= -\left\langle \chi_\alpha\left[\frac{\partial}{\partial\Psi_\alpha}P_L^f(\mathbf{\Psi};\tau)\right]\right\rangle_L = -\frac{\partial}{\partial\Psi_\alpha}\left\langle \chi_\alpha\mathcal{P}_L^f(\mathbf{\Psi};\tau)\right\rangle_L.\end{aligned}$$

Dynamical to PDF Evolution

- **PDF evolution (*linear PDE*)**

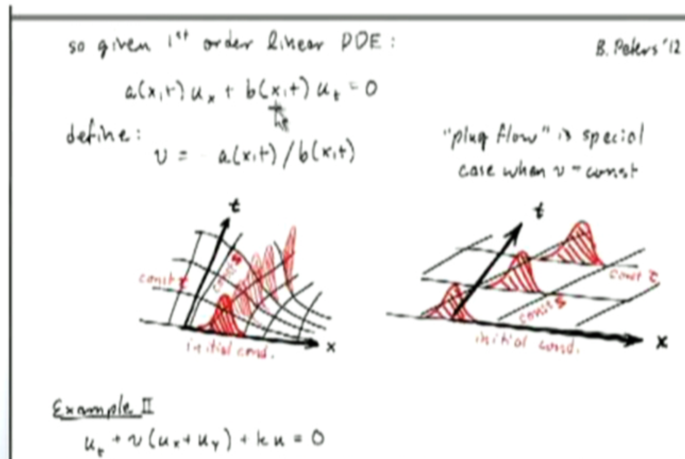
$$\frac{\partial}{\partial \tau}\mathcal{P}_L(\mathbf{\Psi};\tau) + \frac{\partial}{\partial \Psi_\alpha}\left[\langle \chi_\alpha | \mathbf{\Psi};\tau\rangle_L \, \mathcal{P}_L(\mathbf{\Psi};\tau)\right] = 0,$$

- **Method of Characteristics**
  - **PDE -> ODE**
  - **Textbook example**

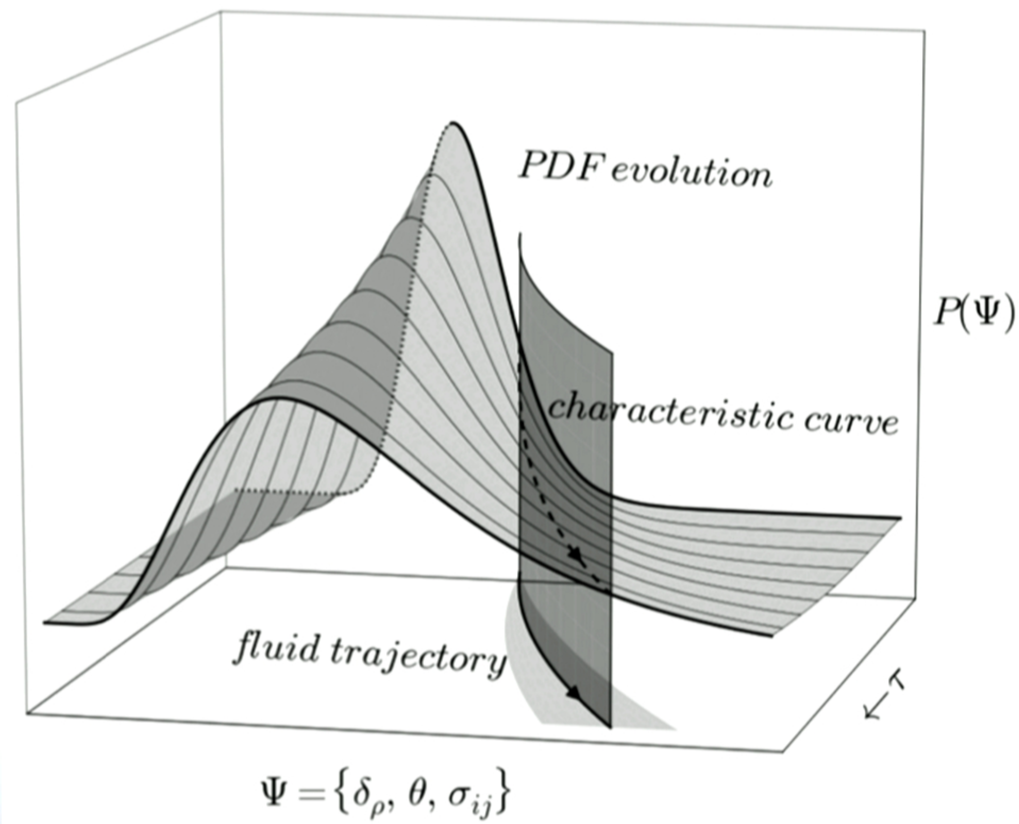
$$a(x,y)u_x + b(x,y)u_y = c(x,y)$$

  - **Integral curves**

$$\frac{dx}{ds} = a(x(s), y(s))$$
$$\frac{dy}{ds} = b(x(s), y(s))$$
$$\frac{dz}{ds} = c(x(s), y(s))$$



PDF Evolution & Characteristic Curve



Dynamical to PDF Evolution

- From statistics

$$\frac{\partial}{\partial \tau}\mathcal{P}_L(\mathbf{\Psi};\tau)+\frac{\partial}{\partial \Psi_\alpha}\left[\langle \chi_\alpha|\mathbf{\Psi};\tau\rangle_L\,\mathcal{P}_L(\mathbf{\Psi};\tau)\right]=0,$$

- Projected characteristic curve

$$\frac{d}{d\tau}\mathbf{\Psi}(\tau)=\langle \boldsymbol{\chi}|\mathbf{\Psi};\tau\rangle_L.$$

- Original dynamics $\frac{d}{d\tau}\psi(\tau)=\boldsymbol{\chi}[\psi,\varepsilon_{ij}]$

