

Title: Random non-commutative geometries

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Abstract:

Results for D^2 action

The simplest cases: type (1, 0) and (0, 1)

The n^2 eigenvalues of Dirac operators can be written in terms of the n eigenvalues μ_i of the matrix H or the eigenvalues $i\mu$ of L . For the (1, 0) case one has eigenvalues

$$\lambda_{ij} = \mu_i + \mu_j$$

while for the (0, 1) case

$$\lambda_{ij} = \mu_i - \mu_j$$

This follows from the fact that eigenvectors of D are of the form $u_i \otimes \bar{u}_j$, with u_i the eigenvectors of H or L .

2. Details

3. Numerical results (with Lisa Glaser)

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1. Geometry \leftrightarrow Dirac operator $D = \gamma^\mu e_\mu^\alpha \partial_\alpha + \omega$

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NC generalisation : algebra of functions \rightarrow NC A

N C gravitation

Application : particle physics : $M \times F$

$M : C$ spacetime

$F : NC$ internal space

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$N \subset$ gravitation + QFT

$M \subset$ spacetime

: NC internal space

Project
Replace $M \sim N.C$ geometry
Truncate to finite # modes A: finite-diml

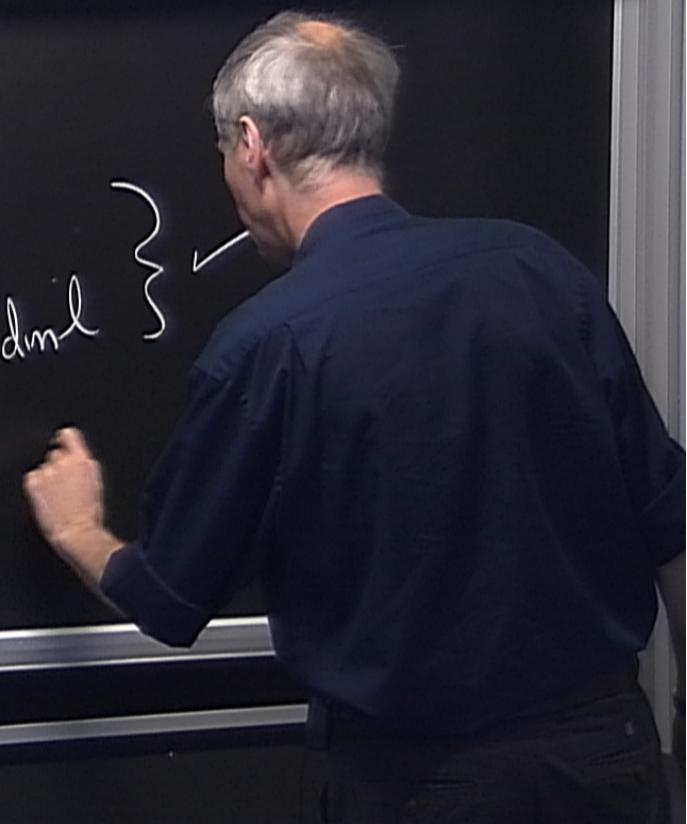
NC gravitation

$M : C^{\mathbb{T} \times \mathbb{S}}$ spacetime

$F : NC$ internal space

Project

- Replace $M \sim NC$ geometry
- Truncate to finite # modes \mathcal{A} : finite-dim
- Compute (something) in random $NC G$
- Couple with F



NC gravitation

Application : particle physics : $M \times F$

$M : C$ spacetime

$F : NC$ internal space

- Project :
- Replace $M \sim NC$ geometry }
• Truncate to finite # modes $A : \text{finite-dim}$
 - Compute (something) in random NCG *in progress*
 - Couple with F - future

$$Z = \int_{D \in \mathcal{G}} e^{-S(D)} dD$$

$$\langle f \rangle = \frac{1}{Z} \int f(D) e^{-S(D)} dD$$

Commutative \mathcal{A}

NC

$$\mathcal{A} = M(n_1, \mathbb{C}) \oplus M(n_2, \mathbb{C}) \oplus \dots$$

$n_i \times n_i$ matrices

Today $\mathcal{A} = M(n, \mathbb{C})$ "fuzzy space"

