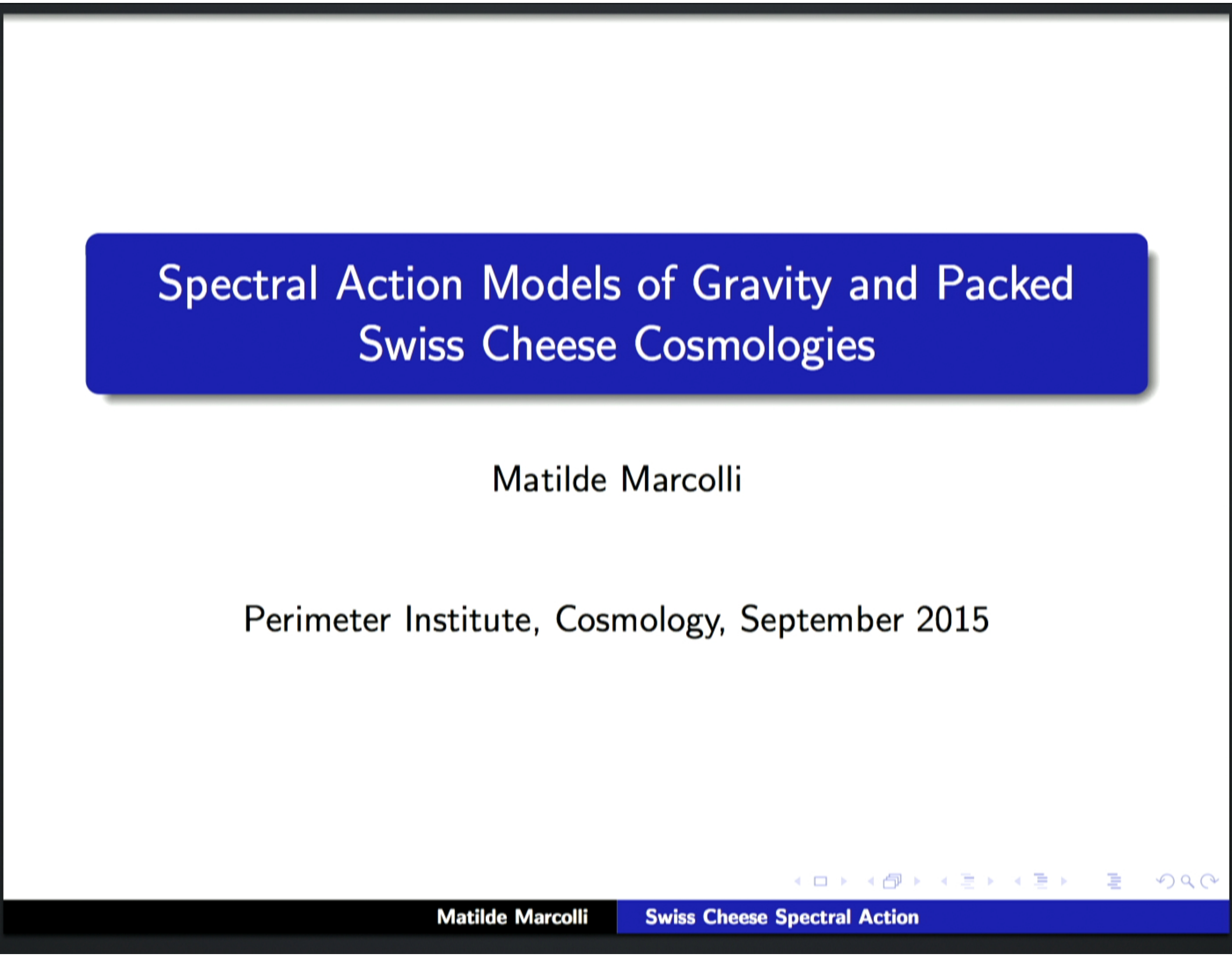
Title: Spectral Action Models of Gravity and Packed Swiss Cheese Cosmology

Date: Sep 12, 2015 02:30 PM

URL: http://pirsa.org/15090060

Abstract: <p>We consider the spectral action as an
action functional for modified gravity on a spacetime
that exhibits a fractal structure modeled on an
Apollonian packing of 3-spheres (packed swiss
cheese) or on a fractal arrangements of dodecahedral
spaces. The contributions in the asymptotic expansion
of the spectral action, that arise from the real poles of
the zeta function, include the Einstein-Hilbert action
with cosmological term and conformal and Gauss-Bonnet
gravity terms. We show that these contributions are
affected by the presence of fractality, which modifies the
corresponding effective gravitational and cosmological
constants, while an additional term appears in the action,
which is entirely due to fractality. This term is further
affected by a contribution of oscillatory terms coming
from the poles of the zeta function that are off the real
line, which are also a property specific to fractals. We
show that the shape of the slow-roll potential obtained
by scalar perturbations of the Dirac operators is also
affected by the presence of fractality.
The talk is based on joint work with Adam Ball.</p>



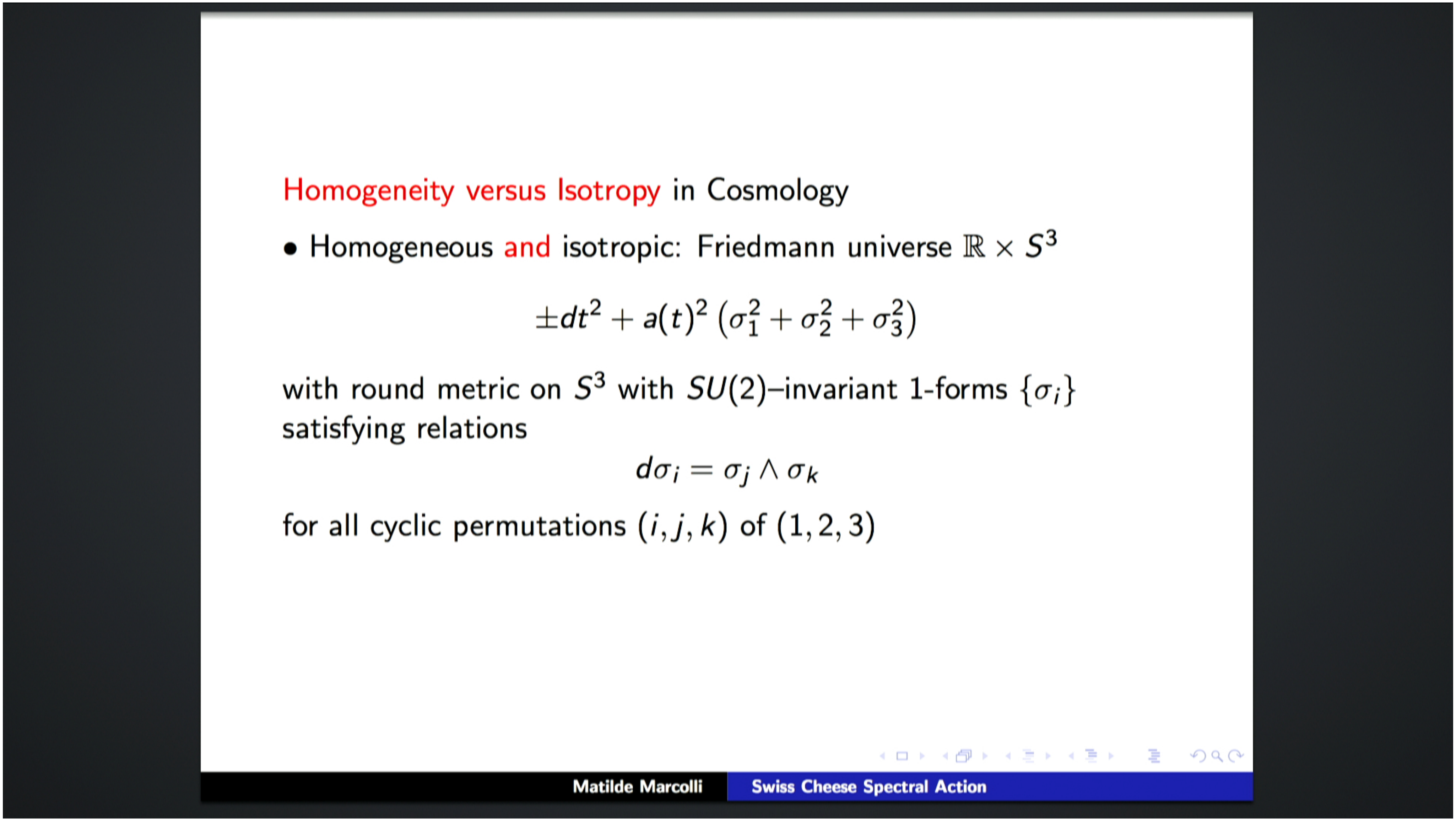
Spectral Action Models of Gravity and Packed Swiss Cheese Cosmologies

Matilde Marcolli

Perimeter Institute, Cosmology, September 2015

Based on:

- Adam Ball, Matilde Marcolli, *Spectral Action Models of Gravity on Packed Swiss Cheese Cosmology*, arXiv:1506.01401



Homogeneity versus Isotropy in Cosmology

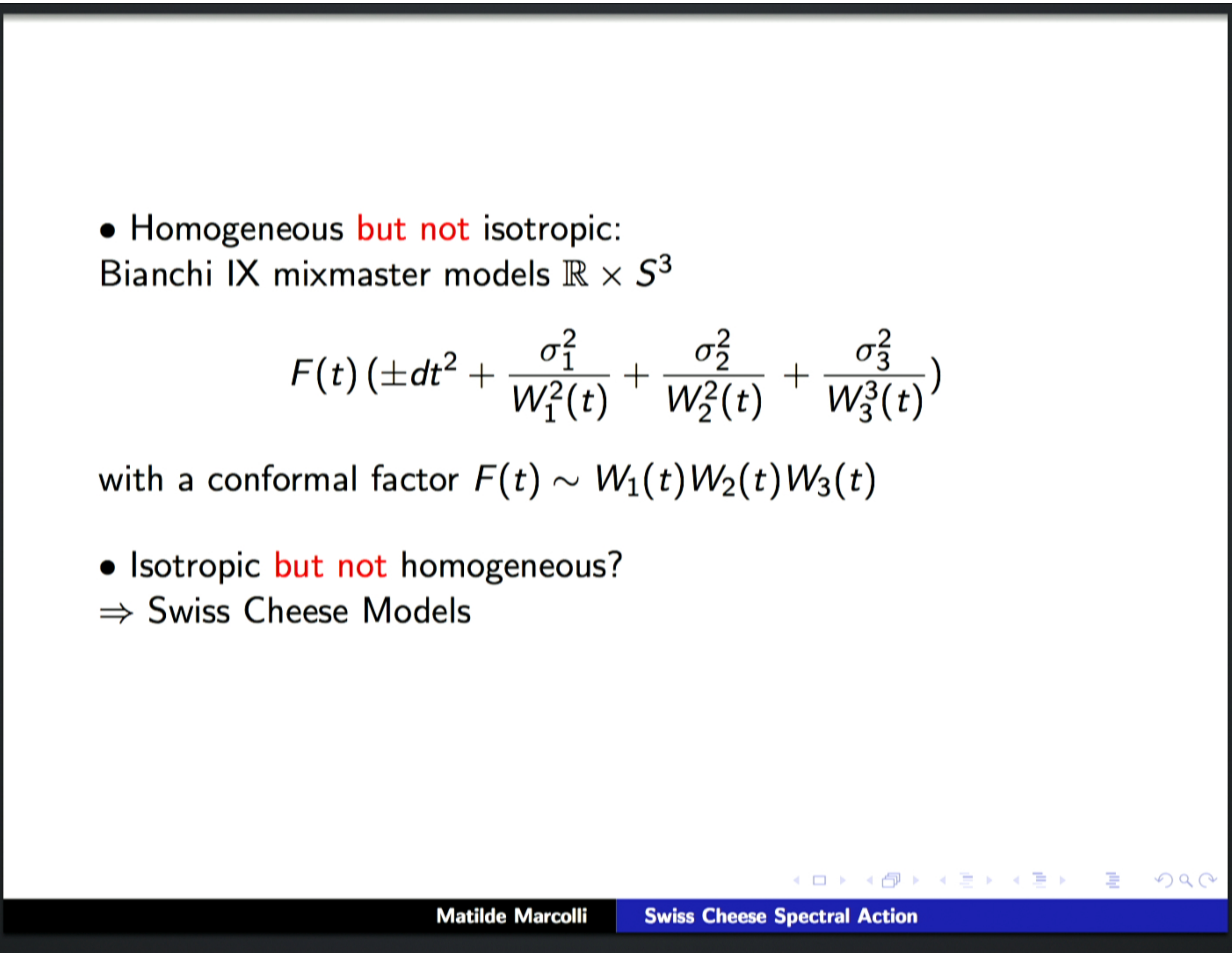
- Homogeneous and isotropic: Friedmann universe $\mathbb{R} \times S^3$

$$\pm dt^2 + a(t)^2 \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)$$

with round metric on S^3 with $SU(2)$–invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$

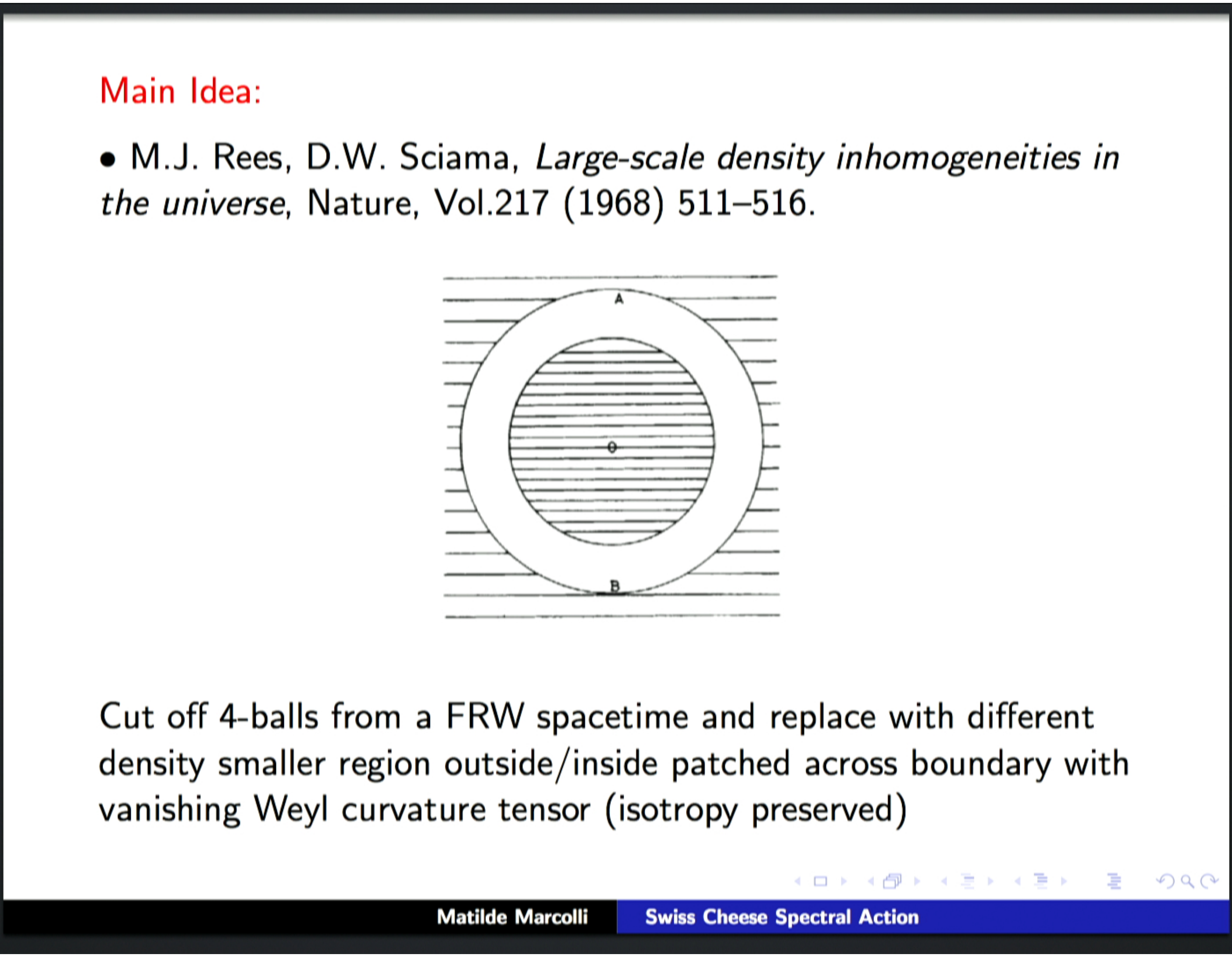


- Homogeneous but not isotropic:
Bianchi IX mixmaster models $\mathbb{R} \times S^3$

$$F(t)\,(\pm dt^2 + \frac{\sigma_1^2}{W_1^2(t)} + \frac{\sigma_2^2}{W_2^2(t)} + \frac{\sigma_3^2}{W_3^3(t)})$$

with a conformal factor $F(t) \sim W_1(t)W_2(t)W_3(t)$

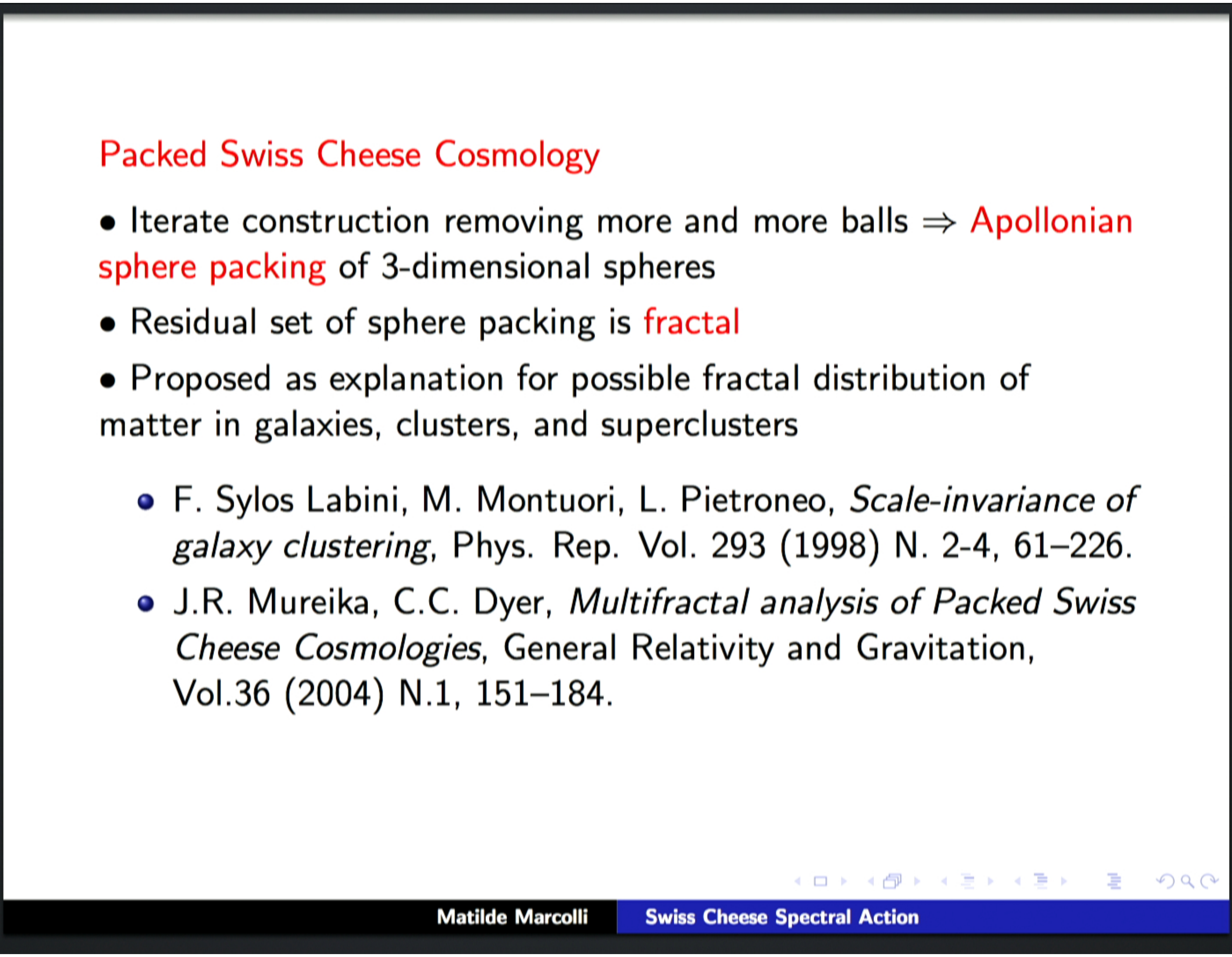
- Isotropic but not homogeneous?
$\Rightarrow$ Swiss Cheese Models



Main Idea:

- M.J. Rees, D.W. Sciama, *Large-scale density inhomogeneities in the universe*, Nature, Vol.217 (1968) 511–516.