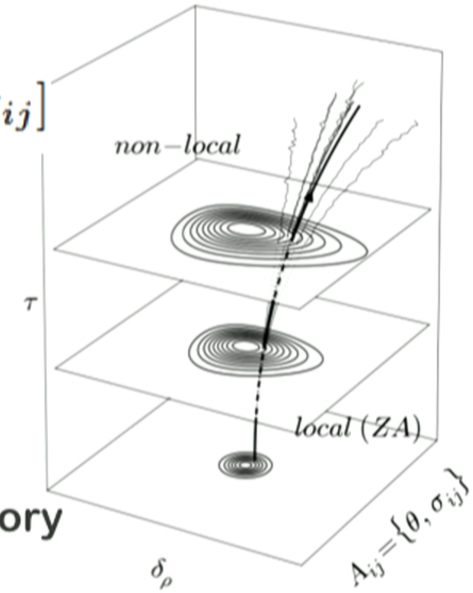
  – The only non-trivial equation:

$$\frac{d}{d\tau}\Sigma_{ij}+\mathcal{H}(\tau)\Sigma_{ij}+\frac{2}{3}\Theta\Sigma_{ij}+\Sigma_{ik}\Sigma^k_j-\frac{1}{3}\Sigma_{mn}\Sigma^{mn}\delta^K_{ij}=-\langle\varepsilon_{ij}|\mathbf{\Psi};\tau\rangle_L.$$

Projected Characteristic Curve

- **Mean evolution for** $\frac{d}{d\tau}\psi(\tau) = \boldsymbol{\chi}[\psi, \varepsilon_{ij}]$

$$\frac{d}{d\tau}\boldsymbol{\Psi}(\tau) = \langle \boldsymbol{\chi} | \boldsymbol{\Psi}; \tau \rangle_L .$$



- **Conserving one-point PDF**
  - Statistical equivalence
  - repeating derivation with mean trajectory
  - Identical PDF evolution equation

$$\frac{\partial}{\partial\tau}\mathcal{P}_L(\boldsymbol{\Psi};\tau) = \left\langle \frac{\partial}{\partial\tau}\mathcal{P}_L^f(\boldsymbol{\Psi};\tau) \right\rangle_L = \left\langle \frac{d\psi_\alpha}{d\tau}\left[\frac{\partial}{\partial\psi_\alpha}\delta_D(\psi(\tau)-\boldsymbol{\Psi})\right]\right\rangle_L$$

$$= -\left\langle \chi_\alpha \left[\frac{\partial}{\partial\Psi_\alpha}P_L^f(\boldsymbol{\Psi};\tau)\right]\right\rangle_L = -\frac{\partial}{\partial\Psi_\alpha}\left\langle \chi_\alpha \mathcal{P}_L^f(\boldsymbol{\Psi};\tau)\right\rangle_L .$$

Statistics Evolution

- Lagrangian Evolution in Eulerian space
  - Tidal tensor should be averaged in Eulerian space
- define density weighted PDF

$$\mathcal{D}(\boldsymbol{\Psi};\tau) = (1+\Delta_\rho(\tau))\mathcal{P}_E(\boldsymbol{\Psi};\tau)$$

- And consider

$$\left\langle (1+\delta_\rho)\frac{d}{d\tau}Q(\psi^{tot}) \right\rangle_E$$

- We obtain

$$\frac{\partial}{\partial\tau}\mathcal{D}(\boldsymbol{\Psi};\tau) + \frac{\partial}{\partial\Psi_\alpha}\left[\langle\chi_\alpha|\boldsymbol{\Psi};\tau\rangle_E\mathcal{D}(\boldsymbol{\Psi};\tau)\right] = 0$$

- and

$$\langle\chi_\alpha|\boldsymbol{\Psi};\tau\rangle_L = \frac{1}{(1+\Delta_\rho)\mathcal{P}_E(\boldsymbol{\Psi})}\int d\boldsymbol{X}\; X_\alpha(1+\Delta_\rho)\mathcal{P}_E(\boldsymbol{X},\boldsymbol{\Psi}) = \langle\chi_\alpha|\boldsymbol{\Psi};\tau\rangle_E$$

Statistical Closure of Tidal Tensor

- **Conditional average of tidal tensor, since**

$$\frac{d}{d\tau}\Sigma_{ij} + \mathcal{H}(\tau)\Sigma_{ij} + \frac{2}{3}\Theta\Sigma_{ij} + \Sigma_{ik}\Sigma^k_j - \frac{1}{3}\Sigma_{mn}\Sigma^{mn}\delta^K_{ij} = \boxed{-\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle}$$

- **By definition,**

$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle \mathcal{P}(\mathbf{\Psi}; \tau) = \int d\boldsymbol{E}\; E_{ij}\mathcal{P}(\boldsymbol{E}, \mathbf{\Psi}; \tau) = \int d\boldsymbol{E}\; E_{ij}\mathcal{P}(\mathbf{\Gamma}; \tau)$$

- **Gaussian variables**

$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle = \xi^{\varepsilon\psi}_{ij,\alpha}\left(\xi^{\psi\psi}\right)^{-1}_{\alpha\beta}\Psi_\beta,$$

$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle = \frac{4\pi G_N \bar{\rho} a^2}{45}\left(\frac{15\sigma^2_{\delta\theta}}{2\sigma^2_{\theta\theta}}\right)\left(3\delta^K_{im}\delta^K_{jn} + 3\delta^K_{in}\delta^K_{jm} - 2\delta^K_{ij}\delta^K_{mn}\right)\mathcal{A}_{mn}$$

$$= 4\pi G_N \bar{\rho} a^2 \left(\frac{\sigma^2_{\delta\theta}}{\sigma^2_{\theta\theta}}\right)\left(\mathcal{A}_{ij} - \frac{\Theta}{3}\delta^K_{ij}\right) = \boxed{4\pi G_N \bar{\rho} a^2 \left(\frac{\sigma^2_{\delta\theta}}{\sigma^2_{\theta\theta}}\right)\Sigma_{ij}}$$

Gaussian Closure

- Check Zel'dovich Approximation first
  - In ZA, $$\mathbf{x}(\tau) = \mathbf{q} + \mathbf{\Psi}(\mathbf{q}, \tau)$$
  - So the particle motion equation
$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}\tau^2} + \mathscr{H}(\tau)\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = -\nabla\Phi$$
  - becomes
$$J(\mathbf{q},\tau)\nabla\cdot\left[\frac{\mathrm{d}^2\mathbf{\Psi}}{\mathrm{d}\tau^2} + \mathscr{H}(\tau)\frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}\tau}\right] = \frac{3}{2}\,\Omega_m\mathscr{H}^2(J-1)$$
  - Which has linear solution (ZA): $\nabla_q\cdot\mathbf{\Psi}^{(1)} = -D_1(\tau)\delta(\mathbf{q})\cdot$
  - On the other hand, we have Poisson equation
$$\nabla^2\Phi = 4\pi G_N\bar{\rho}a^2\delta_\rho$$
  - That means:
$$\varepsilon_{ij} = -\frac{4\pi G_N a^2\bar{\rho}}{\mathcal{H}f}\sigma_{ij}$$