Commutative

NC

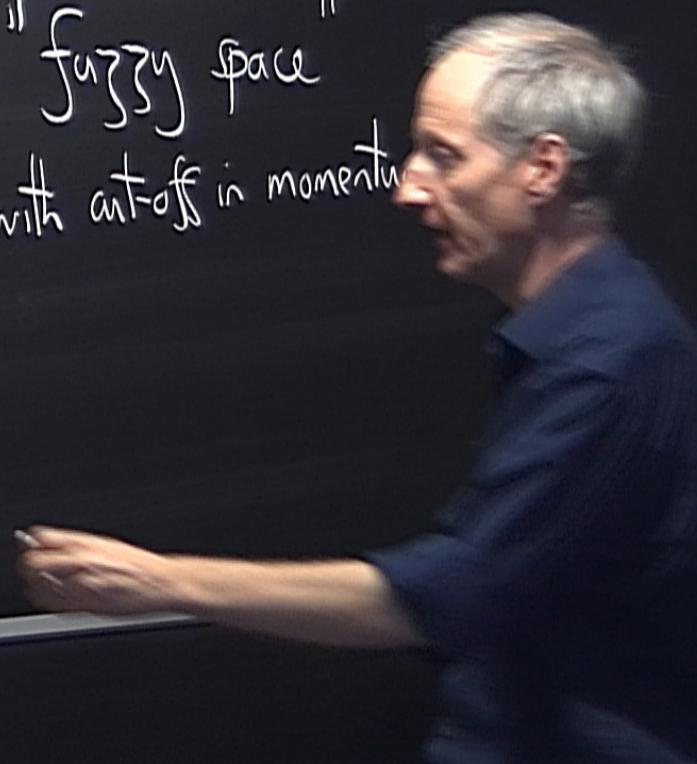
$$A = M(n_1, \mathbb{C}) \oplus M(n_2, \mathbb{C}) \oplus \dots$$

$n_i \times n_i$ matrices

Today

$$A = M(n, \mathbb{C}) \quad \text{"fuzzy space"}$$

\Downarrow
a "function" on space with cut-off in momentum



2. Details Fix $p, q \geq 0$

Gamma matrices $(\gamma^1)^2 = \dots (\gamma^p)^2 = 1$ $d = p + q$ \dim
 $(\gamma^{p+1})^2 = \dots (\gamma^{p+q})^2 = -1$ $S = q - p \bmod 8$ signature

Type (p, q) Dirac operators: $\mathcal{D} = V \otimes M_n(\mathbb{C})$ $V = \mathbb{C}^k$
 $k = \begin{cases} 2^{d/2} & \text{even} \\ 2^{(d-1)/2} & \text{odd} \end{cases}$

d=1

$V = \mathbb{C}$, $\mathcal{R} = M_n(\mathbb{C})$

$(1,0)$ $\mathcal{D} = \{H, \cdot\}$

$(0,1)$ $\mathcal{D} = -i[L, \cdot]$

H Hermitian

L anti-Hermitian

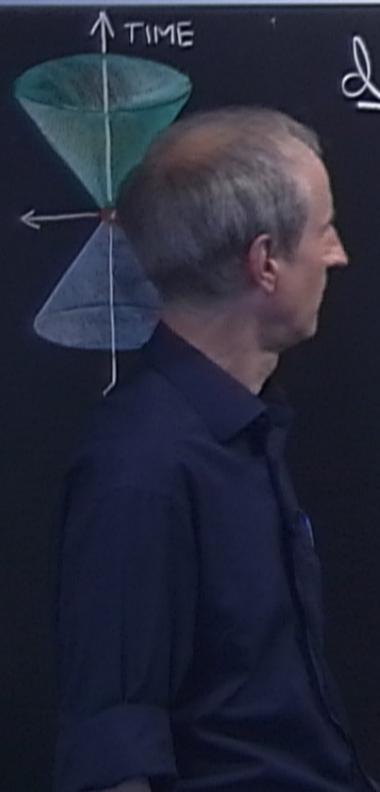
$$K = \begin{cases} \mathbb{Z}^{(d-1)/2} & d \text{ odd} \\ \mathbb{Z}^{(d-1)/2} & d \text{ even} \end{cases}$$



$$\begin{array}{ll}
 \underline{d=1} & V = \mathbb{C}, \mathcal{H} = M_n(\mathbb{C}) \\
 S=7 & (1,0) \quad \mathcal{D} = \{ H, \cdot \} \\
 S=1 & (0,1) \quad \mathcal{D} = -i[L, \cdot]
 \end{array}
 \quad
 \begin{array}{l}
 H \text{ Hermitian} \\
 L \text{ anti-Hermitian}
 \end{array}$$

$$R = \left\{ \begin{array}{l} \alpha \\ 2^{(d-1)/2} \text{ odd} \end{array} \right\}$$