Cut off 4-balls from a FRW spacetime and replace with different density smaller region outside/inside patched across boundary with vanishing Weyl curvature tensor (isotropy preserved)



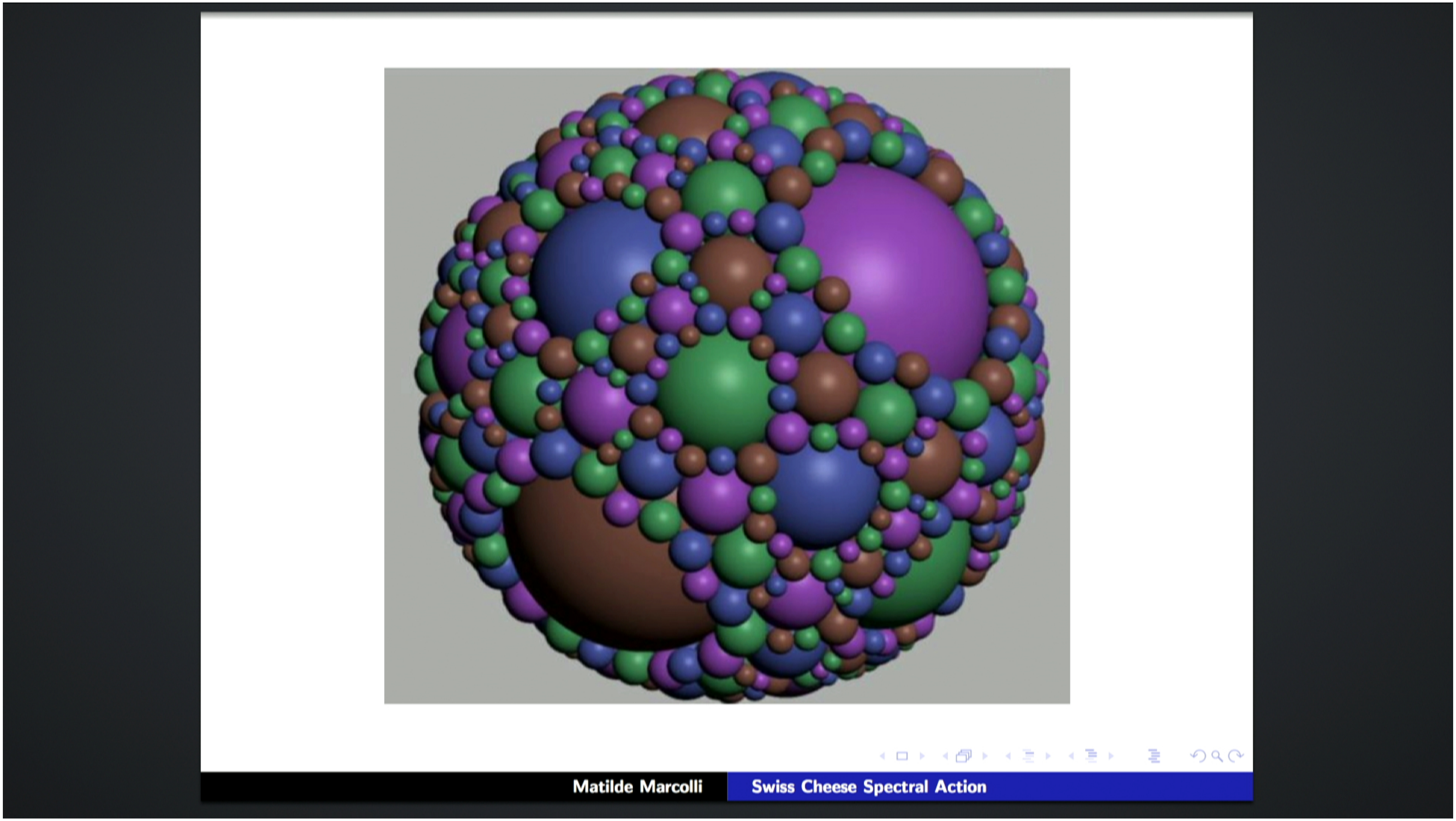
Packed Swiss Cheese Cosmology

• Iterate construction removing more and more balls ⇒ Apollonian sphere packing of 3-dimensional spheres

• Residual set of sphere packing is fractal

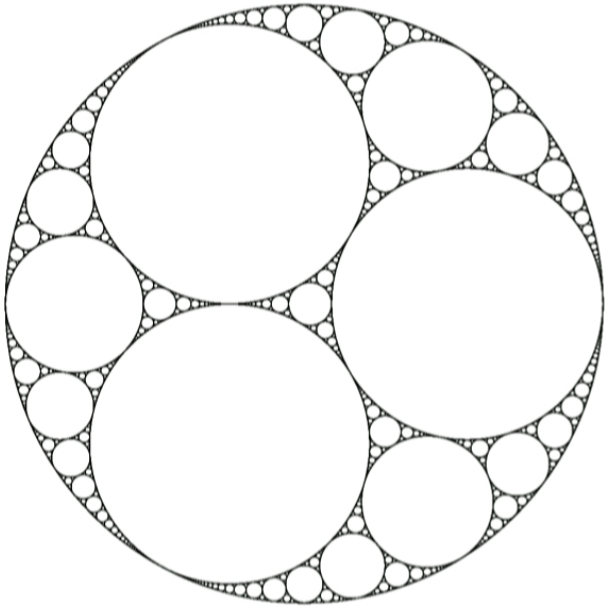
• Proposed as explanation for possible fractal distribution of matter in galaxies, clusters, and superclusters

- F. Sylos Labini, M. Montuori, L. Pietroneo, *Scale-invariance of galaxy clustering*, Phys. Rep. Vol. 293 (1998) N. 2-4, 61–226.
- J.R. Mureika, C.C. Dyer, *Multifractal analysis of Packed Swiss Cheese Cosmologies*, General Relativity and Gravitation, Vol.36 (2004) N.1, 151–184.

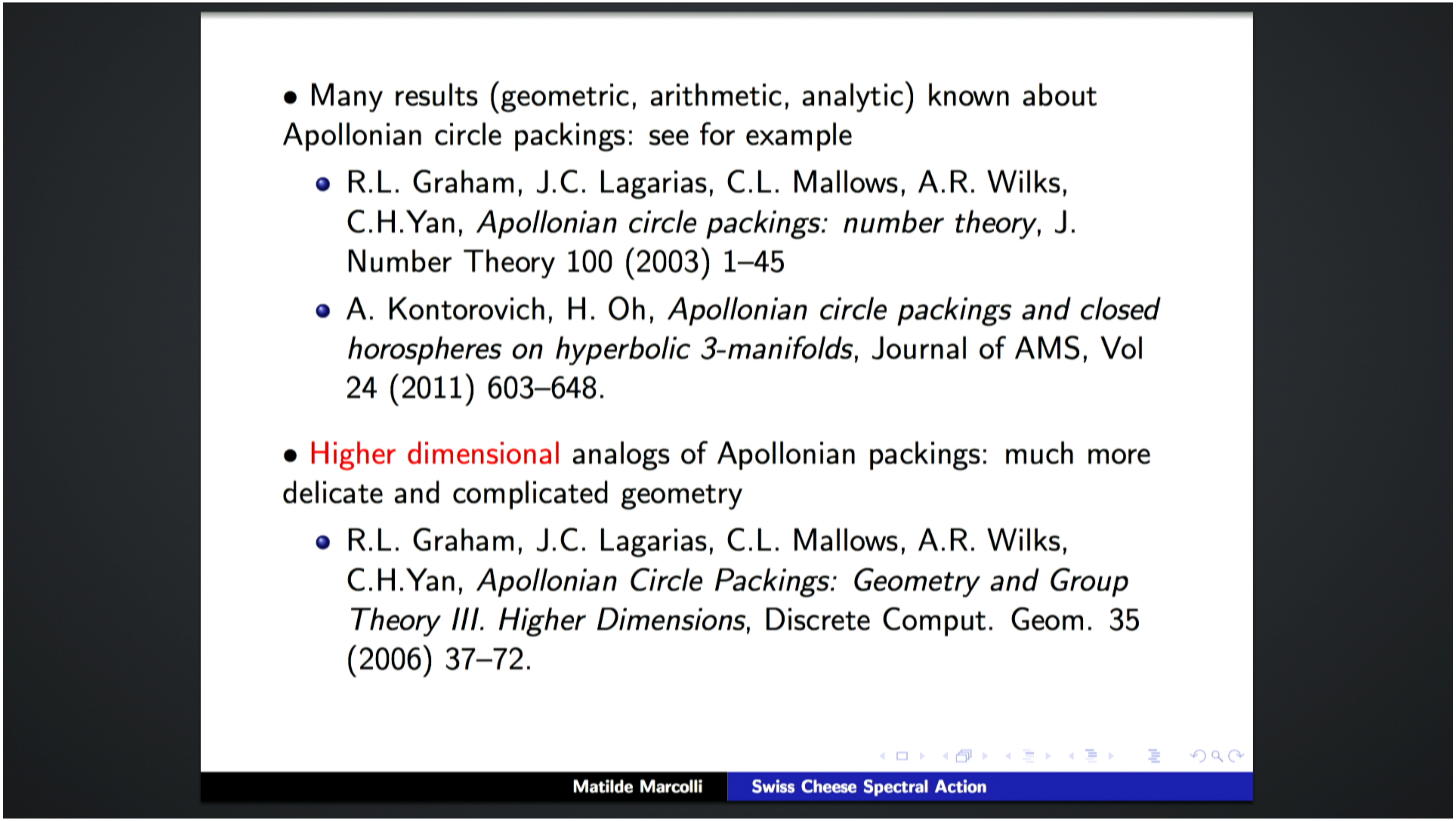


Apollonian sphere packings

- best known and understood case: Apollonian circle packing



Configurations of mutually tanget circles in the plane, iterated on smaller scales filling a full volume region in the unit $2D$ ball: residual set volume zero fractal of Hausdorff dimension 1.30568...



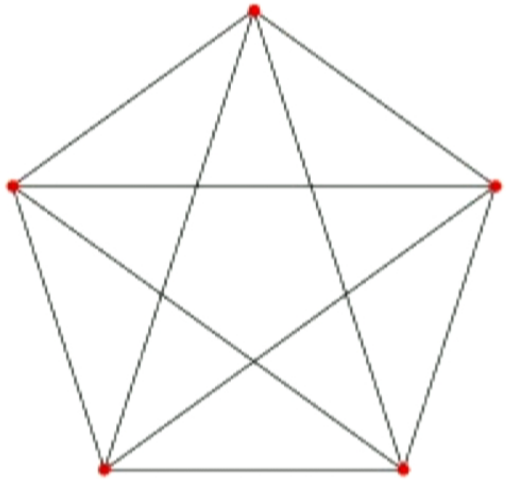
- Many results (geometric, arithmetic, analytic) known about Apollonian circle packings: see for example
  - R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H.Yan, *Apollonian circle packings: number theory*, J. Number Theory 100 (2003) 1–45
  - A. Kontorovich, H. Oh, *Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds*, Journal of AMS, Vol 24 (2011) 603–648.

- Higher dimensional analogs of Apollonian packings: much more delicate and complicated geometry
  - R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H.Yan, *Apollonian Circle Packings: Geometry and Group Theory III. Higher Dimensions*, Discrete Comput. Geom. 35 (2006) 37–72.

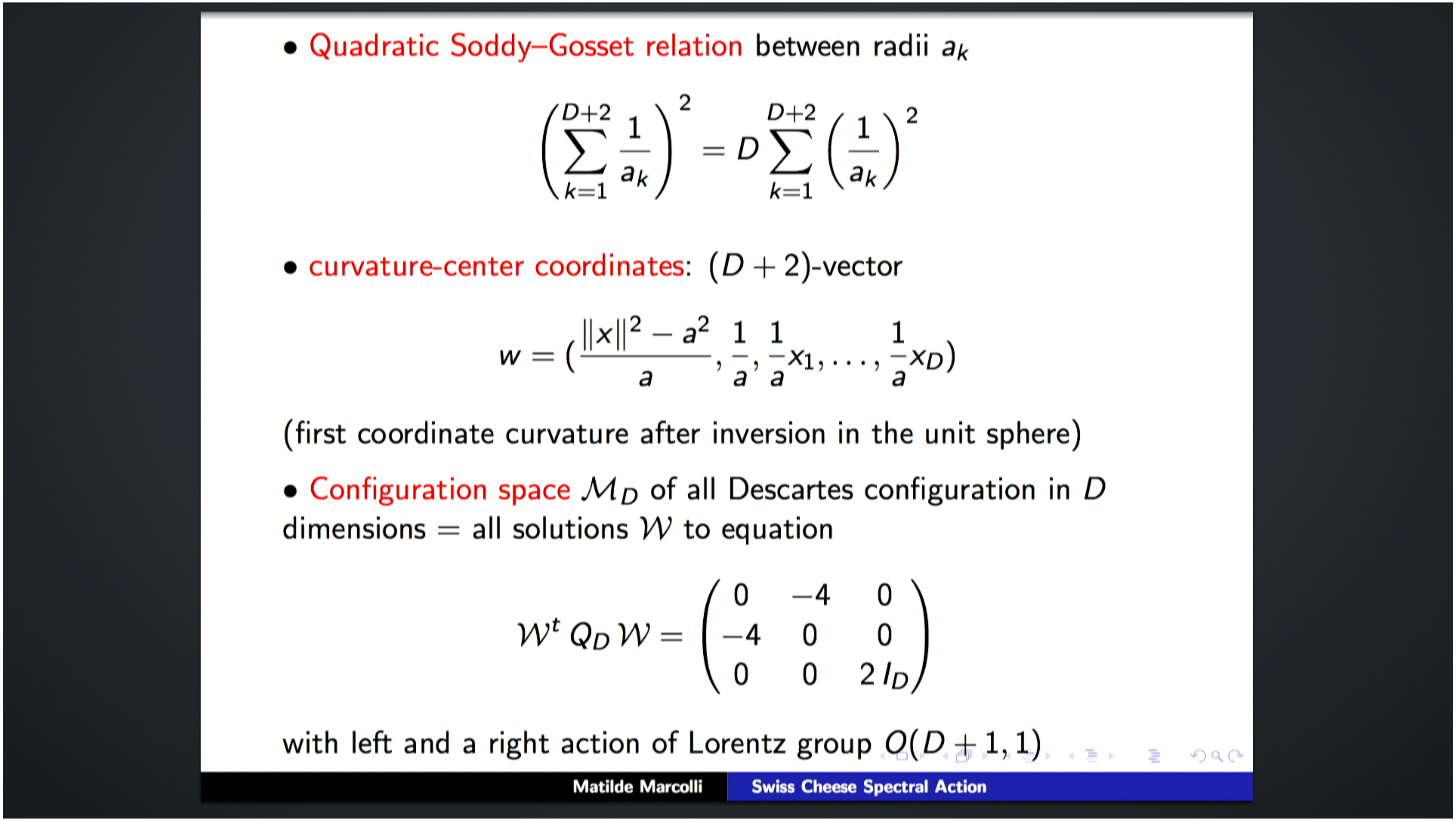
Some known facts on Apollonian sphere packings

- **Descartes configuration** in D dimensions: $D+2$ mutually tangent $(D-1)$-dimensional spheres

- Example: start with $D+1$ equal size mutually tangent S^{D-1} centered at the vertices of D-simplex and one more smaller sphere in the center tangent to all



4-dimensional simplex



- Quadratic Soddy–Gosset relation between radii a_k

$$\left(\sum_{k=1}^{D+2} \frac{1}{a_k}\right)^2 = D \sum_{k=1}^{D+2} \left(\frac{1}{a_k}\right)^2$$

- curvature-center coordinates: $(D+2)$-vector

$$w = \left(\frac{\|x\|^2 - a^2}{a}, \frac{1}{a}, \frac{1}{a}x_1, \ldots, \frac{1}{a}x_D\right)$$

(first coordinate curvature after inversion in the unit sphere)