Gaussian Closure

- For Gaussian closure, assume $\theta = -\mathcal{H}(\tau)f(\tau)\delta_\rho$

$$\sigma^2_{\delta\theta}/\sigma^2_{\theta\theta} = -1/\mathcal{H}(\tau)f(\tau)$$

- so that

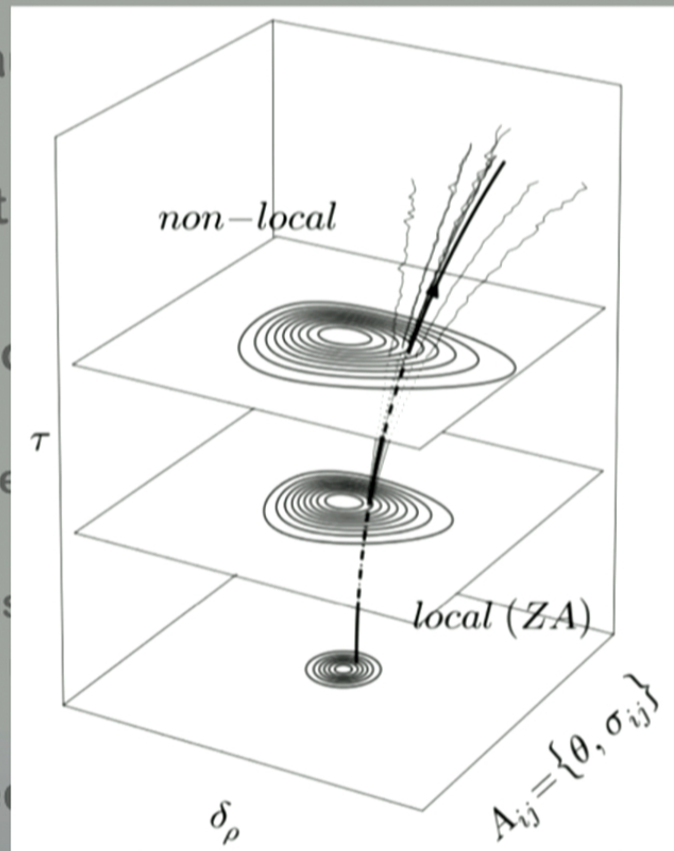
$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle = 4\pi G_N \bar{\rho} a^2 \left(\frac{\sigma^2_{\delta\theta}}{\sigma^2_{\theta\theta}} \right) \sigma_{ij} = \frac{-4\pi G_N \bar{\rho} a^2}{\mathcal{H} f} \sigma_{ij}$$

- ZA? too trivial?
  - Since in ZA $\varepsilon_{ij} = -\frac{4\pi G_N a^2 \bar{\rho}}{\mathcal{H} f} \sigma_{ij}$, it's also local.
  - Statistical closure = ***Localization procedure***
  - So it is **required** that **Gaussian closure = ZA**

- Verified the method !

Gaussian Closure

- For Ga … $-\mathcal{H}(\tau)f(\tau)\delta_\rho$
- so that … $\frac{4\pi G_N\bar{\rho}a^2}{\mathcal{H}f}\sigma_{ij}$
- ZA? to
  - Since … o local.
  - Statis … *edure*
  - So it … e = ZA
- Verifie



Gaussian Closure

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- ZA? too trivial?
  - Since in ZA $\varepsilon_{ij} = -\frac{4\pi G_N a^2 \bar{\rho}}{\mathcal{H} f} \sigma_{ij}$, it's also local.
  - Statistical closure = ***Localization procedure***
  - So it is required that Gaussian closure = ZA

- Verified the method !

Gaussian Closure

- For linearity assumption (also ZA)

$$\frac{d}{d\tau}\theta + \mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij} = \frac{4\pi G_N \bar{\rho} a^2}{\mathcal{H} f}\theta$$

- Not $\frac{d}{d\tau}\theta = -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij}\right] - 4\pi G_N \bar{\rho} a^2 \delta_\rho$

- ZA != spherical collapse (SC)

- Gaussian closure:
  - No approximation until linear continuity equation
  - Consistent with both ZA & SC

$$\frac{d}{d\tau}\delta_\rho = -(1+\delta_\rho)\theta,$$

$$\frac{d}{d\tau}\theta = -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \cancel{\sigma^{ij}\sigma_{ij}}\right] - 4\pi G_N \bar{\rho} a^2 \delta_\rho,$$

Weakly Non-Gaussian Closure

- Non-linearity & non-Guassianity
  - statistical closure: ***keep all nonlinearity***

$$\frac{d}{d\tau}\mathbf{\Psi}(\tau) = \langle \chi | \mathbf{\Psi}; \tau \rangle_L .$$

  - Except for the tidal tensor ...
  - By definition

$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle \mathcal{P}(\mathbf{\Psi}; \tau) = \int d\boldsymbol{E}\ E_{ij} \mathcal{P}(\boldsymbol{E}, \mathbf{\Psi}; \tau) = \int d\boldsymbol{E}\ E_{ij} \mathcal{P}(\mathbf{\Gamma}; \tau)$$

  - In general, PDF expansion

$$\mathcal{P}(\mathbf{\Gamma}) = \exp\left[\sum_{n\geq 3} \frac{(-1)^n}{n!} \xi^{(n)}_{\alpha_1\cdots\alpha_n} \frac{\partial^n}{\partial\Gamma_{\alpha_1}\cdots\partial\Gamma_{\alpha_n}}\right] \mathcal{P}_G(\mathbf{\Gamma})$$

Weakly Non-Gaussian Closure

- **Expand to the 3rd order cumulant:**

$$\langle \varepsilon_{ij}|\boldsymbol{\Psi}\rangle = \left[1+\frac{1}{3!}\mathcal{P}_G^{-1}(\boldsymbol{\Psi})\xi^{\psi\psi\psi}_{\alpha\beta\gamma}\left(\frac{\partial^3}{\partial\Psi_\alpha\partial\Psi_\beta\partial\Psi_\gamma}\mathcal{P}_G(\boldsymbol{\Psi})\right)\right]\langle\varepsilon_{ij}|\boldsymbol{\Psi}\rangle_G - \frac{1}{3!}\mathcal{P}_G^{-1}(\boldsymbol{\Psi})\xi^{\Gamma\Gamma\Gamma}_{\alpha\beta\gamma} \times \int dE\; E_{ij}\left(\frac{\partial^3}{\partial\Gamma_\alpha\partial\Gamma_\beta\partial\Gamma_\gamma}\mathcal{P}_G(\boldsymbol{\Gamma})\right).$$