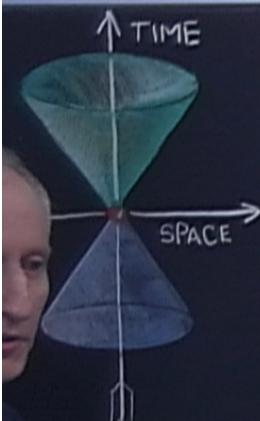
$$\begin{array}{ll}
 \underline{d=2} & V = \mathbb{C}^2 \quad \mathcal{H} = \mathbb{C}^2 \otimes M_n(\mathbb{C}) \\
 (2,0) & \mathcal{D} = \gamma^1 \otimes \{ H_1, \cdot \} + \gamma^2 \otimes \{ H_2, \cdot \} \\
 (1,1) & \mathcal{D} = \gamma^1 \otimes \{ H, \cdot \} + \gamma^2 \otimes [L, \cdot] \\
 (0,2) & \mathcal{D} = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot]
 \end{array}$$



$$\begin{array}{ll} d=1 & V = \mathbb{C}, \mathcal{H} = M_n(\mathbb{C}) \\ S=7 & (1,0) \quad \mathcal{D} = \{ H, \cdot \} \\ S=1 & (0,1) \quad \mathcal{D} = -i [L, \cdot] \end{array}$$

H Hermitian
L anti-Hermitian

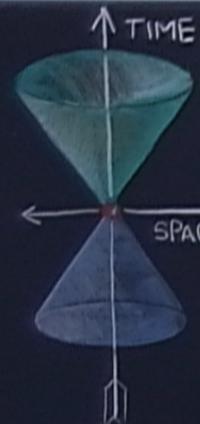
$$R = \left\{ \begin{array}{l} \alpha \\ 2^{(d-1)/2} \text{ odd} \end{array} \right.$$



$$\begin{array}{ll} d=2 & V = \mathbb{C}^2 \quad \mathcal{H} = \mathbb{C}^2 \otimes M_n(\mathbb{C}) \\ (2,0) & \mathcal{D} = \gamma^1 \otimes \{ H_1, \cdot \} + \gamma^2 \otimes \{ H_2, \cdot \} \\ (1,1) & \mathcal{D} = \gamma^1 \otimes \{ H_1, \cdot \} + \gamma^2 \otimes [L, \cdot] \\ (0,2) & \mathcal{D} = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot] \\ d=3 & \mathcal{D} = \{ H, \cdot \} + \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot] + \gamma^3 \otimes [L_3, \cdot] \end{array}$$

$\underline{d=1}$ $S=7$ $V = \mathbb{C}, \mathcal{H} = M_n(\mathbb{C})$ $D = \{H, \cdot\}$ $S=1 \quad (0,1) \quad D = -i[H, \cdot]$ $H \text{ Hermitian}$ $L \text{ anti-Hermitian}$

$$R = \begin{cases} \alpha \\ 2^{(d-1)/2} \end{cases} \text{ odd}$$

 $\underline{d=2}$ $S=6$ $V = \mathbb{C}^2, \mathcal{H} = \mathbb{C}^2 \otimes M_n(\mathbb{C})$ $D = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$ $(1,1) \quad D = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes [L_1, \cdot]$ $(0,2) \quad D = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot]$ $\underline{\exists (0,3)}$ $D = \{H, \cdot\} + \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot] + \gamma^3 \otimes [L_3, \cdot]$

$$\underline{d=1} \quad V = \mathbb{C}, \quad \mathcal{H} = M_n(\mathbb{C})$$

$$S=7 \quad (1,0) \quad \mathcal{D} = \{ H, \cdot \}$$

$$S=1 \quad (0,1) \quad \mathcal{D} = -i \Gamma_1 \cdot \gamma$$

H Hermitian
L anti-Hermitian

$$R = \begin{cases} 2 & d \text{ even} \\ 2^{(d-1)/2} & d \text{ odd} \end{cases}$$