- Configuration space $\mathcal{M}_D$ of all Descartes configuration in D dimensions = all solutions $\mathcal{W}$ to equation

$$\mathcal{W}^t \, Q_D \, \mathcal{W} = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2\, I_D \end{pmatrix}$$

with left and a right action of Lorentz group $O(D+1,1)$

- Dual Apollonian group $\mathcal{G}_D^\perp$ generated by reflections: inversion with respect to the j-th sphere

$$S_j^\perp = I_{D+2} + 2\,1_{D+2} e_j^t - 4\, e_j e_j^t$$

$e_j = j$-th unit coordinate vector

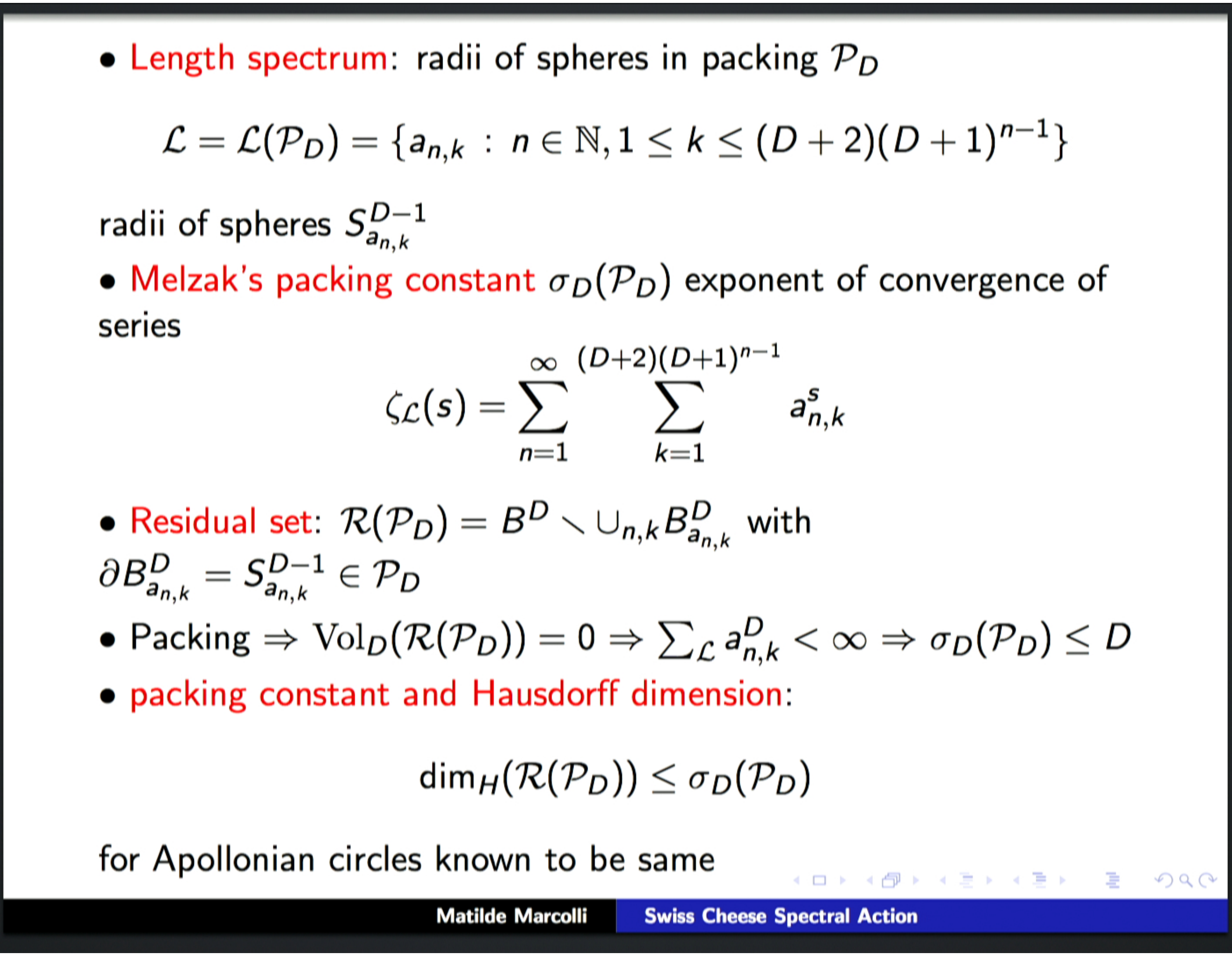
- $D \neq 3$: only relations in $\mathcal{G}_D^\perp$ are $(S_j^\perp)^2 = 1$
- $\mathcal{G}_D^\perp$ discrete subgroup of $\mathrm{GL}(D+2, \mathbb{R})$
- Apollonian packing $\mathcal{P}_D =$ an orbit of $\mathcal{G}_D^\perp$ on $\mathcal{M}_D$

$\Rightarrow$ iterative construction: at n-th step add spheres obtained from initial Descartes configuration via all possible

$$S_{j_1}^\perp S_{j_2}^\perp \cdots S_{j_n}^\perp, \quad j_k \neq j_{k+1}, \forall k$$

there are N_n spheres in the n-th level

$$N_n = (D+2)(D+1)^{n-1}$$



- Length spectrum: radii of spheres in packing $\mathcal{P}_D$

$$\mathcal{L} = \mathcal{L}(\mathcal{P}_D) = \{a_{n,k} \,:\, n \in \mathbb{N}, 1 \leq k \leq (D+2)(D+1)^{n-1}\}$$

radii of spheres $S^{D-1}_{a_{n,k}}$

- Melzak's packing constant $\sigma_D(\mathcal{P}_D)$ exponent of convergence of series

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s$$

- Residual set: $\mathcal{R}(\mathcal{P}_D) = B^D \setminus \cup_{n,k} B^D_{a_{n,k}}$ with $\partial B^D_{a_{n,k}} = S^{D-1}_{a_{n,k}} \in \mathcal{P}_D$
- Packing $\Rightarrow \mathrm{Vol}_D(\mathcal{R}(\mathcal{P}_D)) = 0 \Rightarrow \sum_{\mathcal{L}} a^D_{n,k} < \infty \Rightarrow \sigma_D(\mathcal{P}_D) \leq D$
- packing constant and Hausdorff dimension:

$$\dim_H(\mathcal{R}(\mathcal{P}_D)) \leq \sigma_D(\mathcal{P}_D)$$

for Apollonian circles known to be same

- Sphere counting function: spheres with given curvature bound

$$\mathcal{N}_\alpha(\mathcal{P}_D) = \#\{S^{D-1}_{a_{n,k}} \in \mathcal{P}_D \, : \, a_{n,k} \geq \alpha\}$$

curvatures $c_{n,k} = a_{n,k}^{-1} \leq \alpha^{-1}$

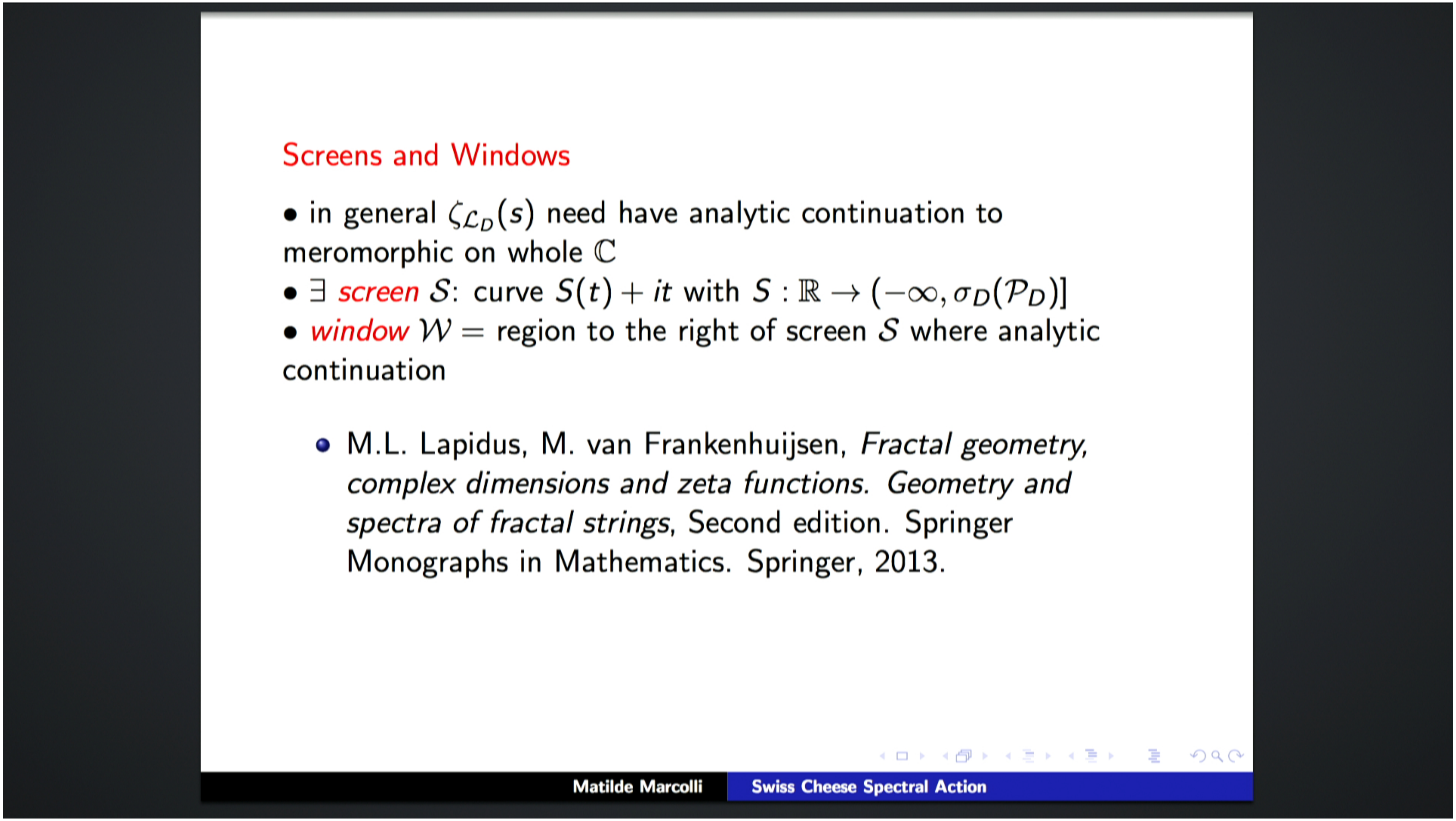
- for Apollonian circles power law (Kontorovich–Oh)

$$\mathcal{N}_\alpha(\mathcal{P}_2) \sim_{\alpha\to 0} \alpha^{-\dim_H(\mathcal{R}(\mathcal{P}_2))}$$

- for higher dimensions (Boyd): packing constant

$$\limsup_{\alpha\to 0} \, -\frac{\log \mathcal{N}_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$$

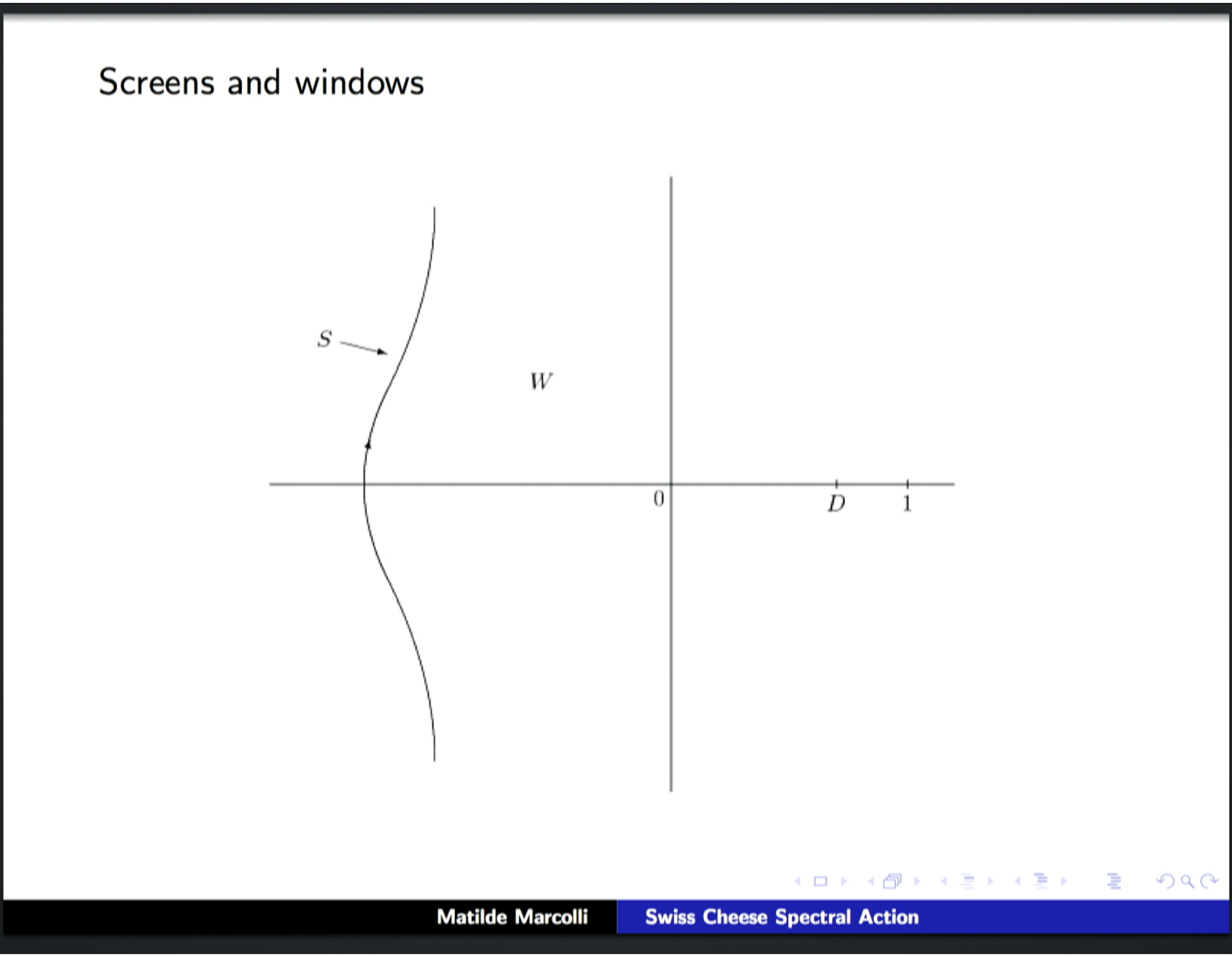
if limit exists $\mathcal{N}_\alpha(\mathcal{P}_D) \sim_{\alpha\to 0} \alpha^{-(\sigma_D(\mathcal{P}_D)+o(1))}$



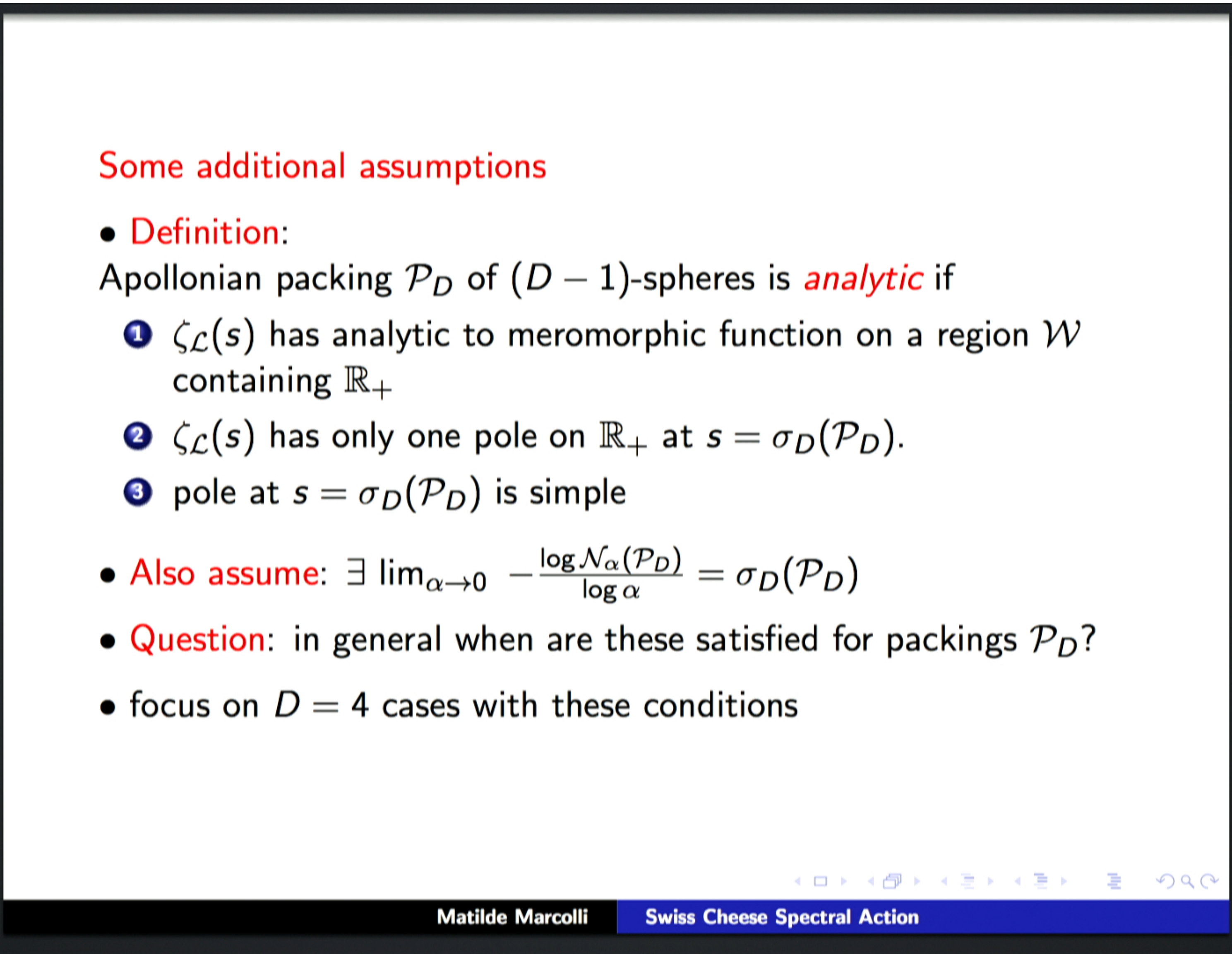
Screens and Windows

- in general $\zeta_{\mathcal{L}_D}(s)$ need have analytic continuation to meromorphic on whole $\mathbb{C}$
- $\exists$ *screen* $\mathcal{S}$: curve $S(t) + it$ with $S : \mathbb{R} \to (-\infty, \sigma_D(\mathcal{P}_D)]$
- *window* $\mathcal{W} =$ region to the right of screen $\mathcal{S}$ where analytic continuation

  - M.L. Lapidus, M. van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions. Geometry and spectra of fractal strings*, Second edition. Springer Monographs in Mathematics. Springer, 2013.