  - Where $\boldsymbol{\Gamma} = \{\boldsymbol{E}, \boldsymbol{\Psi}\}$.
  - The corrections expressed as

$$\langle \varepsilon_{ij}|\boldsymbol{\Psi}\rangle = \langle\varepsilon_{ij}|\boldsymbol{\Psi}\rangle_G + \frac{1}{2}\left[\xi^{\varepsilon\psi\psi}_{ij,\alpha\beta} - \xi^{\psi\psi\psi}_{\alpha\beta\gamma}\xi^{\varepsilon\psi}_{ij,\delta}(\xi^{\psi\psi})^{-1}_{\delta\gamma}\right]\left[\boxed{-\left(\xi^{\psi\psi}\right)^{-1}_{\alpha\beta}} + \left(\xi^{\psi\psi}\right)^{-1}_{\alpha\lambda}\left(\xi^{\psi\psi}\right)^{-1}_{\beta\tau}\Psi_\lambda\Psi_\tau\right]$$

  - Therefore,

$$\Delta_{\langle\varepsilon_{ij}|\boldsymbol{\Psi}\rangle} = \langle\varepsilon_{ij}|\boldsymbol{\Psi}\rangle - \langle\varepsilon_{ij}|\boldsymbol{\Psi}\rangle_G = \frac{1}{2}\xi^{(\varepsilon-\sigma)\psi\psi}_{ij,\alpha\beta}(\xi^{\psi\psi})^{-1}_{\alpha\delta}\;(\xi^{\psi\psi})^{-1}_{\beta\lambda}\Psi_\delta\Psi_\lambda,$$

Weakly Non-Gaussian Closure

- Eventually the next order correction

$$\Delta_{\langle \varepsilon_{ij} | \Psi \rangle} = (Q_\rho \Delta_\rho + Q_\theta \Theta)\, \Sigma_{ij} + Q_{\Sigma^2} (\widetilde{\Sigma^2})_{ij},$$

  - Where $(\widetilde{\Sigma^2})_{ij} = \Sigma_i{}^m \Sigma_{mj} - \frac{1}{3} (\Sigma^{mn} \Sigma_{mn}) \delta^K_{ij}$
  - and coefficients

$$Q_\rho = D_2 D_3 \left[\frac{1}{5} \tilde{\xi}_1^{(\varepsilon-\sigma)AA} + 6 \left(\frac{D_1}{D_2} \right) \xi^{(\varepsilon-\sigma)\delta A} \right]$$
$$Q_\theta = D_3 D_5 \left[\frac{1}{5} \tilde{\xi}_1^{(\varepsilon-\sigma)AA} + 6 \left(\frac{D_2}{D_5} \right) \xi^{(\varepsilon-\sigma)\delta A} \right]$$
$$Q_{\Sigma^2} = 4 D_3^2 \xi_4^{(\varepsilon-\sigma)AA}$$

- In general, as higher order polynomials

Generalization

- Up to now, only fluid description
  - possible to generalize?

- Effective trajectory:
  - Only statistics evolution is important
  - Many sources of stochasticity
  - Shell-crossing, internal dynamics, halo formation, accretion, merger, fragmentation, etc.

- e.g. Shell crossing:
  - Extra source of Eulerian equation

$$\varsigma_i(\mathbf{x}) = \frac{1}{\rho}\nabla_j(\rho\pi_{ij})$$

  - where

$$\pi_{ij}(\mathbf{x},\tau) = \frac{1}{\rho}\int \frac{d^3p}{a^2m^2}\,(p_i-\bar{p}_i)(p_j-\bar{p}_j)f(\mathbf{x},\mathbf{p},\tau) = \frac{1}{\rho}\sum_s \rho^{(s)}\left[u_i^{(s)}-\bar{u}_i\right]\left[u_j^{(s)}-\bar{u}_j\right]$$

Stochastic Generalization

- **Stochastic noise**

$$\frac{d}{d\tau}\psi_\alpha(\tau) = \chi_\alpha(\boldsymbol{\psi},\tau) + \zeta_\alpha(\tau)$$

  - **Where** $\langle \zeta_\alpha(\mathbf{x},\tau)\zeta_\beta(\mathbf{x},\tau')\rangle = \xi^{\zeta}_{\alpha\beta}(\tau)\delta_D(\tau-\tau')$

- **PDF evolution: Fokker-Planck eq.**

$$\frac{\partial}{\partial\tau}\mathcal{D} + \frac{\partial}{\partial x_i}U_i\mathcal{D} + \frac{\partial}{\partial\Psi_\alpha}\langle\chi_\alpha|\boldsymbol{\Psi}\rangle\mathcal{D} = \frac{1}{2}\xi^{\zeta}_{\alpha\beta}(\tau)\frac{\partial^2}{\partial\Psi_\alpha\partial\Psi_\beta}\mathcal{D}$$

- **Langevin equation**

$$\frac{d}{d\tau}\Psi_\alpha(\tau) = \langle\chi_\alpha(\tau)|\boldsymbol{\Psi}(\tau)\rangle + \zeta_\alpha(\tau)$$

  - **In particular ?**

$$\frac{d}{d\tau}\omega_i + \mathcal{H}(\tau)\omega_i + \frac{2}{3}\theta\omega_i - \sigma_i^j\omega_j = (stochastic\ term...).$$

Summary

- Summary:
  - Statistical closure method
  - PDF conserved
  - Gaussian level, recover ZA
  - Non-Gaussian level, tidal tensor expands as polynomials of dynamical variables