$$\underline{d=2} \quad V = \mathbb{C}^2 \quad \mathcal{H} = \mathbb{C}^2 \otimes M_n(\mathbb{C})$$

$$S=6 \quad (2,0) \quad \mathcal{D} = \gamma^1 \otimes \{ H_1, \cdot \} + \gamma^2 \otimes \{ H_2, \cdot \}$$

$$S=0 \quad (1,1) \quad \mathcal{D} = \gamma^1 \otimes \{ H_1, \cdot \} + \gamma^2 \otimes [L, \cdot]$$

$$S=2 \quad (0,2) \quad \mathcal{D} = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot]$$

$$\underline{d=3} \quad \mathcal{D} = \{ H, \cdot \} + \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot] + \gamma^3 \otimes [L_3, \cdot]$$

$$S=3 \quad (0,3)$$

$$\underline{d=1}$$

$$V = \mathbb{C}, \mathcal{H} = M_n(\mathbb{C})$$

$$S=7$$

$$(1,0)$$

$$\mathcal{D} = \{ H, \cdot \}$$

H Hermitian

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$$S=6$$

$$(2,0)$$

$$\mathcal{D} = \gamma^1 \otimes \{ H_1, \cdot \} + \gamma^2 \otimes \{ H_2, \cdot \}$$

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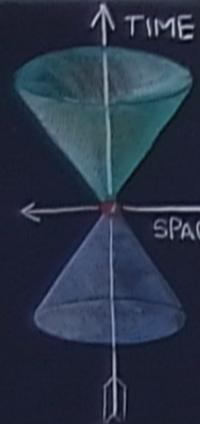
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$$\underline{d=3}$$

$$\mathcal{D} = \{ H, \cdot \} + \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot] + \gamma^3 \otimes [L_3, \cdot]$$

$$S=3$$

$$(0,3)$$



SPACE

$$S=0 \quad (1,1) \quad D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L_1, \cdot]$$

$$S=2 \quad (0,2) \quad D = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot]$$

$$d=3 \quad S=3 \quad (0,3) \quad D = \{H, \cdot\} + \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot] + \gamma^3 \otimes [L_3, \cdot]$$

Action $S(D) = \text{tr } V(D) = \sum_i V(\lambda_i)$ "spectral"

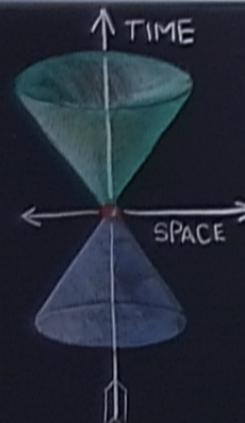
$$= D^2, D^4 + gD^2.$$



$$\begin{array}{ll} \underline{d=1} & V = \mathbb{C}, \mathcal{H} = M_n(\mathbb{C}) \\ S=7 & (1,0) \quad \mathcal{D} = \{ H, \cdot \} \\ S=1 & (0,1) \quad \mathcal{D} = -i [L, \cdot] \end{array}$$

$$\begin{array}{l} H \text{ Hermitian} \\ L \text{ anti-Hermitian} \\ \text{trace-free} \end{array} \left\{ \begin{array}{l} \text{free} \\ \text{data} \end{array} \right.$$

$$R = \left\{ \begin{array}{l} \alpha \\ 2^{(d-1)h} \text{ odd} \end{array} \right\}$$



$$\begin{array}{ll} \underline{d=2} & V = \mathbb{C}^2 \quad \mathcal{H} = \mathbb{C}^2 \otimes M_n(\mathbb{C}) \\ S=6 & (2,0) \quad \mathcal{D} = \gamma^1 \otimes \{ H_1, \cdot \} + \gamma^2 \otimes \{ H_2, \cdot \} \\ S=0 & (0,2) \quad \mathcal{D} = \gamma^1 \otimes \{ H_1, \cdot \} + \gamma^2 \otimes [L, \cdot] \\ S=2 & (1,1) \quad \mathcal{D} = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot] \end{array}$$