Screens and windows



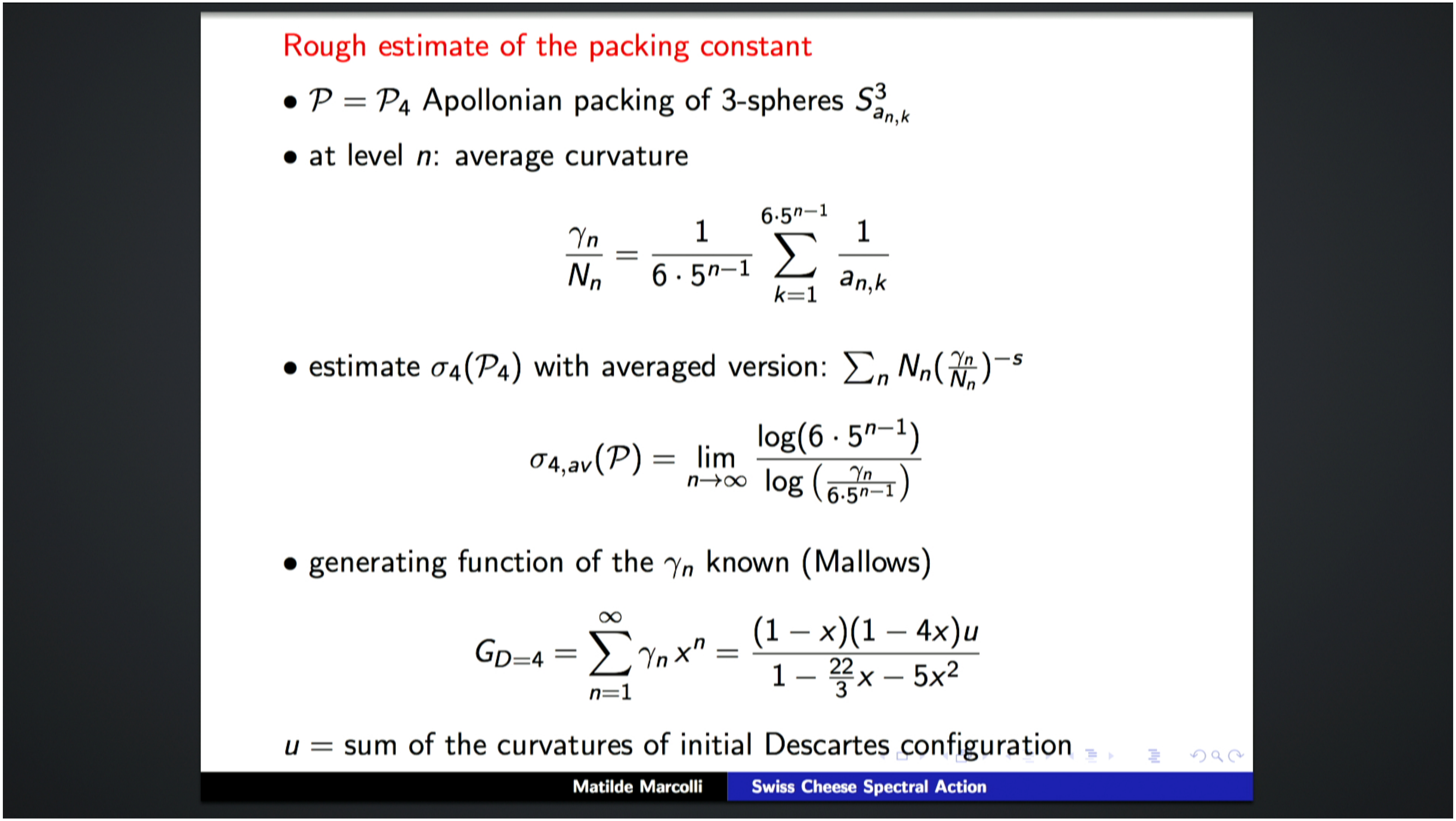
Some additional assumptions

- **Definition**:

Apollonian packing $\mathcal{P}_D$ of $(D-1)$-spheres is *analytic* if

1. $\zeta_{\mathcal{L}}(s)$ has analytic to meromorphic function on a region $\mathcal{W}$ containing $\mathbb{R}_+$
2. $\zeta_{\mathcal{L}}(s)$ has only one pole on $\mathbb{R}_+$ at $s = \sigma_D(\mathcal{P}_D)$.
3. pole at $s = \sigma_D(\mathcal{P}_D)$ is simple

- Also assume: $\exists \lim_{\alpha\to 0} -\frac{\log \mathcal{N}_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$
- Question: in general when are these satisfied for packings $\mathcal{P}_D$?
- focus on $D = 4$ cases with these conditions



Rough estimate of the packing constant

- $\mathcal{P} = \mathcal{P}_4$ Apollonian packing of 3-spheres $S^3_{a_{n,k}}$
- at level n: average curvature

$$\frac{\gamma_n}{N_n} = \frac{1}{6 \cdot 5^{n-1}} \sum_{k=1}^{6 \cdot 5^{n-1}} \frac{1}{a_{n,k}}$$

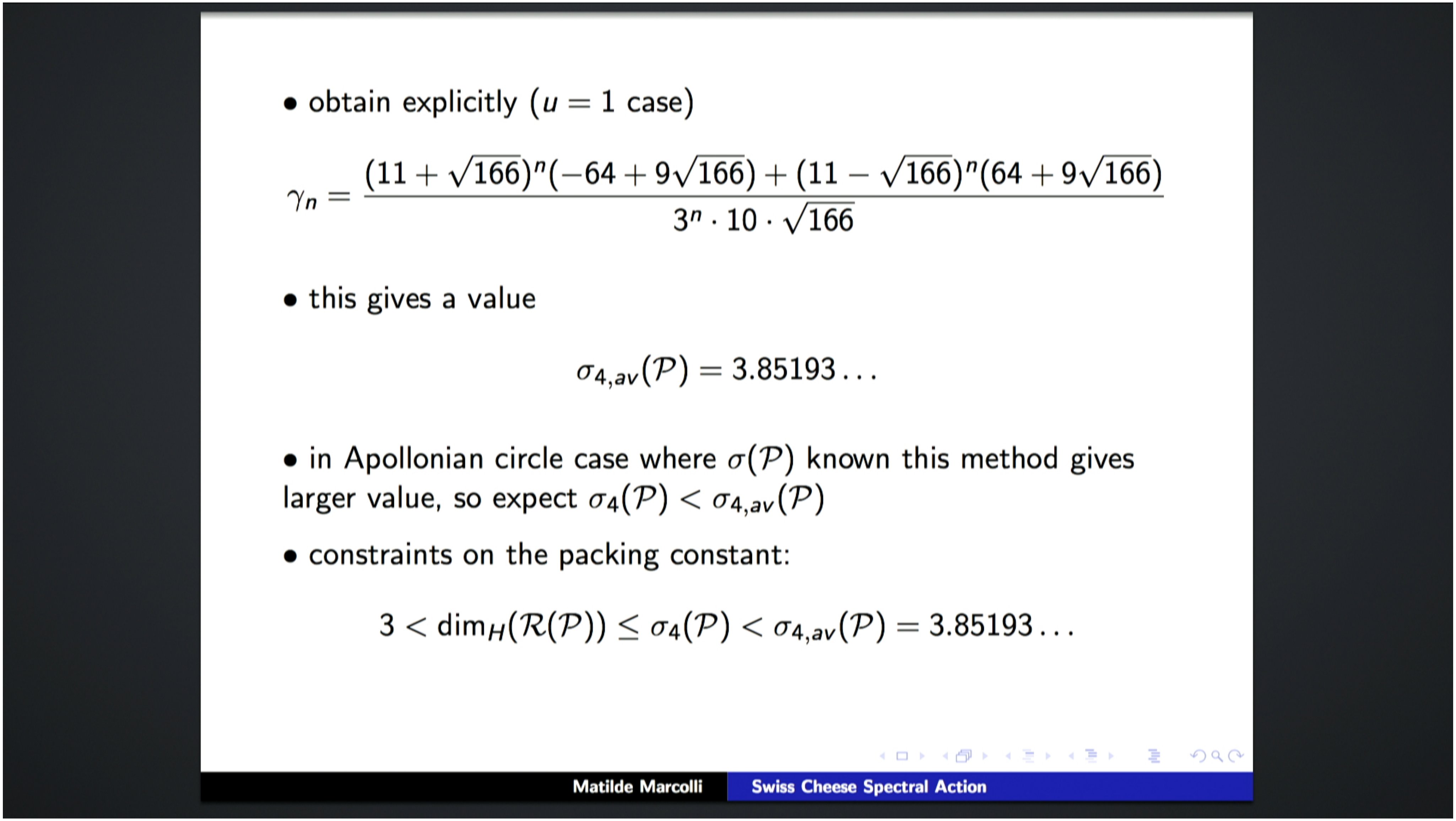
- estimate $\sigma_4(\mathcal{P}_4)$ with averaged version: $\sum_n N_n(\frac{\gamma_n}{N_n})^{-s}$

$$\sigma_{4,av}(\mathcal{P}) = \lim_{n\to\infty} \frac{\log(6 \cdot 5^{n-1})}{\log\left(\frac{\gamma_n}{6 \cdot 5^{n-1}}\right)}$$

- generating function of the γ_n known (Mallows)

$$G_{D=4} = \sum_{n=1}^{\infty} \gamma_n x^n = \frac{(1-x)(1-4x)u}{1 - \frac{22}{3}x - 5x^2}$$

$u =$ sum of the curvatures of initial Descartes configuration



- obtain explicitly ($u = 1$ case)

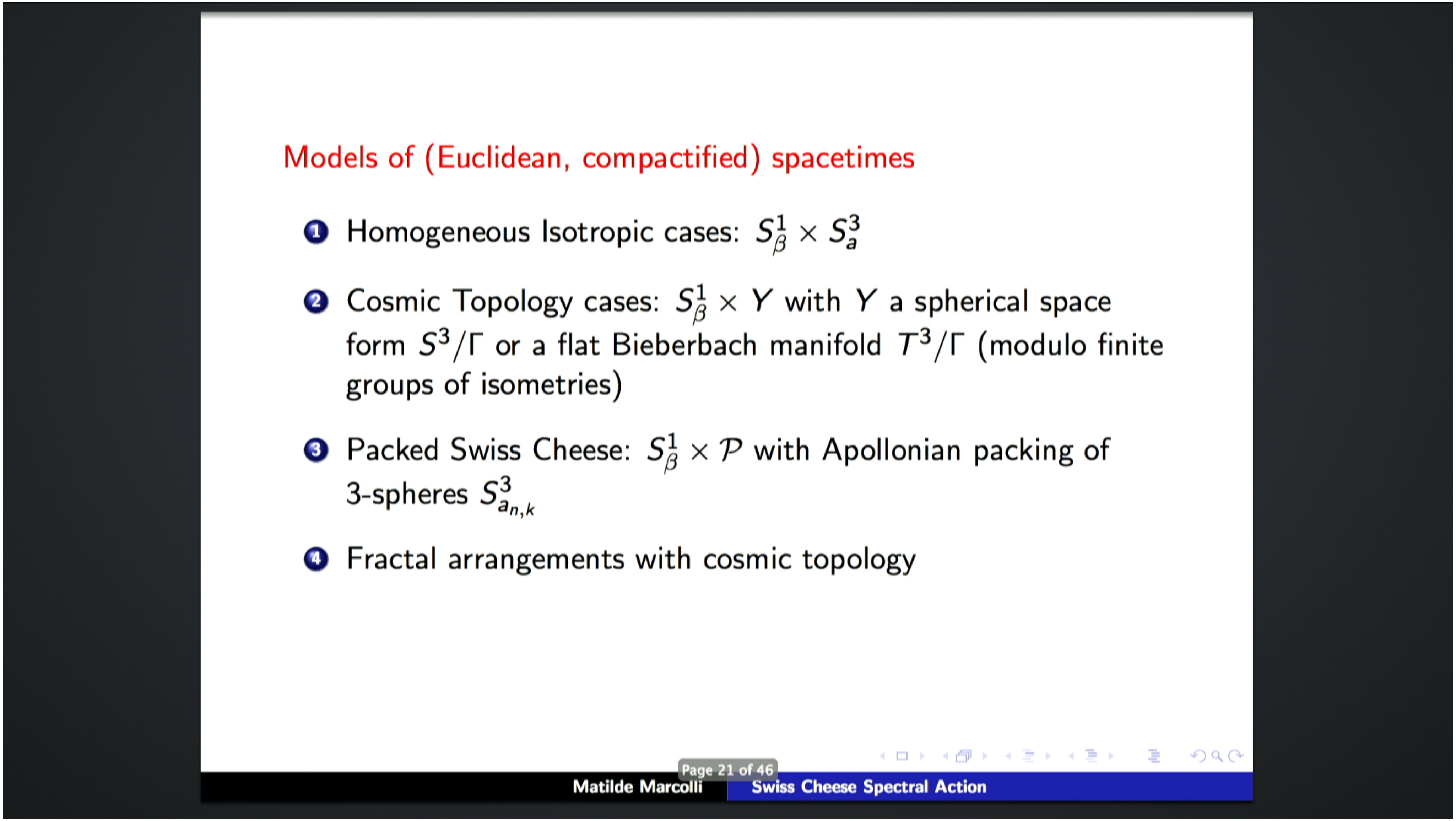
$$\gamma_n = \frac{(11+\sqrt{166})^n(-64+9\sqrt{166})+(11-\sqrt{166})^n(64+9\sqrt{166})}{3^n\cdot 10\cdot\sqrt{166}}$$

- this gives a value

$$\sigma_{4,av}(\mathcal{P}) = 3.85193\dots$$

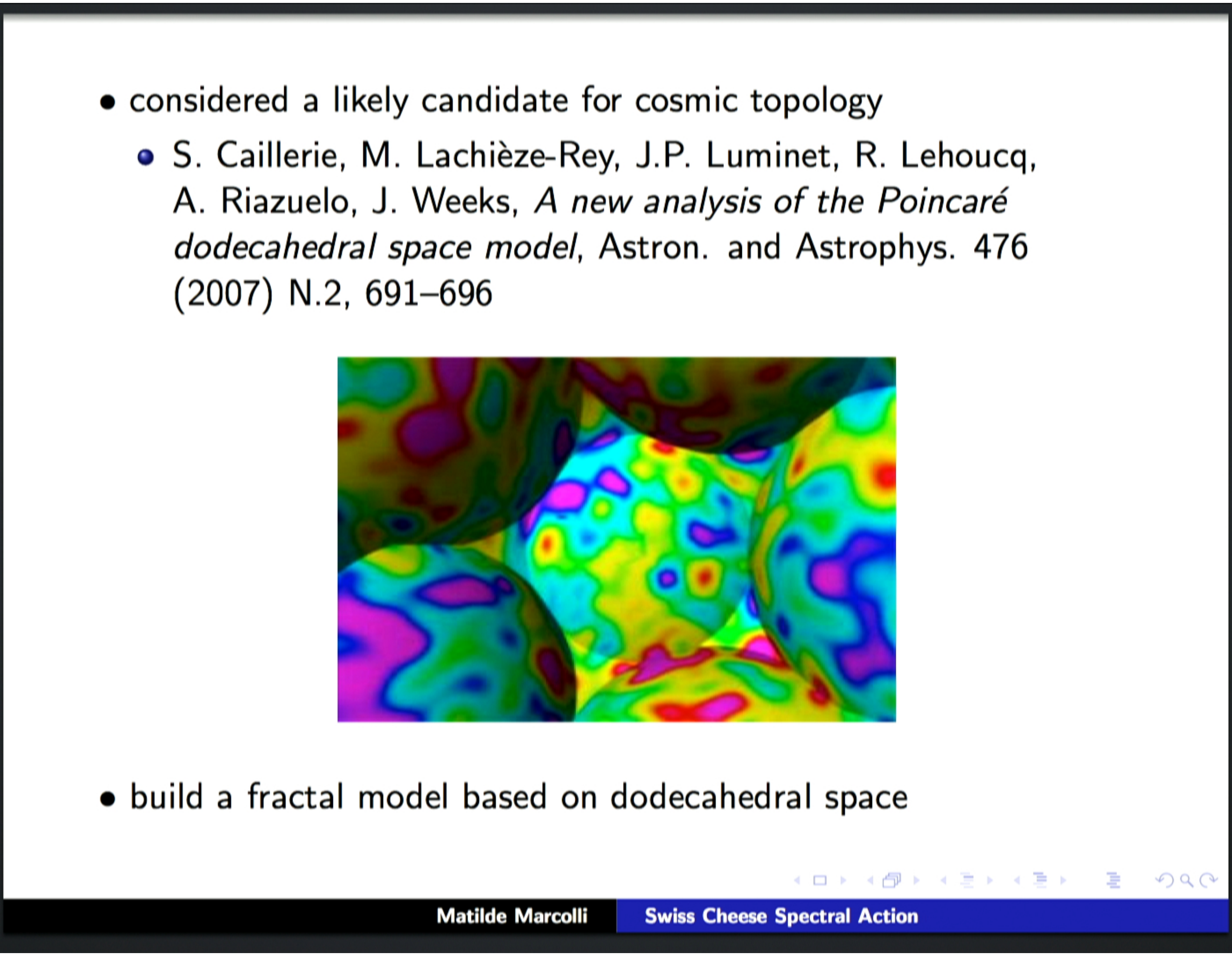
- in Apollonian circle case where $\sigma(\mathcal{P})$ known this method gives larger value, so expect $\sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P})$
- constraints on the packing constant:

$$3 < \dim_H(\mathcal{R}(\mathcal{P})) \leq \sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P}) = 3.85193\dots$$



Models of (Euclidean, compactified) spacetimes

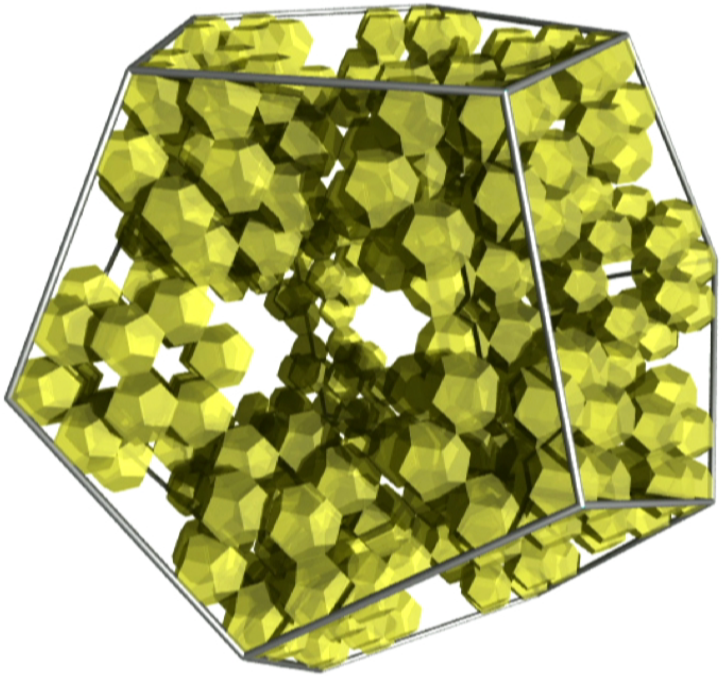
1. Homogeneous Isotropic cases: $S^1_\beta \times S^3_a$
2. Cosmic Topology cases: $S^1_\beta \times Y$ with Y a spherical space form S^3/Γ or a flat Bieberbach manifold T^3/Γ (modulo finite groups of isometries)
3. Packed Swiss Cheese: $S^1_\beta \times \mathcal{P}$ with Apollonian packing of 3-spheres $S^3_{a_{n,k}}$
4. Fractal arrangements with cosmic topology