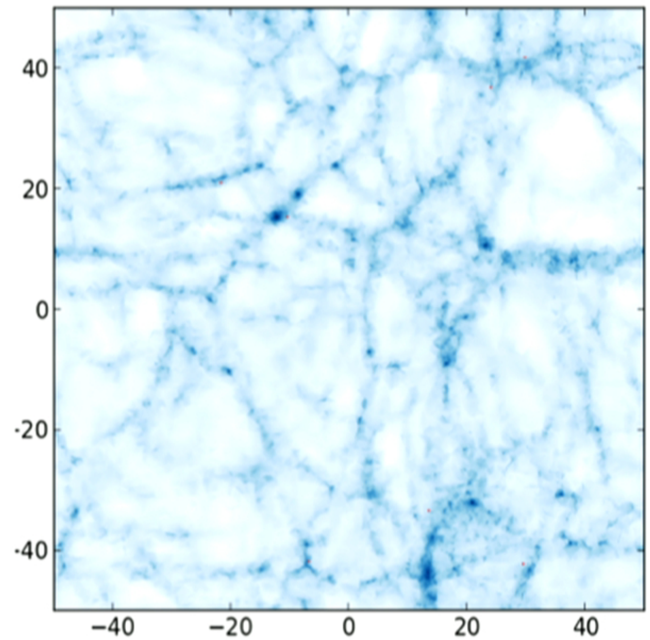
- How about realistic halos (or other objects) ?
  - everything is effective (in this picture)
  - carefully choose your ensemble
  - e.g. conditioning on peaks
  - evolution of scattering as well

Example: *Kinematics of Cosmic Web*

- **Cosmic web classification**
  - **Velocity gradient tensor**

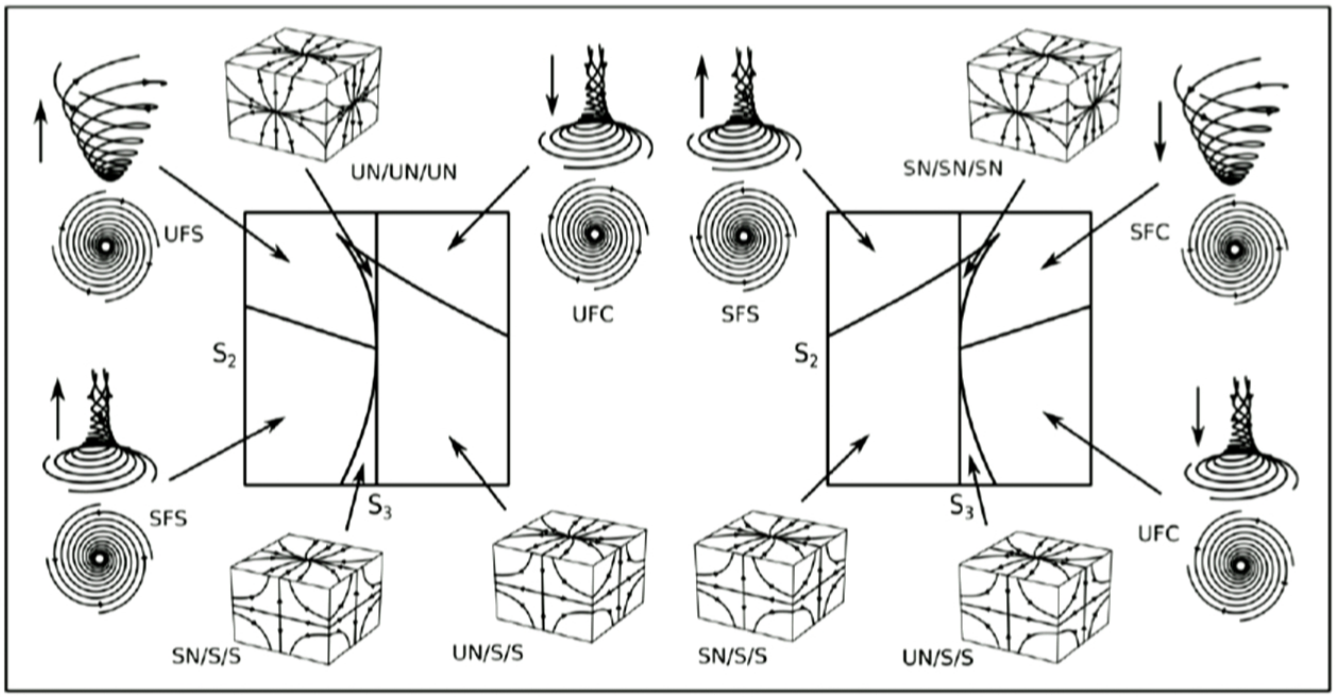
- **where** $$\mathbf{A} = \mathbf{R}^{-1}\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}\mathbf{R},$$
  - **halo:** $\lambda_i < 0,$
  - **void:** $\lambda_i > 0$
  - **filament /wall:** indefinite

$$\det[\mathbf{A} - \lambda\mathbf{I}] = \lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = 0,$$