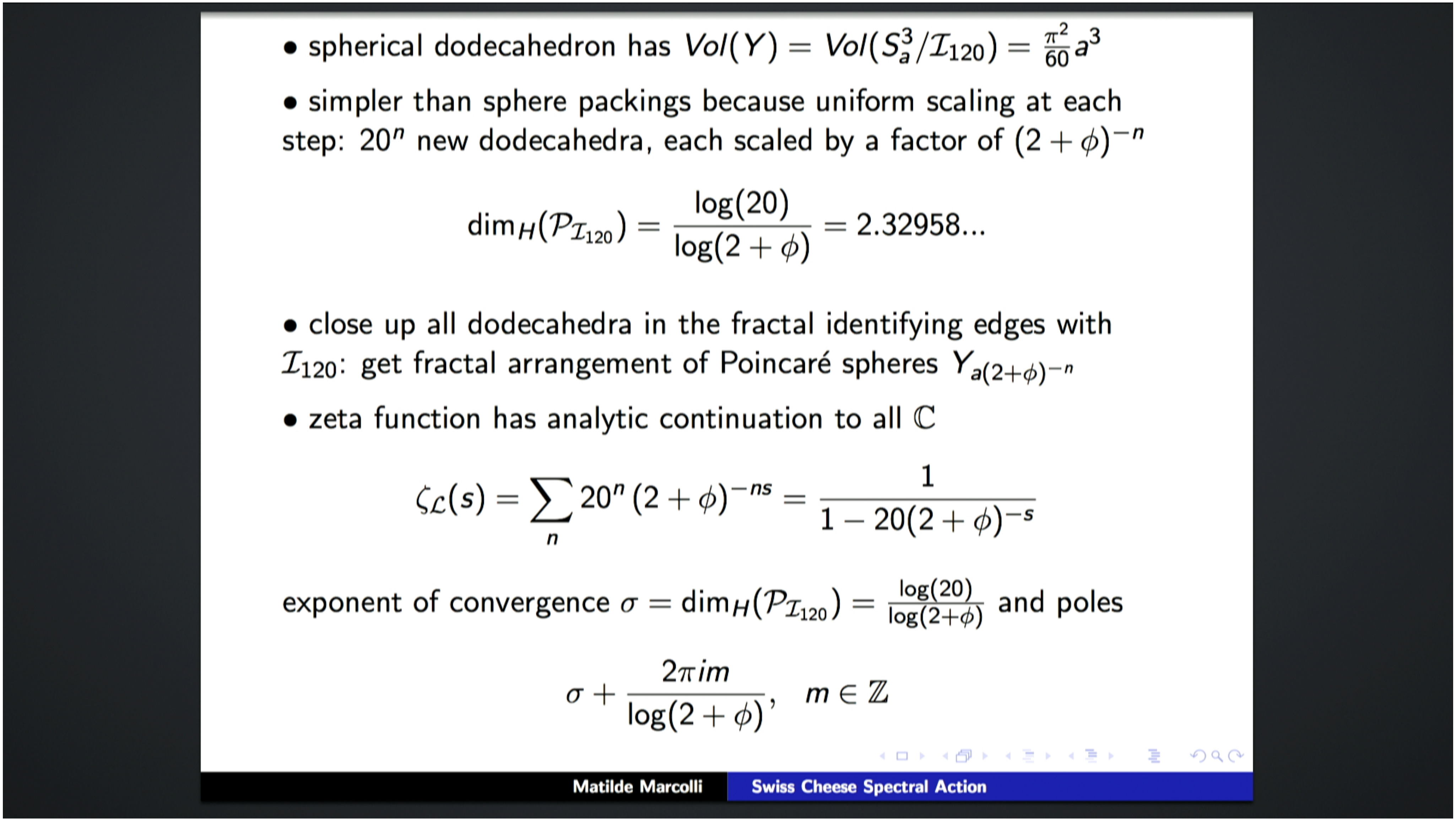


- considered a likely candidate for cosmic topology
  - S. Caillerie, M. Lachièze-Rey, J.P. Luminet, R. Lehoucq, A. Riazuelo, J. Weeks, *A new analysis of the Poincaré dodecahedral space model*, Astron. and Astrophys. 476 (2007) N.2, 691–696

- build a fractal model based on dodecahedral space

Fractal configurations of dodecahedra (Sierpinski dodecahedra)





- spherical dodecahedron has $Vol(Y) = Vol(S^3_a/\mathcal{I}_{120}) = \frac{\pi^2}{60}a^3$
- simpler than sphere packings because uniform scaling at each step: 20^n new dodecahedra, each scaled by a factor of $(2+\phi)^{-n}$

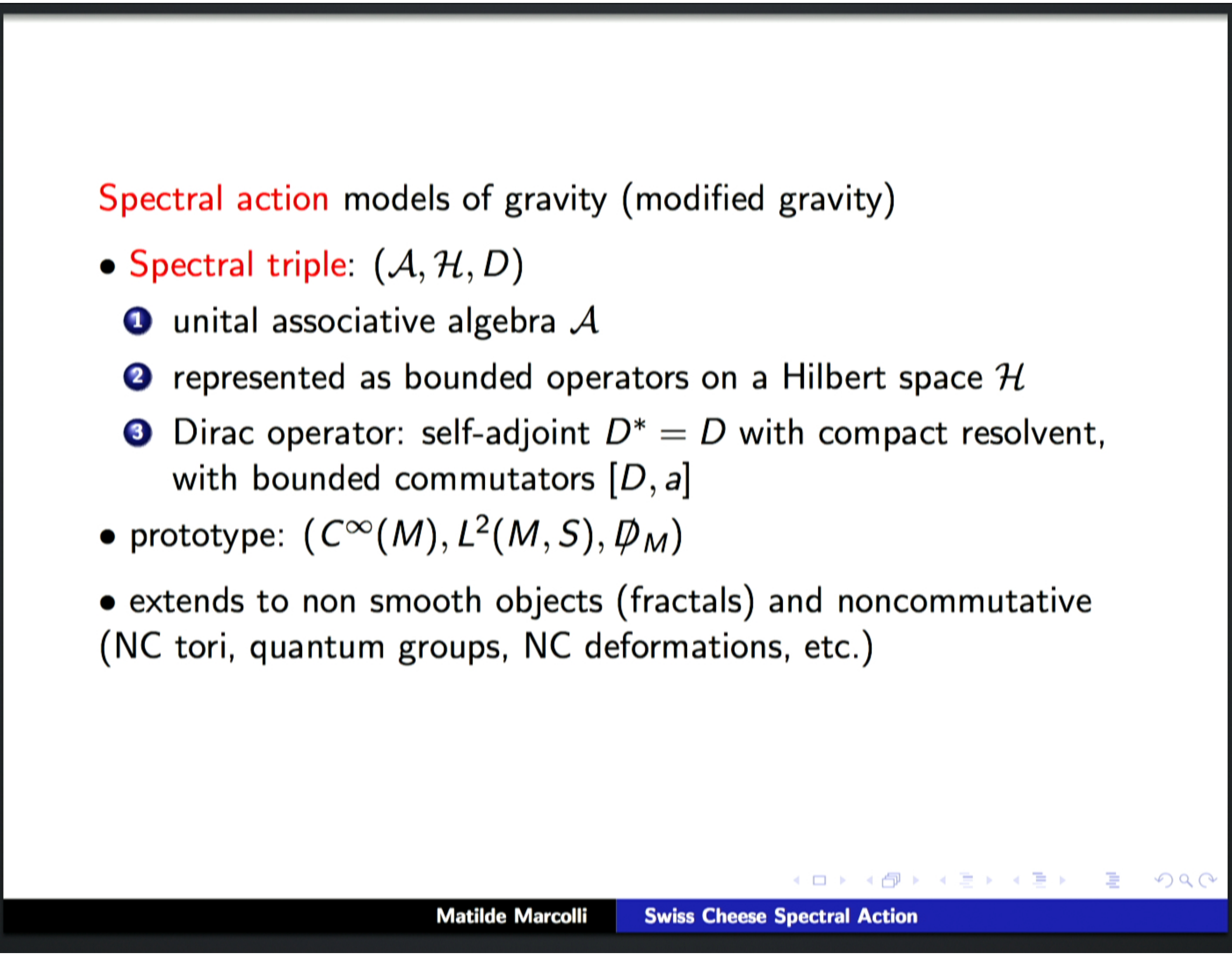
$$\dim_H(\mathcal{P}_{\mathcal{I}_{120}}) = \frac{\log(20)}{\log(2+\phi)} = 2.32958...$$

- close up all dodecahedra in the fractal identifying edges with $\mathcal{I}_{120}$: get fractal arrangement of Poincaré spheres $Y_{a(2+\phi)^{-n}}$
- zeta function has analytic continuation to all $\mathbb{C}$

$$\zeta_{\mathcal{L}}(s) = \sum_n 20^n\,(2+\phi)^{-ns} = \frac{1}{1-20(2+\phi)^{-s}}$$

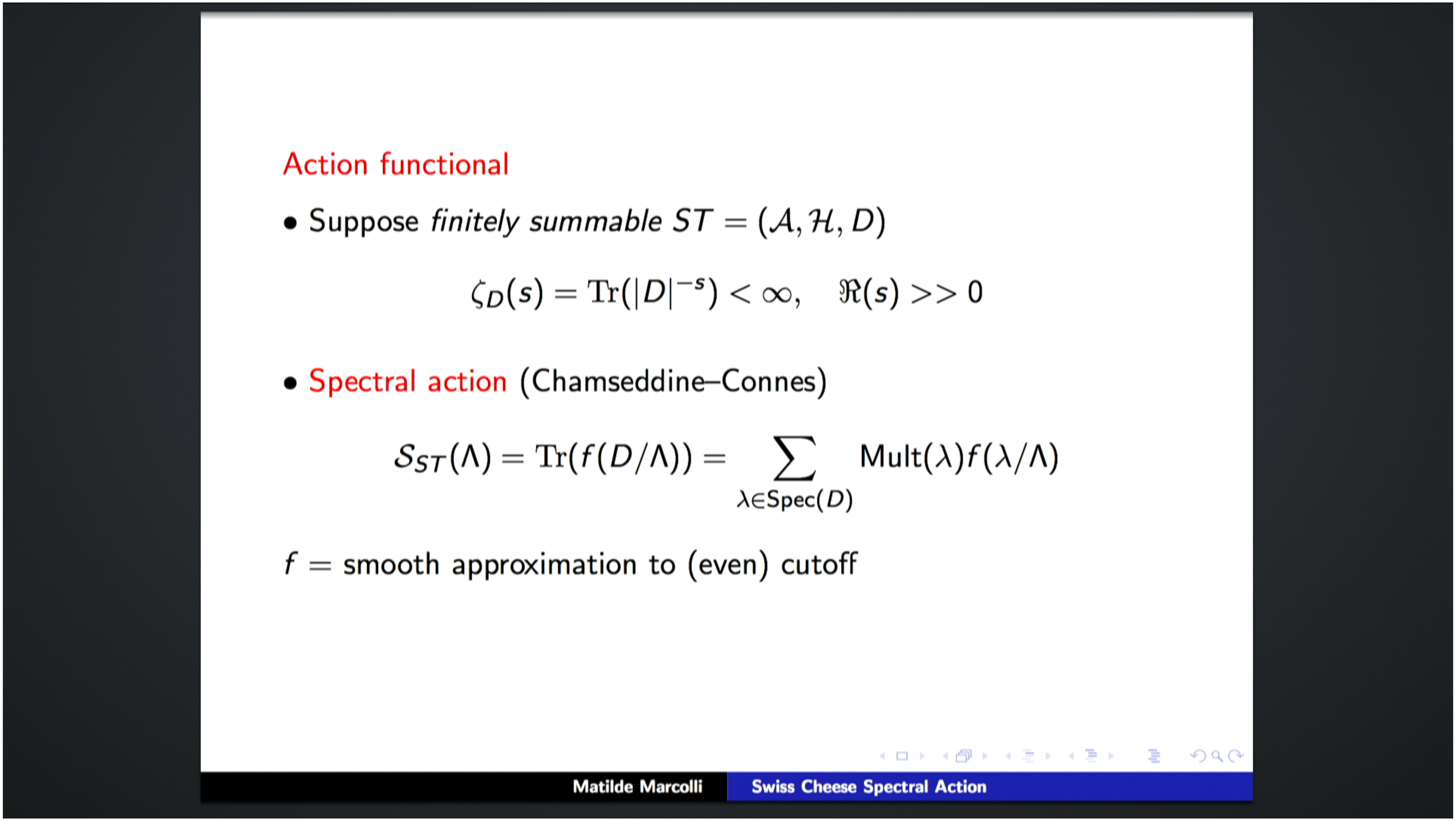
exponent of convergence $\sigma = \dim_H(\mathcal{P}_{\mathcal{I}_{120}}) = \frac{\log(20)}{\log(2+\phi)}$ and poles

$$\sigma + \frac{2\pi i m}{\log(2+\phi)}, \quad m \in \mathbb{Z}$$



Spectral action models of gravity (modified gravity)

- Spectral triple: $(\mathcal{A}, \mathcal{H}, D)$
  1. unital associative algebra $\mathcal{A}$
  2. represented as bounded operators on a Hilbert space $\mathcal{H}$
  3. Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$
- prototype: $(C^\infty(M), L^2(M, S), \not{D}_M)$
- extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.)



Action functional

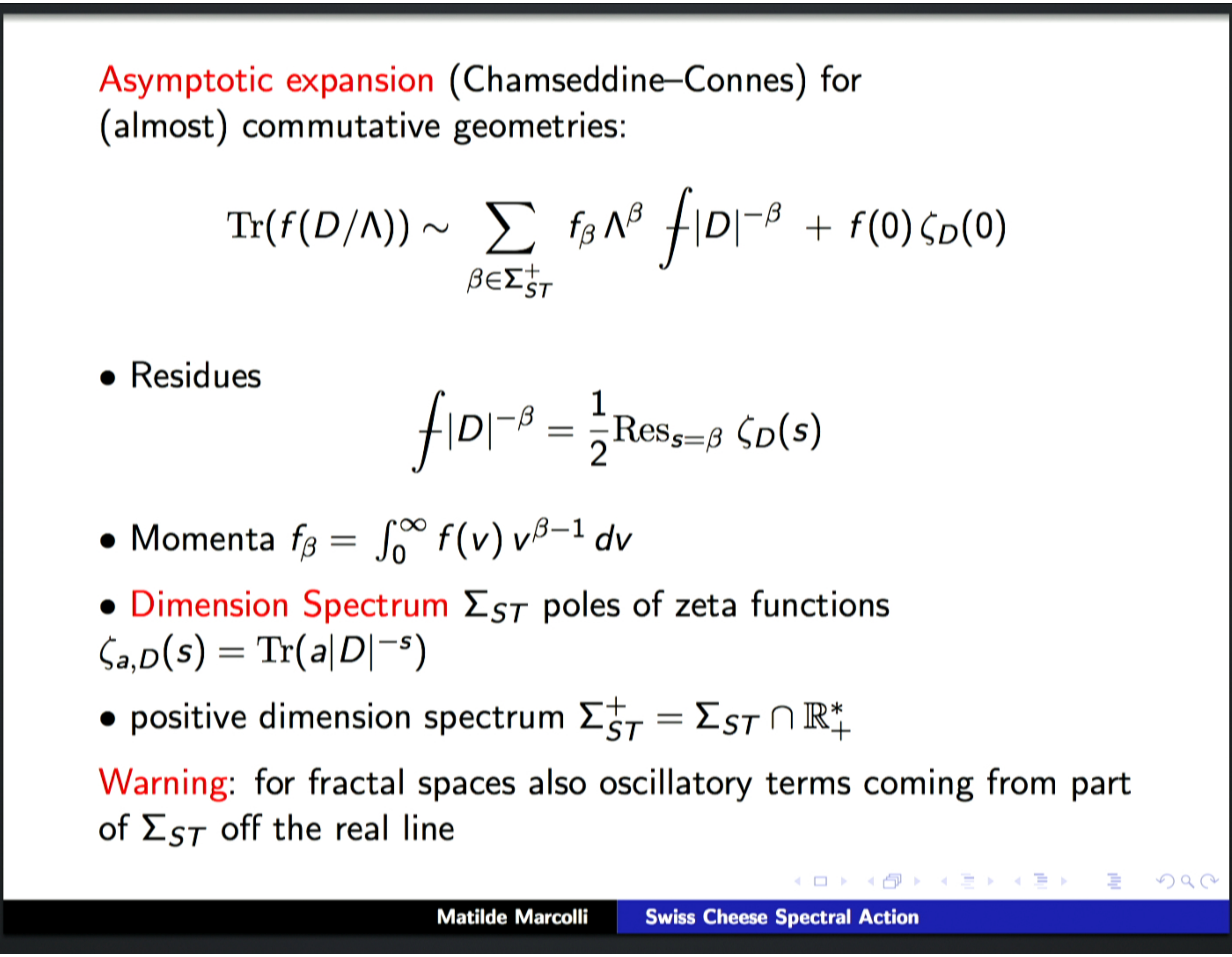
- Suppose *finitely summable* $ST = (\mathcal{A}, \mathcal{H}, D)$

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) < \infty, \quad \Re(s) >> 0$$

- Spectral action (Chamseddine–Connes)

$$\mathcal{S}_{ST}(\Lambda) = \mathrm{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda)$$

f = smooth approximation to (even) cutoff



Asymptotic expansion (Chamseddine–Connes) for (almost) commutative geometries:

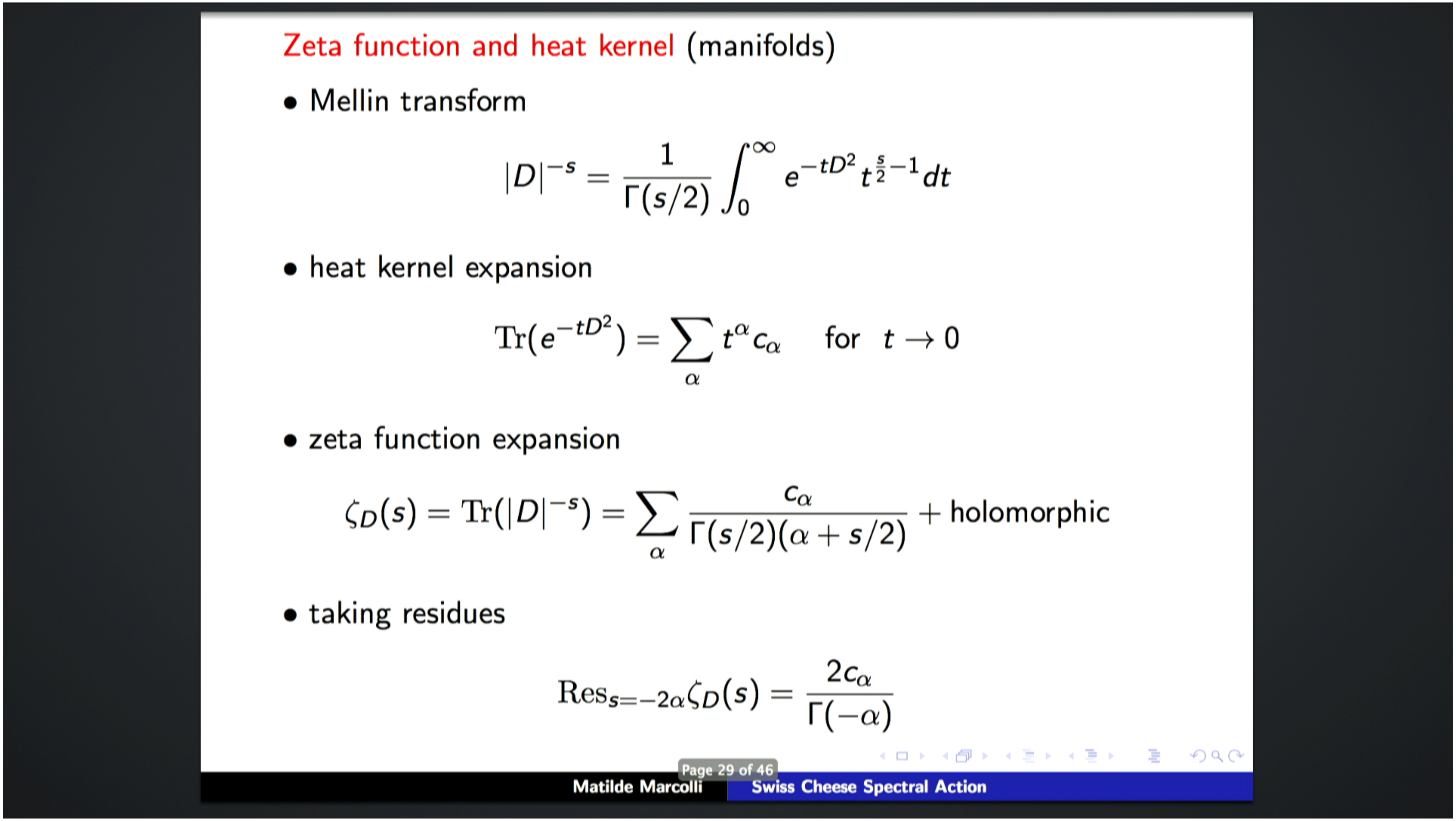
$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta \in \Sigma^+_{ST}} f_\beta \, \Lambda^\beta \, ⨍ |D|^{-\beta} \; + \; f(0)\, \zeta_D(0)$$

- Residues

$$⨍ |D|^{-\beta} = \frac{1}{2} \mathrm{Res}_{s=\beta} \, \zeta_D(s)$$

- Momenta $f_\beta = \int_0^\infty f(v)\, v^{\beta-1}\, dv$
- Dimension Spectrum Σ_{ST} poles of zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$
- positive dimension spectrum $\Sigma^+_{ST} = \Sigma_{ST} \cap \mathbb{R}^*_+$

Warning: for fractal spaces also oscillatory terms coming from part of Σ_{ST} off the real line



Zeta function and heat kernel (manifolds)

- Mellin transform

$$|D|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tD^2} t^{\frac{s}{2}-1} dt$$

- heat kernel expansion

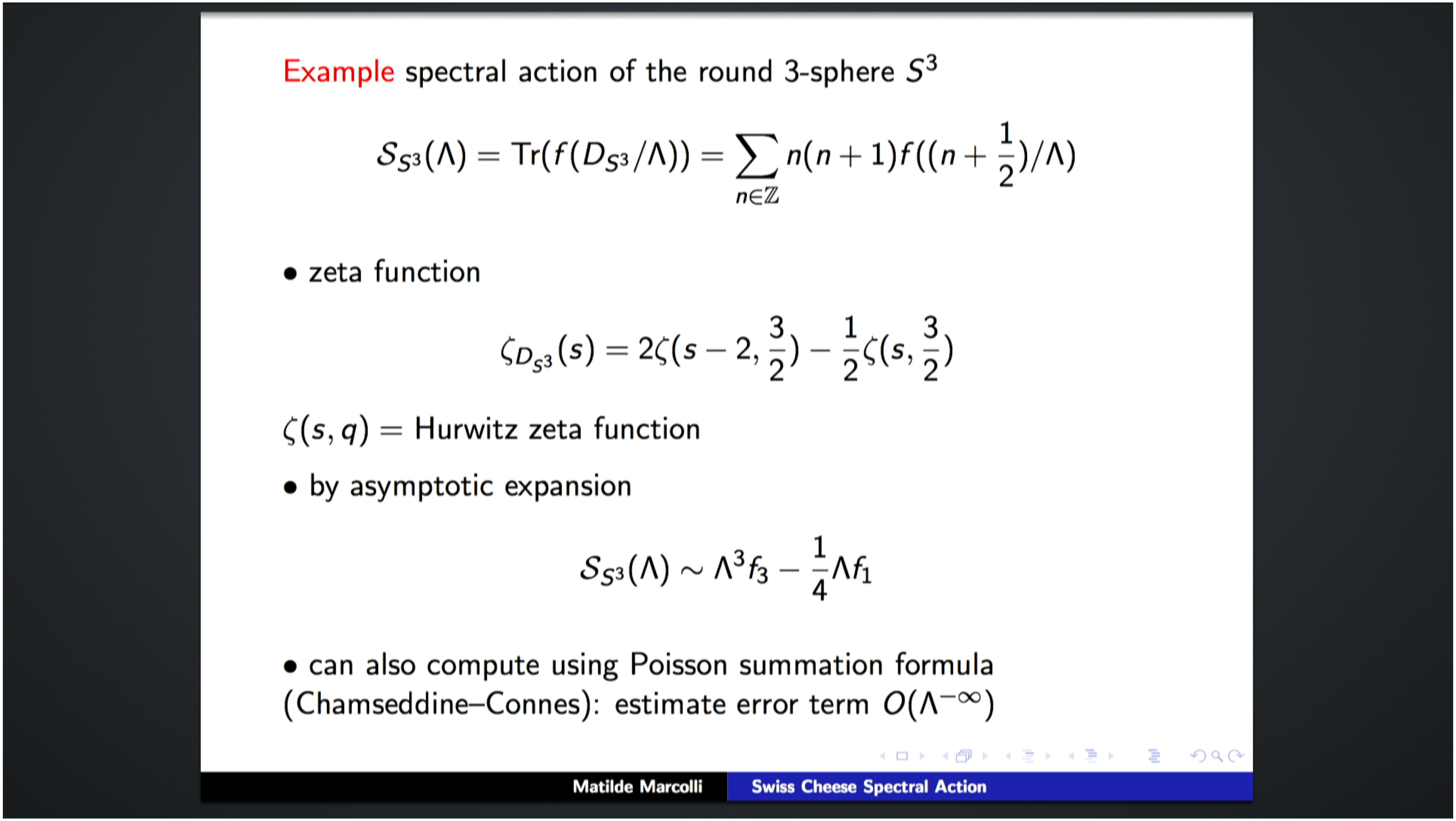
$$\mathrm{Tr}(e^{-tD^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for} \quad t \to 0$$

- zeta function expansion

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) = \sum_\alpha \frac{c_\alpha}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}$$

- taking residues

$$\mathrm{Res}_{s=-2\alpha} \zeta_D(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$



Example spectral action of the round 3-sphere S^3

$$\mathcal{S}_{S^3}(\Lambda) = \mathrm{Tr}(f(D_{S^3}/\Lambda)) = \sum_{n\in\mathbb{Z}} n(n+1)f((n+\frac{1}{2})/\Lambda)$$

- zeta function

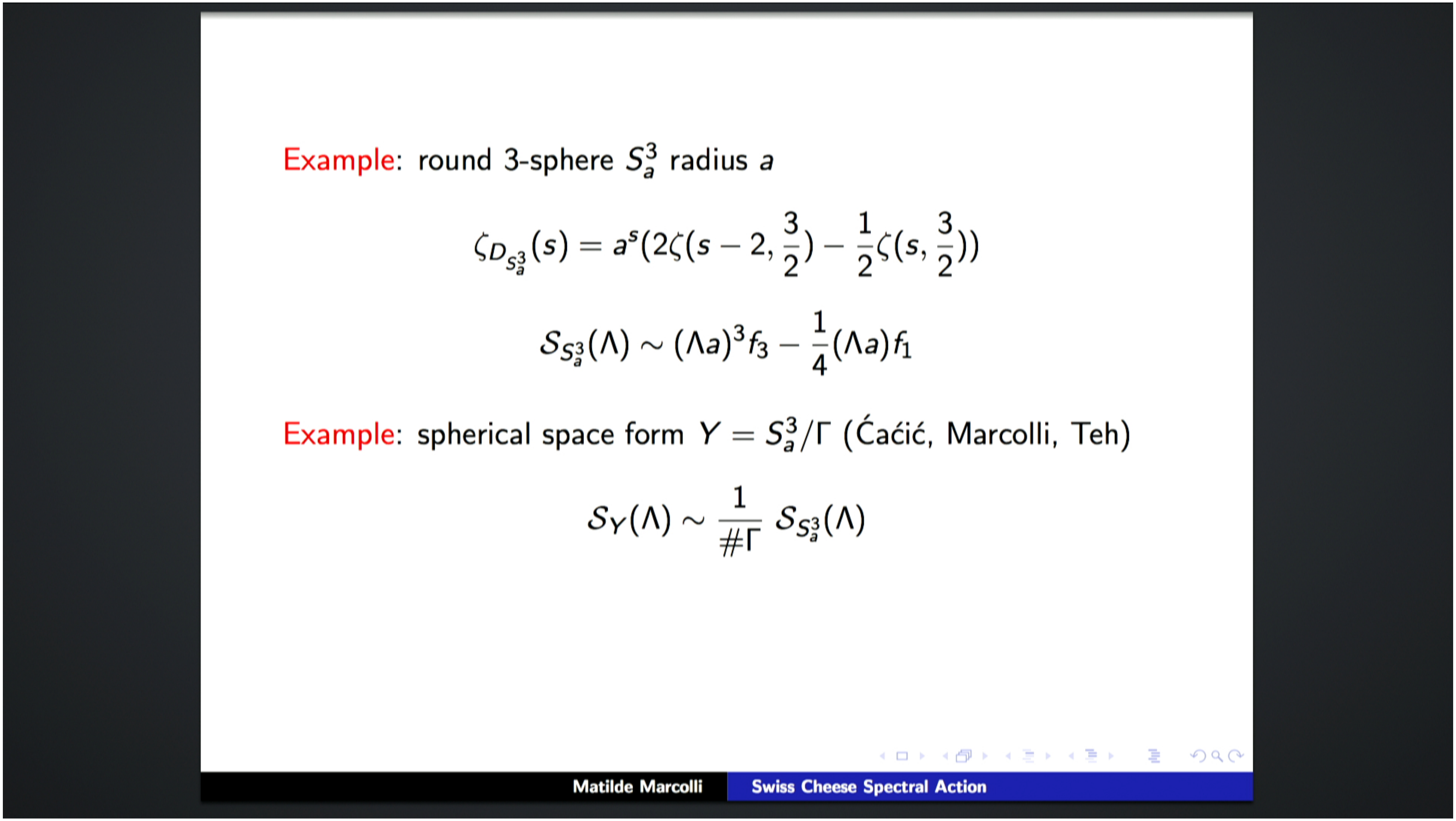
$$\zeta_{D_{S^3}}(s) = 2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2})$$

$\zeta(s,q) =$ Hurwitz zeta function

- by asymptotic expansion

$$\mathcal{S}_{S^3}(\Lambda) \sim \Lambda^3 f_3 - \frac{1}{4}\Lambda f_1$$

- can also compute using Poisson summation formula (Chamseddine–Connes): estimate error term $O(\Lambda^{-\infty})$



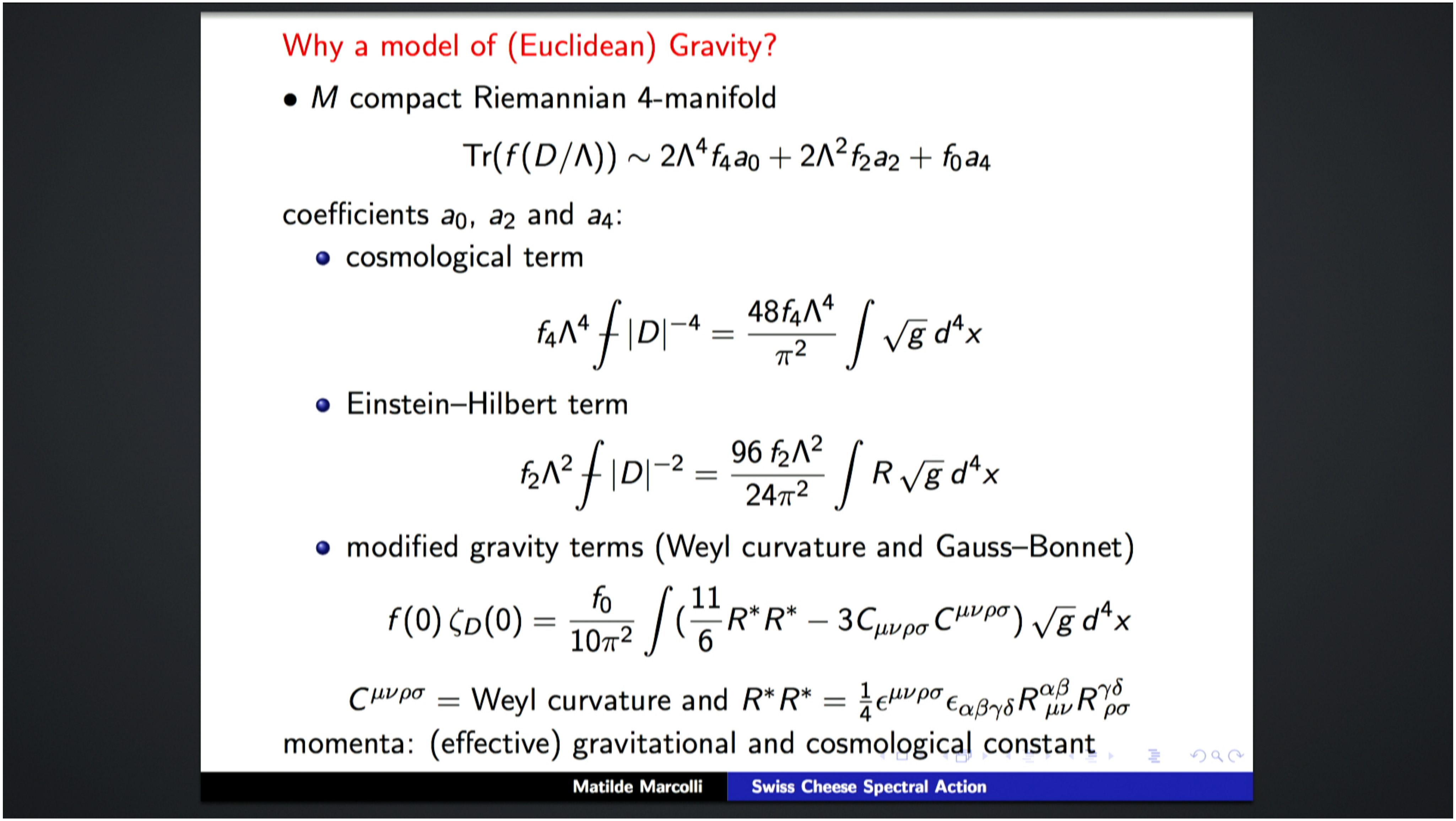
Example: round 3-sphere S^3_a radius a

$$\zeta_{D_{S^3_a}}(s) = a^s\left(2\zeta(s-2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2})\right)$$

$$\mathcal{S}_{S^3_a}(\Lambda) \sim (\Lambda a)^3 f_3 - \frac{1}{4}(\Lambda a) f_1$$

Example: spherical space form $Y = S^3_a/\Gamma$ (Ćaćić, Marcolli, Teh)

$$\mathcal{S}_Y(\Lambda) \sim \frac{1}{\#\Gamma}\, \mathcal{S}_{S^3_a}(\Lambda)$$



Why a model of (Euclidean) Gravity?