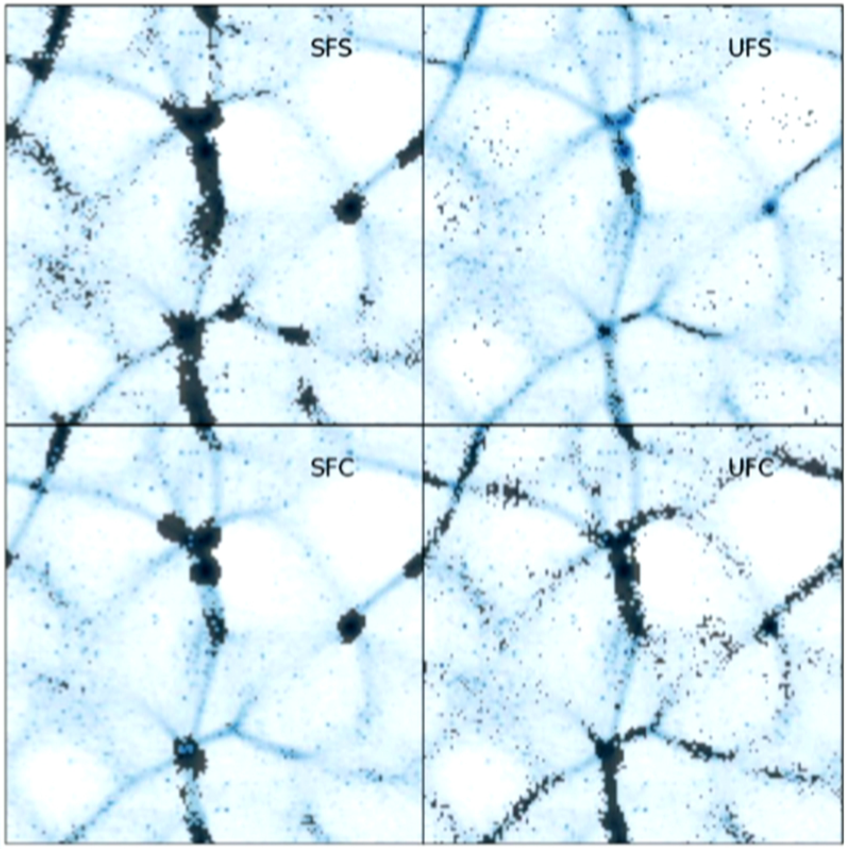
Wang et al. 2014

Example: *Kinematics of Cosmic Web*



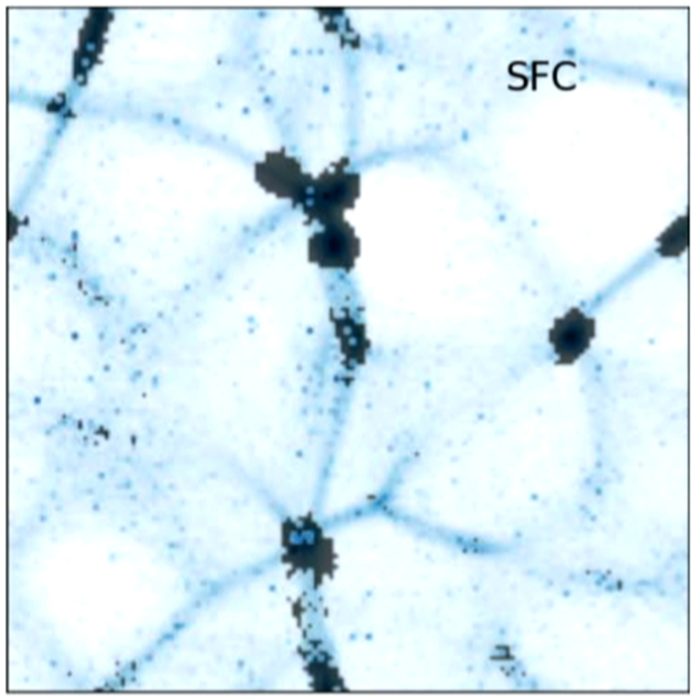
Wang et al. 2014

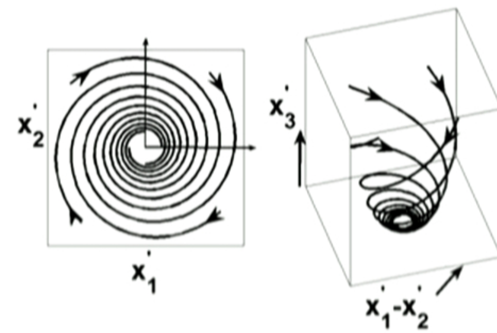
Rotational flow – *spatial distribution*