- M compact Riemannian 4-manifold

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

coefficients a_0, a_2 and a_4:

- cosmological term

$$f_4 \Lambda^4 \fint |D|^{-4} = \frac{48 f_4 \Lambda^4}{\pi^2} \int \sqrt{g}\, d^4x$$

- Einstein–Hilbert term

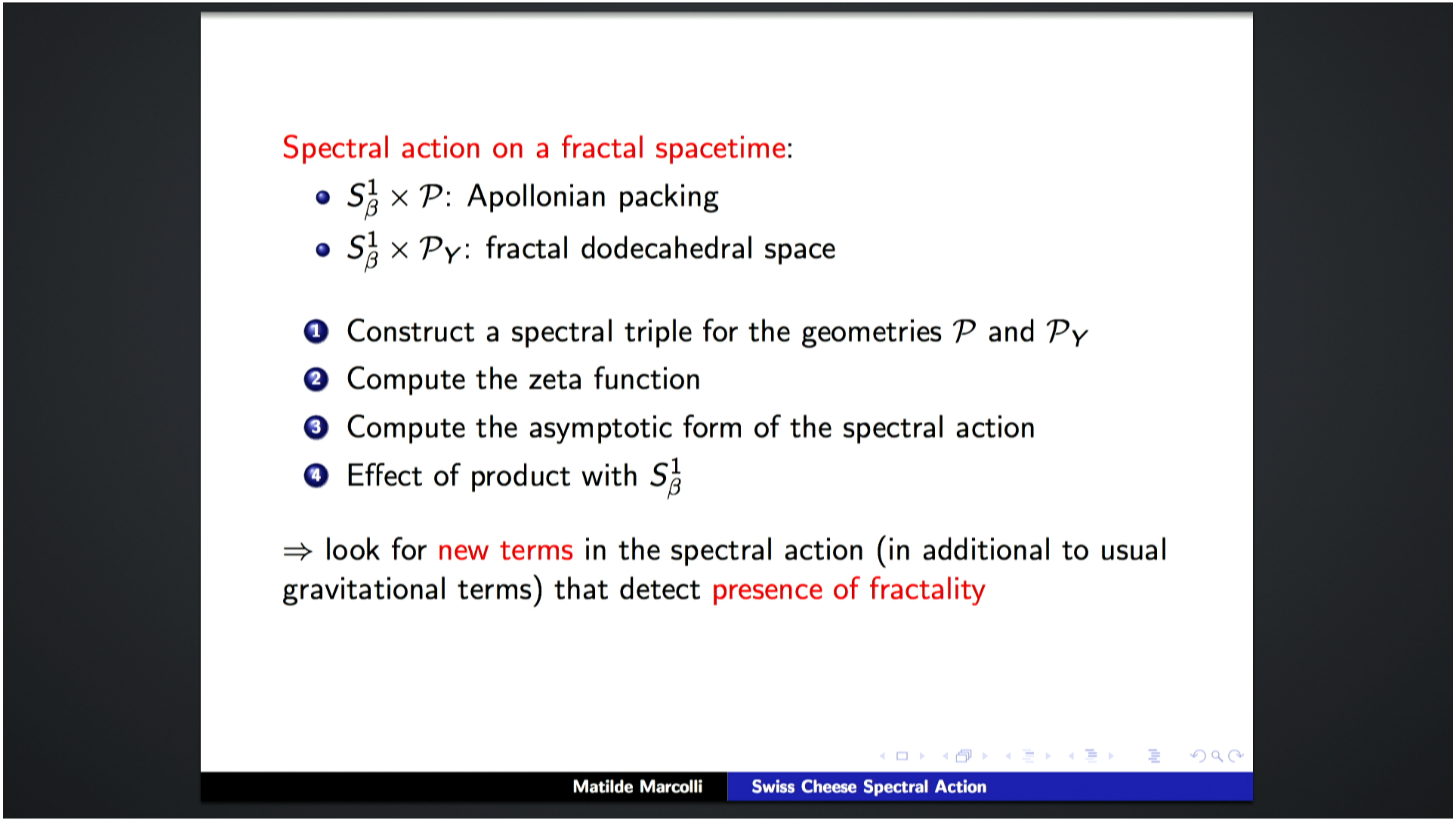
$$f_2 \Lambda^2 \fint |D|^{-2} = \frac{96\, f_2 \Lambda^2}{24\pi^2} \int R \sqrt{g}\, d^4x$$

- modified gravity terms (Weyl curvature and Gauss–Bonnet)

$$f(0)\,\zeta_D(0) = \frac{f_0}{10\pi^2} \int \left(\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}\right) \sqrt{g}\, d^4x$$

$$C^{\mu\nu\rho\sigma} = \text{Weyl curvature and } R^* R^* = \tfrac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\rho\sigma}$$

momenta: (effective) gravitational and cosmological constant

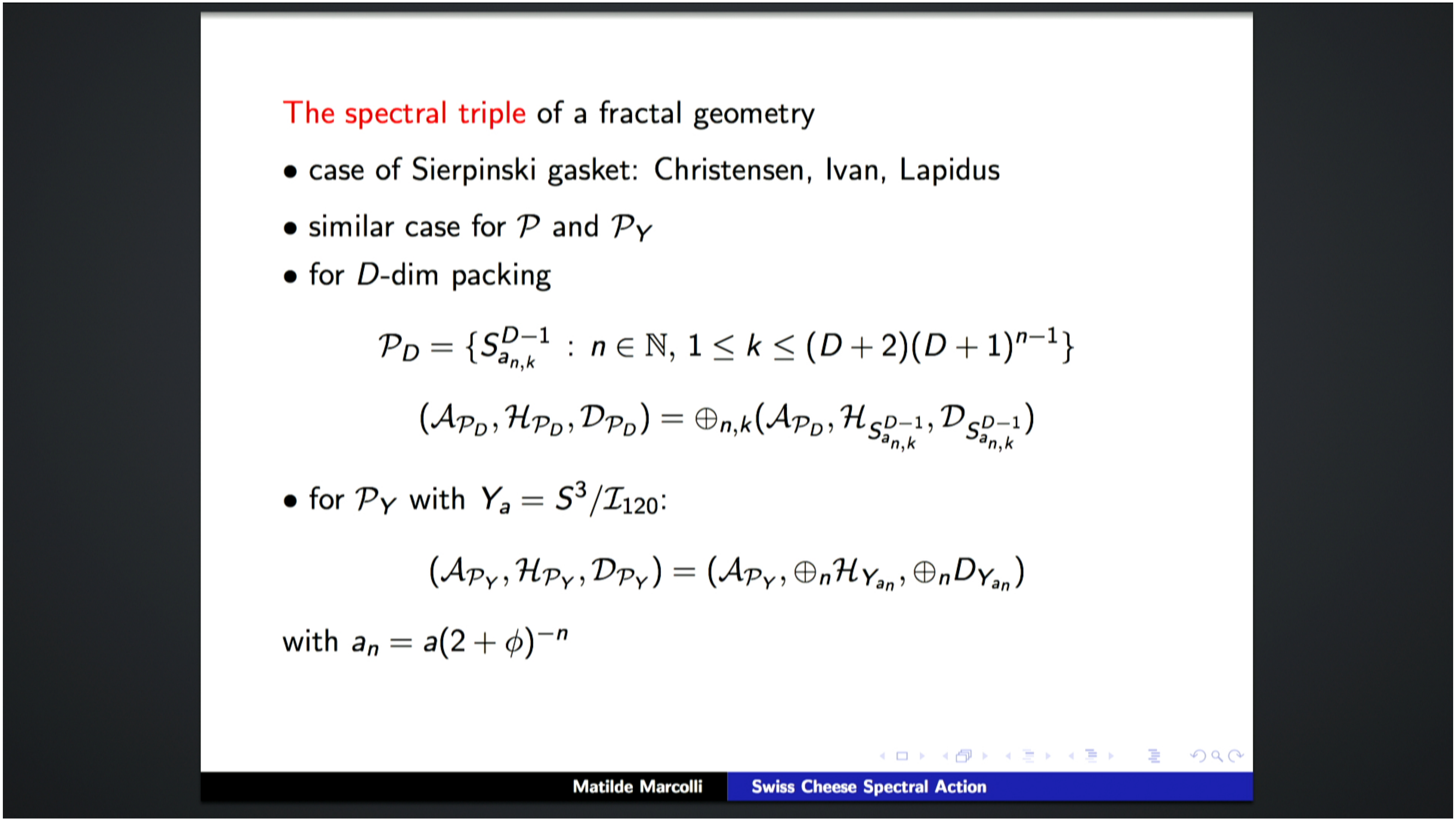


Spectral action on a fractal spacetime:

- $S^1_\beta \times \mathcal{P}$: Apollonian packing
- $S^1_\beta \times \mathcal{P}_Y$: fractal dodecahedral space

1. Construct a spectral triple for the geometries $\mathcal{P}$ and $\mathcal{P}_Y$
2. Compute the zeta function
3. Compute the asymptotic form of the spectral action
4. Effect of product with S^1_β

$\Rightarrow$ look for new terms in the spectral action (in additional to usual gravitational terms) that detect presence of fractality



The spectral triple of a fractal geometry

- case of Sierpinski gasket: Christensen, Ivan, Lapidus
- similar case for $\mathcal{P}$ and $\mathcal{P}_Y$
- for D-dim packing

$$\mathcal{P}_D = \{ S^{D-1}_{a_{n,k}} \,:\, n \in \mathbb{N},\ 1 \leq k \leq (D+2)(D+1)^{n-1} \}$$

$$(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{\mathcal{P}_D}, \mathcal{D}_{\mathcal{P}_D}) = \oplus_{n,k} (\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{S^{D-1}_{a_{n,k}}}, \mathcal{D}_{S^{D-1}_{a_{n,k}}})$$

- for $\mathcal{P}_Y$ with $Y_a = S^3/\mathcal{I}_{120}$:

$$(\mathcal{A}_{\mathcal{P}_Y}, \mathcal{H}_{\mathcal{P}_Y}, \mathcal{D}_{\mathcal{P}_Y}) = (\mathcal{A}_{\mathcal{P}_Y}, \oplus_n \mathcal{H}_{Y_{a_n}}, \oplus_n D_{Y_{a_n}})$$

with $a_n = a(2+\phi)^{-n}$

Zeta functions for Apollonian packing of 3-spheres:

- Lengths zeta function (fractal string)

$$\zeta_{\mathcal{L}}(s) := \sum_{n\in\mathbb{N}} \sum_{k=1}^{6\cdot 5^{n-1}} a_{n,k}^s$$

with $\mathcal{L} = \mathcal{L}_4 = \{a_{n,k} \,|\, n \in \mathbb{N},\ k \in \{1, \dots, 6\cdot 5^{n-1}\}\}$

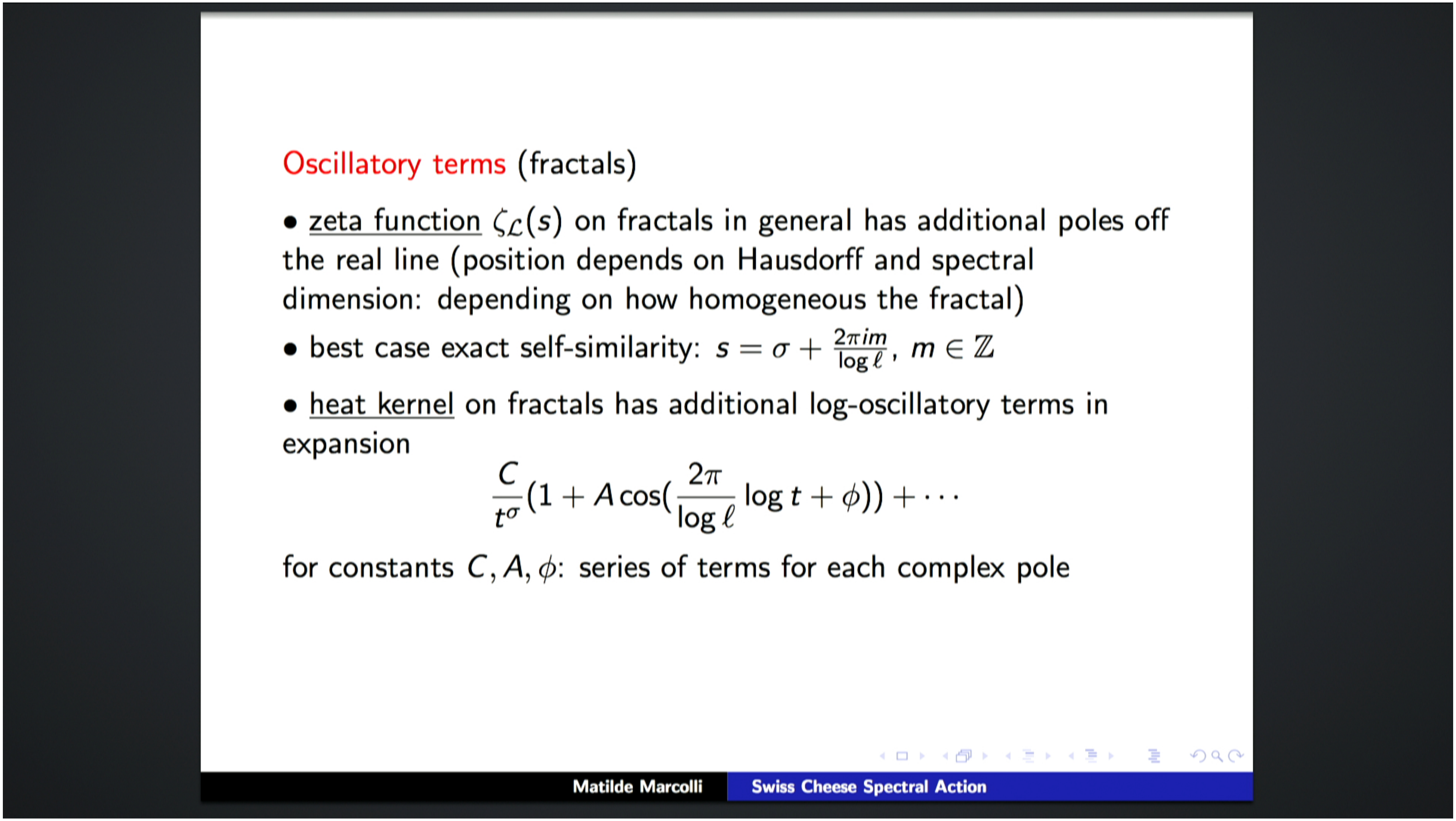
- zeta function of Dirac operator of the spectral triple

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n=1}^{\infty} \sum_{k=1}^{6\cdot 5^{n-1}} \mathrm{Tr}(|D_{S^3_{a_{n,k}}}|^{-s})$$

each term $\mathrm{Tr}(|D_{S^3_{a_{n,k}}}|^{-s}) = a_{n,k}^s(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2}))$ gives

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \left(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2})\right) \sum_{n,k} a_{n,k}^s$$

$$= \left(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2})\right) \zeta_{\mathcal{L}}(s)$$

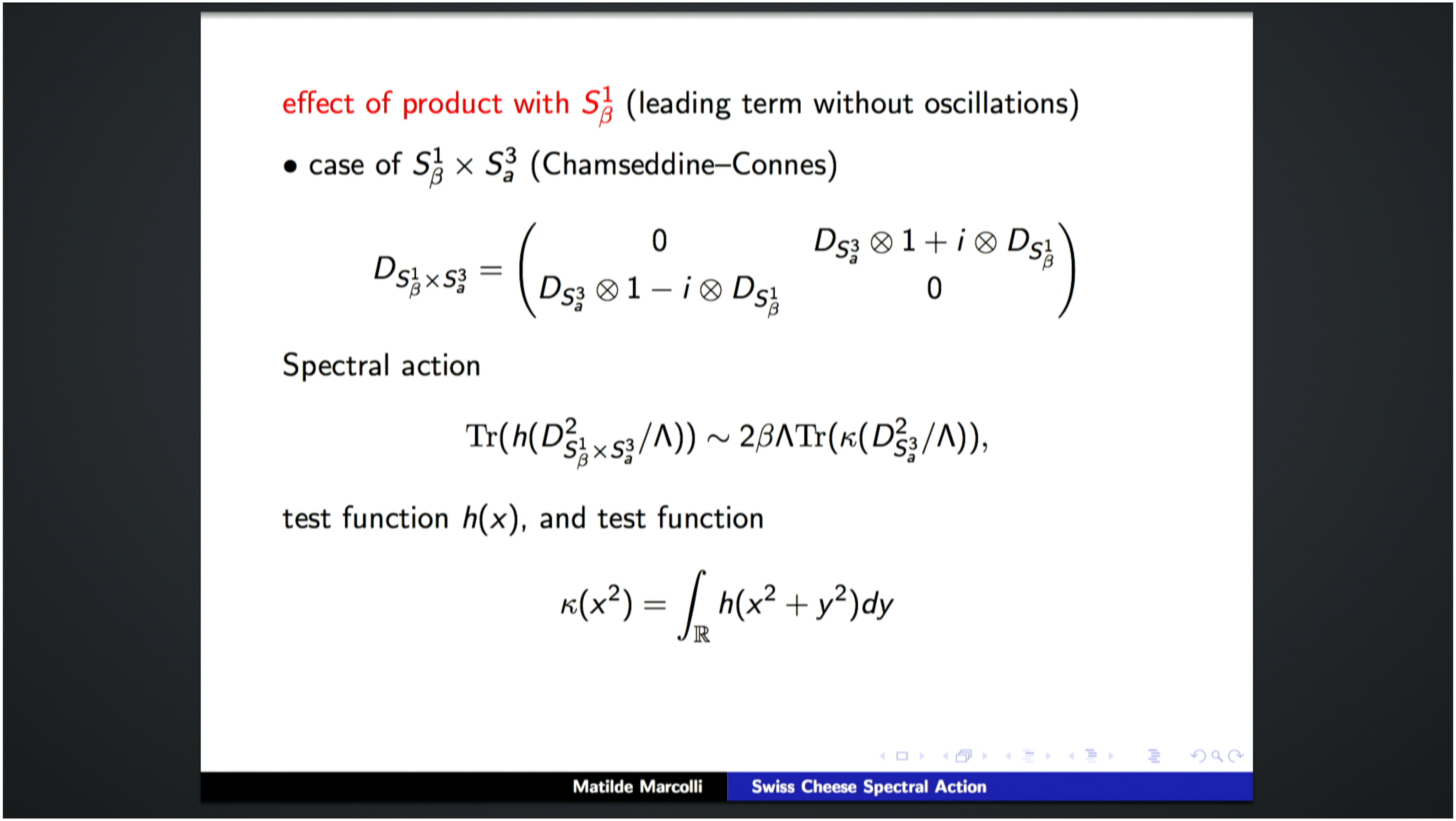


Oscillatory terms (fractals)

- zeta function $\zeta_{\mathcal{L}}(s)$ on fractals in general has additional poles off the real line (position depends on Hausdorff and spectral dimension: depending on how homogeneous the fractal)
- best case exact self-similarity: $s = \sigma + \frac{2\pi i m}{\log \ell}$, $m \in \mathbb{Z}$
- heat kernel on fractals has additional log-oscillatory terms in expansion

$$\frac{C}{t^{\sigma}}\left(1 + A\cos\left(\frac{2\pi}{\log \ell}\log t + \phi\right)\right) + \cdots$$

for constants C, A, ϕ: series of terms for each complex pole



effect of product with S^1_β (leading term without oscillations)

- case of $S^1_\beta \times S^3_a$ (Chamseddine–Connes)