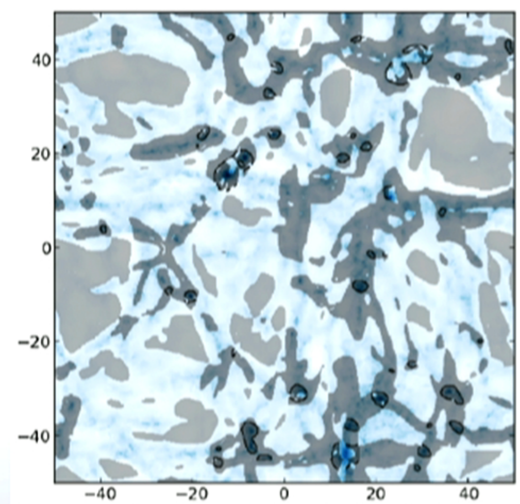


Wang et. al. 2014

Emergence of Vorticity

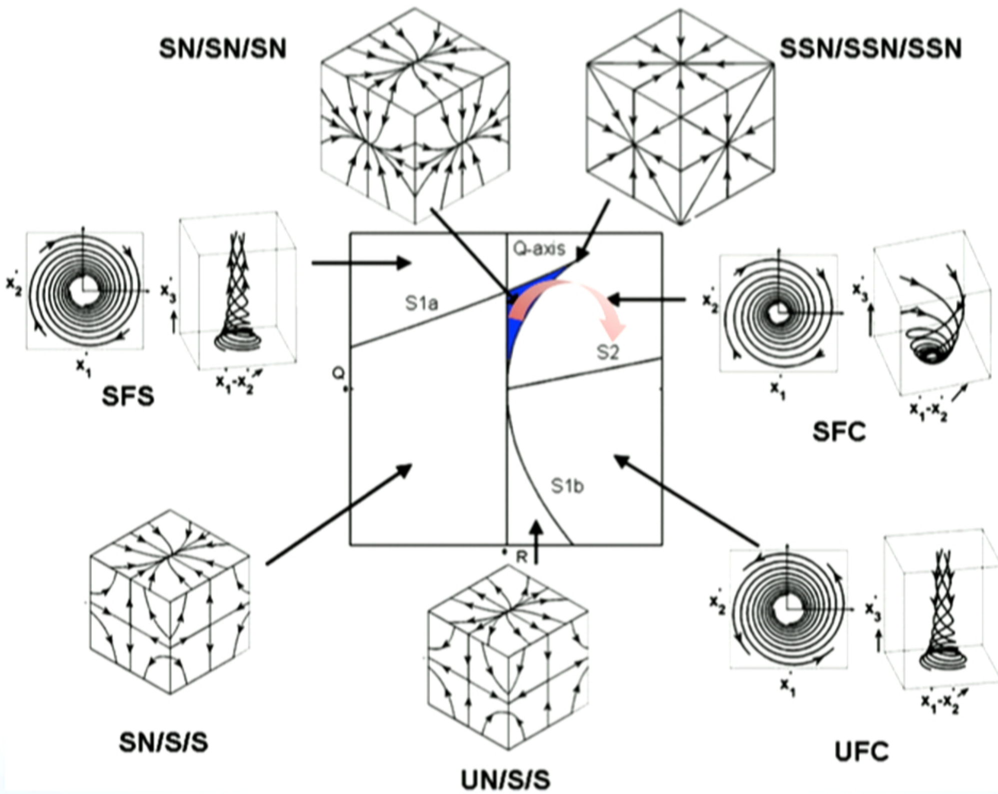






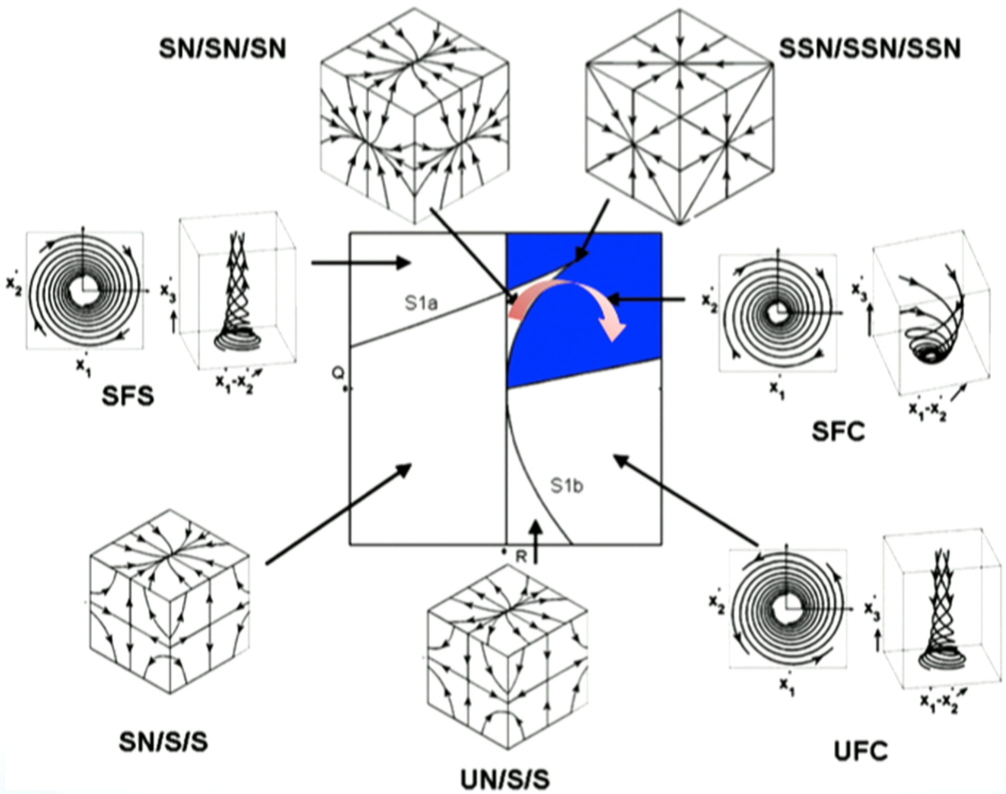
Potential – Vortical transition

$s_1 > 0,$



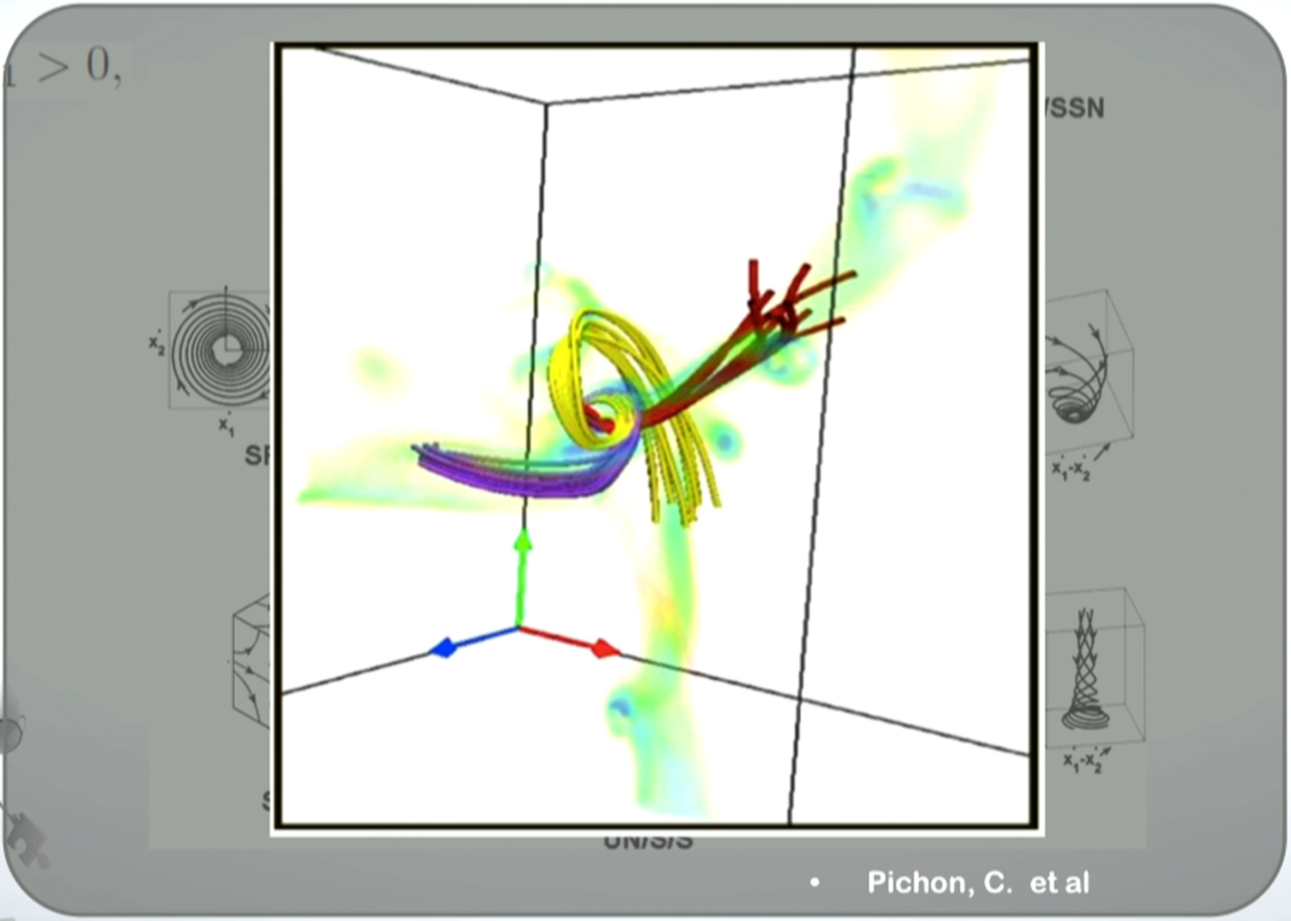
Potential – Vortical transition

$s_1 > 0,$



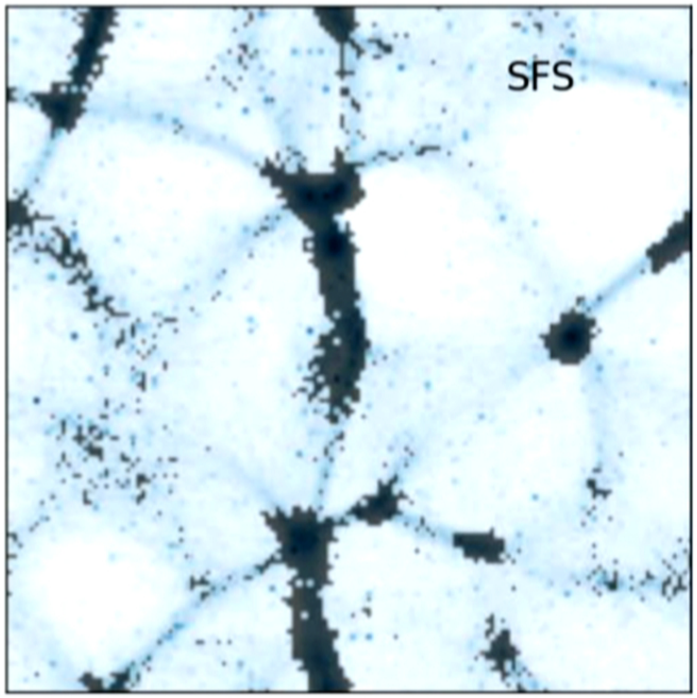
Potential – Vortical transition

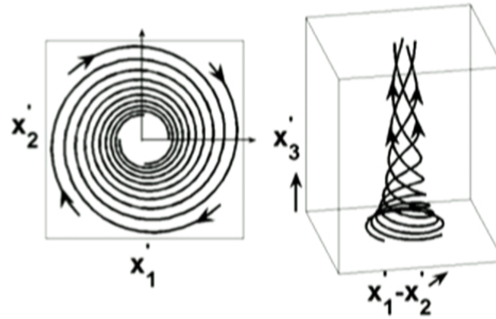
$s_1 > 0,$

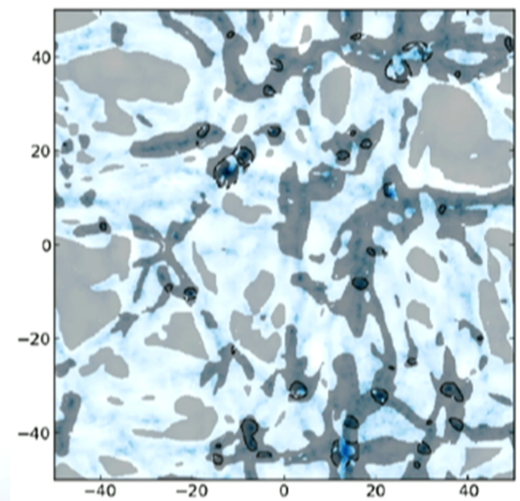


- Pichon, C. et al