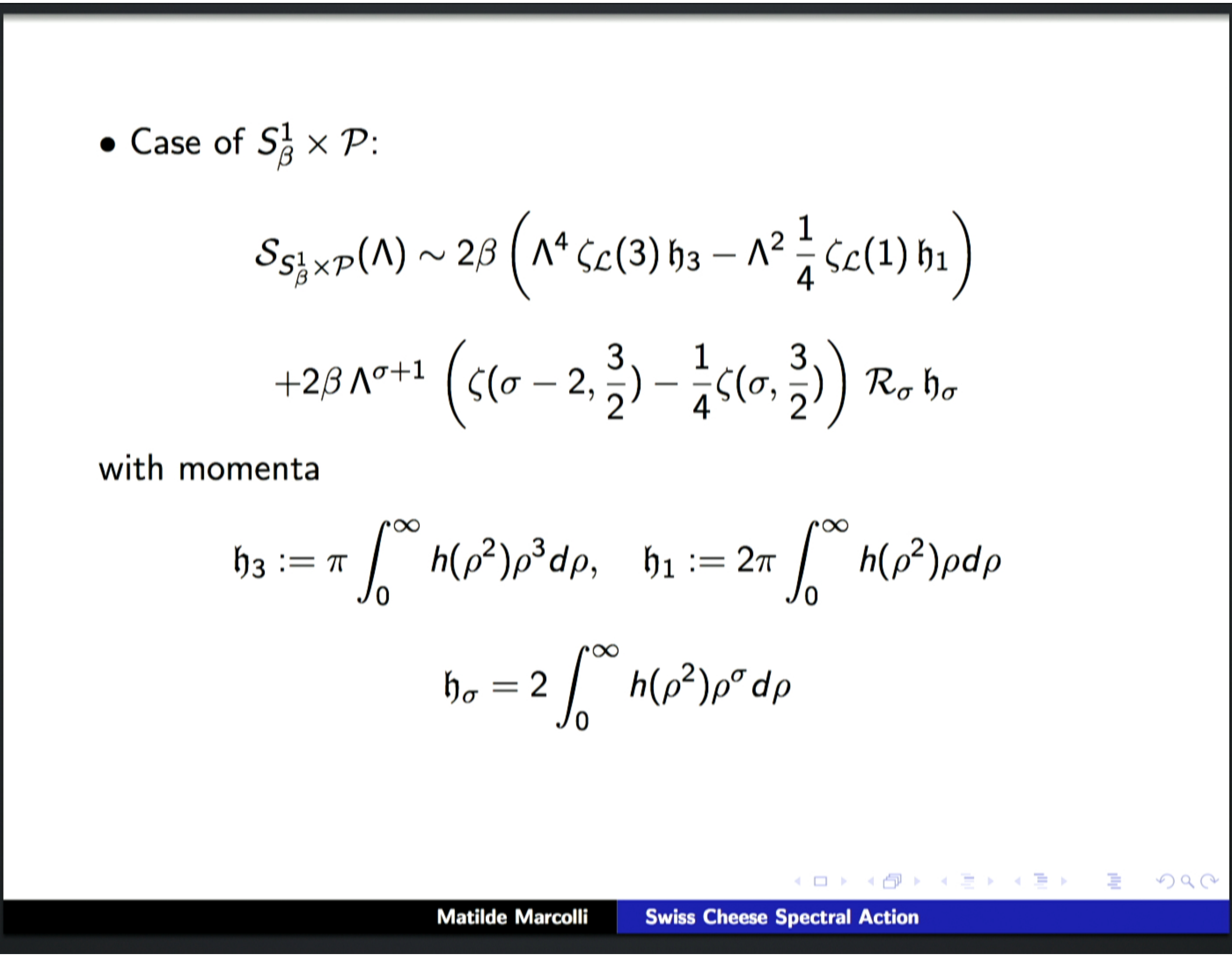
$$D_{S^1_\beta \times S^3_a} = \begin{pmatrix} 0 & D_{S^3_a} \otimes 1 + i \otimes D_{S^1_\beta} \\ D_{S^3_a} \otimes 1 - i \otimes D_{S^1_\beta} & 0 \end{pmatrix}$$

Spectral action

$$\mathrm{Tr}(h(D^2_{S^1_\beta \times S^3_a}/\Lambda)) \sim 2\beta\Lambda \mathrm{Tr}(\kappa(D^2_{S^3_a}/\Lambda)),$$

test function $h(x)$, and test function

$$\kappa(x^2) = \int_{\mathbb{R}} h(x^2 + y^2) dy$$



- Case of $S^1_\beta \times \mathcal{P}$:

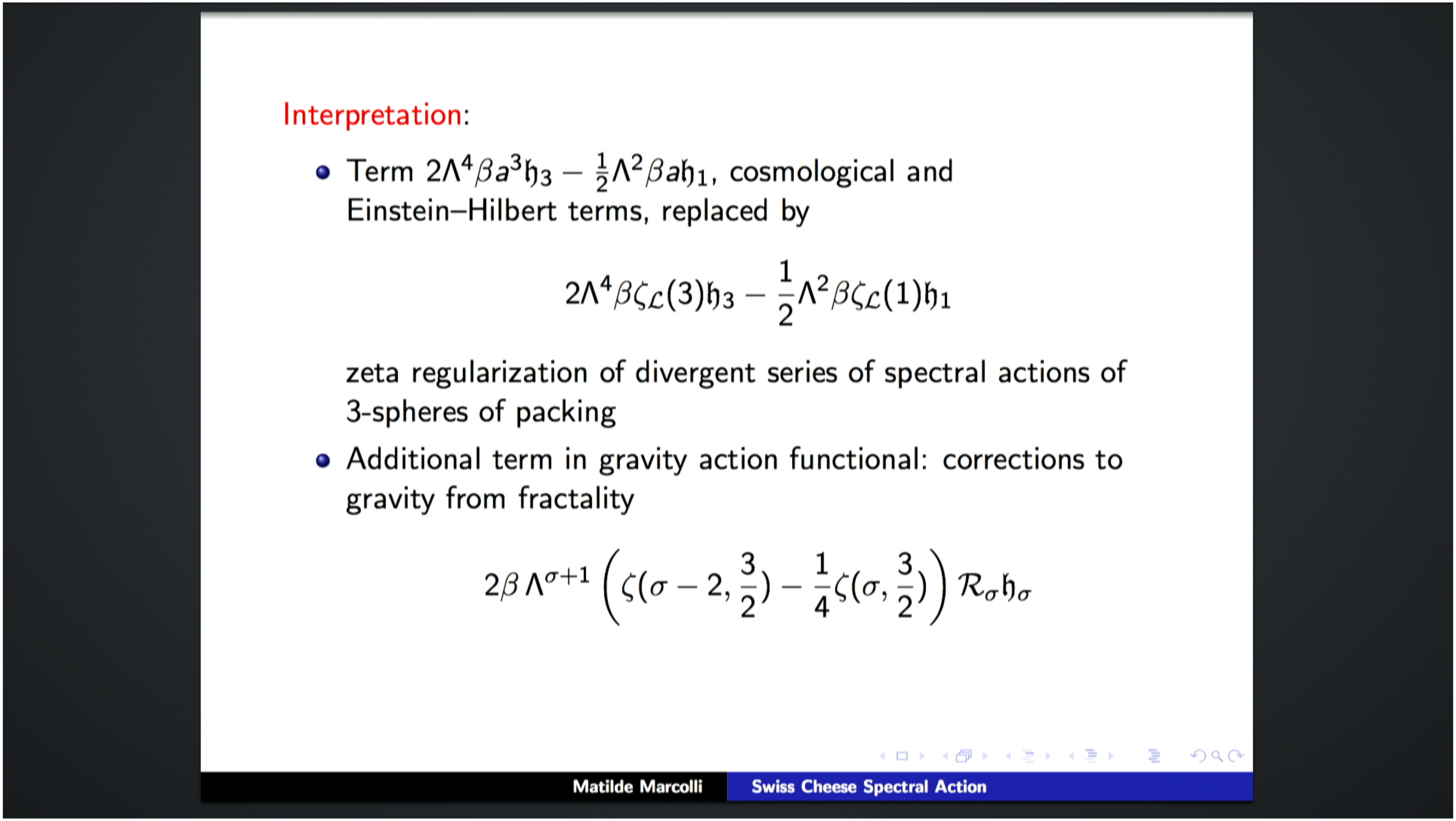
$$\mathcal{S}_{S^1_\beta \times \mathcal{P}}(\Lambda) \sim 2\beta \left(\Lambda^4 \, \zeta_{\mathcal{L}}(3) \, \mathfrak{h}_3 - \Lambda^2 \, \frac{1}{4} \, \zeta_{\mathcal{L}}(1) \, \mathfrak{h}_1 \right)$$

$$+ 2\beta \, \Lambda^{\sigma+1} \left(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2}) \right) \mathcal{R}_\sigma \, \mathfrak{h}_\sigma$$

with momenta

$$\mathfrak{h}_3 := \pi \int_0^\infty h(\rho^2) \rho^3 d\rho, \quad \mathfrak{h}_1 := 2\pi \int_0^\infty h(\rho^2) \rho d\rho$$

$$\mathfrak{h}_\sigma = 2 \int_0^\infty h(\rho^2) \rho^\sigma \, d\rho$$



Interpretation:

- Term $2\Lambda^4\beta a^3\mathfrak{h}_3 - \frac{1}{2}\Lambda^2\beta a\mathfrak{h}_1$, cosmological and Einstein–Hilbert terms, replaced by

$$2\Lambda^4\beta\zeta_{\mathcal{L}}(3)\mathfrak{h}_3 - \frac{1}{2}\Lambda^2\beta\zeta_{\mathcal{L}}(1)\mathfrak{h}_1$$

  zeta regularization of divergent series of spectral actions of 3-spheres of packing

- Additional term in gravity action functional: corrections to gravity from fractality

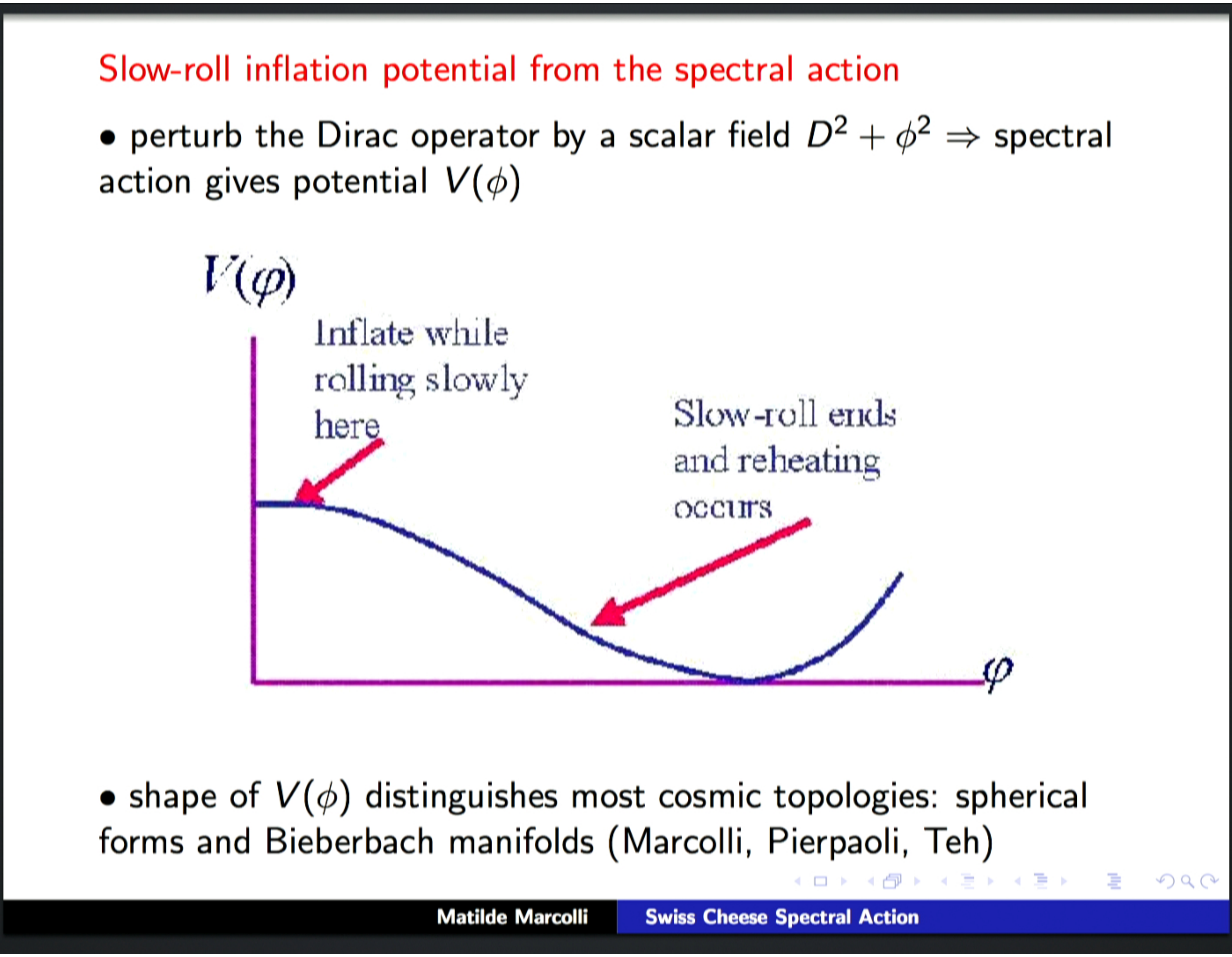
$$2\beta\,\Lambda^{\sigma+1}\left(\zeta(\sigma-2,\frac{3}{2})-\frac{1}{4}\zeta(\sigma,\frac{3}{2})\right)\mathcal{R}_\sigma\mathfrak{h}_\sigma$$

- on product geometry $S^1_\beta \times \mathcal{P}_Y$

$$\mathcal{S}_{S^1_\beta \times \mathcal{P}_Y}(\Lambda) \sim 2\beta \left(\Lambda^4 \frac{a^3 \zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} \mathfrak{h}_3 - \Lambda^2 \frac{a \zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} \mathfrak{h}_1 \right)$$

$$+2\beta \Lambda^{\sigma+1} \frac{a^\sigma (\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}))}{120 \log(2+\phi)} \mathfrak{h}_\sigma + \mathcal{S}^{osc}_{S^1_\beta \times Y, \Lambda}$$

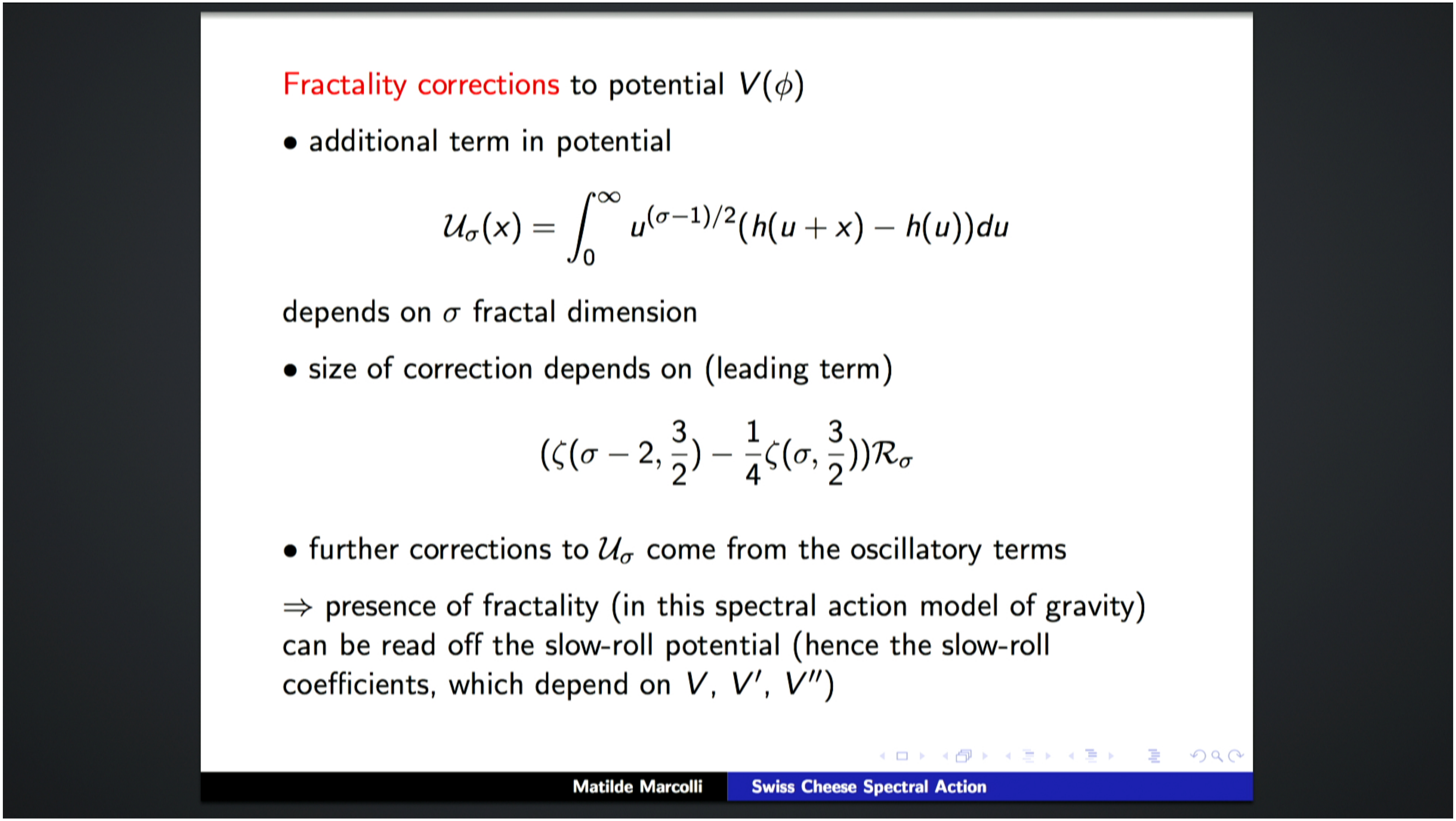
- Note: correction term now at different σ than Apollonian $\mathcal{P}$
- oscillatory terms $\mathcal{S}^{osc}_{Y,\Lambda}$ more explicit than in the Apollonian case



Slow-roll inflation potential from the spectral action

- perturb the Dirac operator by a scalar field $D^2 + \phi^2 \Rightarrow$ spectral action gives potential $V(\phi)$

- shape of $V(\phi)$ distinguishes most cosmic topologies: spherical forms and Bieberbach manifolds (Marcolli, Pierpaoli, Teh)



Fractality corrections to potential $V(\phi)$

- additional term in potential

$$\mathcal{U}_\sigma(x) = \int_0^\infty u^{(\sigma-1)/2}(h(u+x) - h(u))du$$

depends on σ fractal dimension

- size of correction depends on (leading term)

$$(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}))\mathcal{R}_\sigma$$

- further corrections to $\mathcal{U}_\sigma$ come from the oscillatory terms

$\Rightarrow$ presence of fractality (in this spectral action model of gravity) can be read off the slow-roll potential (hence the slow-roll coefficients, which depend on V, V', V'')