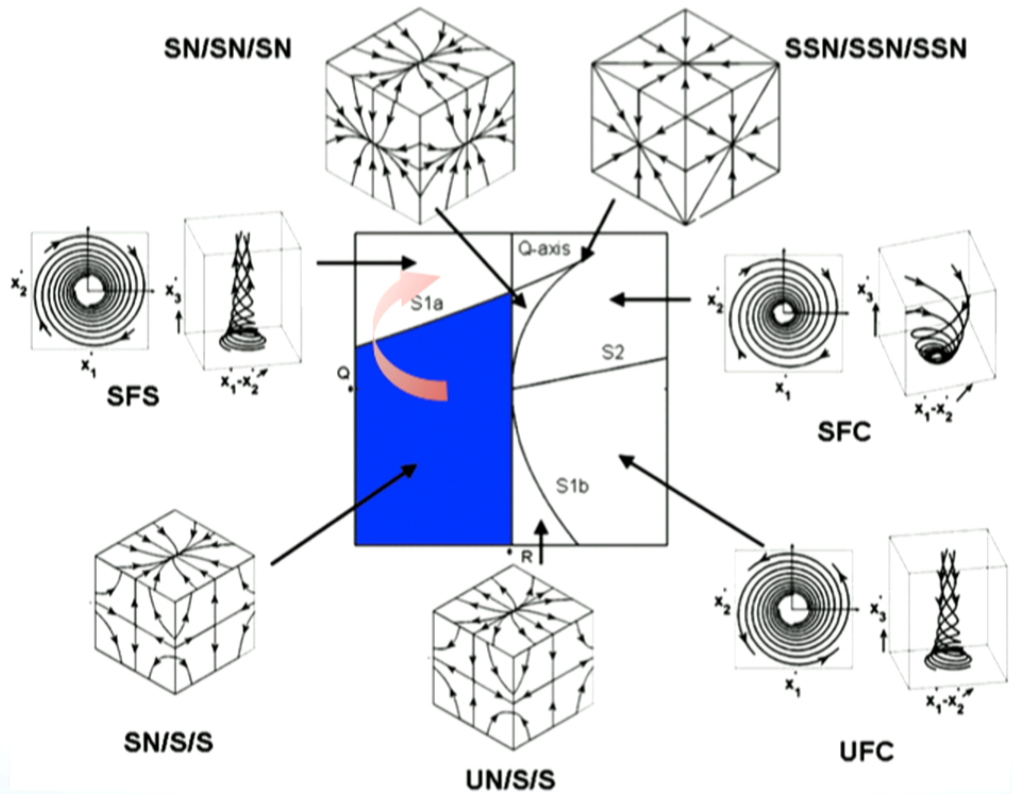
Emergence of Vorticity







Potential – Vortical transition



Duffusion

- Explanation??
  - ***Diffusive process ?***

$$\frac{d}{d\tau}\Psi_\alpha(\tau) = \langle \chi_\alpha(\tau)|\mathbf{\Psi}(\tau)\rangle + \zeta_\alpha(\tau)$$

  - vorticity generation = ***effective diffusive process***

$$\frac{d}{d\tau}\omega_i + \mathcal{H}(\tau)\omega_i + \frac{2}{3}\theta\omega_i - \sigma_i^j\omega_j = (stochastic\ term...).$$

Thank you !