Observables

$$\begin{aligned} f_i(D) &= \lambda_i \\ f_\lambda &= \frac{1}{\#\text{ev}} \sum_i \delta(\lambda - \lambda_i) \end{aligned}$$

$$\begin{array}{ll} \underline{d=3} & S=3 (0,0,1) \\ & \{ H, \cdot \} + \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot] + \gamma^3 \otimes [L_3, \cdot] \\ \text{Action} & \text{tr } V(D) = \sum V(\lambda_i) \quad \text{"spectral"} \\ V(D) & D^4 + g D^2. \quad \text{Connes-Chamseddine} \quad V \int \end{array}$$

A type (p, q) geometry has a signature $s = q - p \bmod 8$ which determines some of the characteristics of the spectrum of D . These properties are well-known, holding also for the case of a Riemannian geometry in dimension d , which is a type $(0, d)$ spectral triple with signature $s = d \bmod 8$. The properties can be seen in the Monte Carlo simulations below.

Symmetry For $s \neq 3, 7$, if λ is an eigenvalue then so is $-\lambda$.

Doubling For $s = 2, 3$ or 4 , each eigenvalue appears with an even multiplicity. 

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Doubling For $s = 2, 3$ or 4 , each eigenvalue appears with an even multiplicity.



A Monte Carlo algorithm for matrix geometries

An observable $f(D)$ is a real- or complex-valued function of Dirac operators. The expectation value of f is defined to be

$$\langle f \rangle = \frac{1}{Z} \int f(D) e^{-S(D)} dD. \quad (1)$$

The integral can be approximated as a sum over a discrete ensemble $\{D_j, j = 1, \dots, N\}$.

$$\langle f \rangle_N = \frac{\sum_j f(D_j) e^{-S(D_j)}}{\sum_j e^{-S(D_j)}} \quad (2)$$

In the limit taking $N \rightarrow \infty$, the average obtained through this discrete sum will converge towards the continuum value. This convergence can be improved by using a Markov Chain Monte Carlo algorithm.

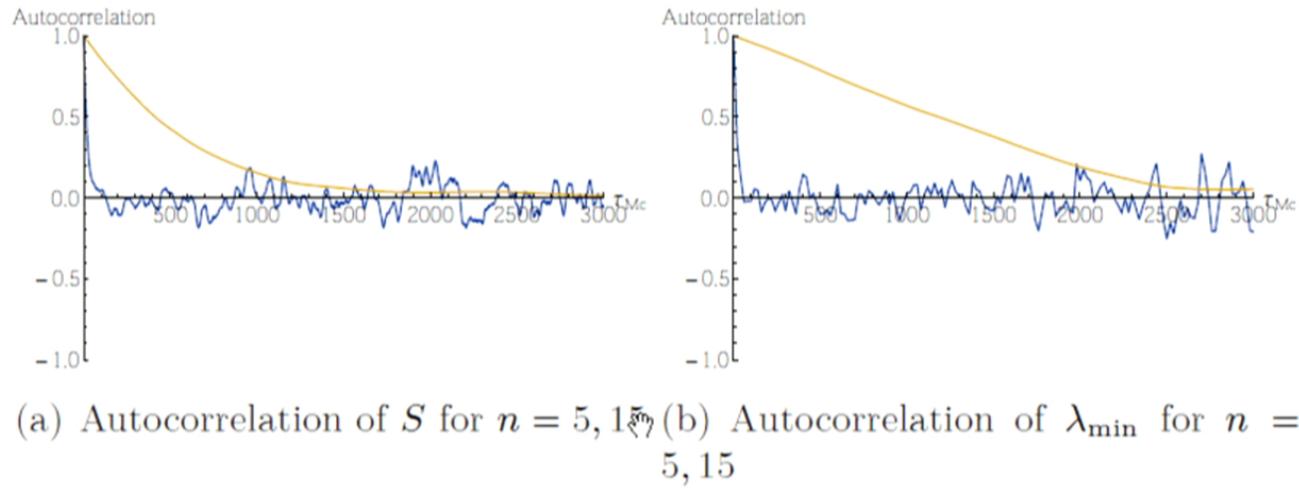


Figure 1: Fall-off of the autocorrelation for the action and the minimum eigenvalue for a type $(1, 0)$ geometry with $S = \text{Tr} (D^4)$. The blue line is $n = 5$ and the yellow line is $n = 15$. The horizontal axis is Monte Carlo time.

Results for D^2 action

The simplest cases: type (1, 0) and (0, 1)

The n^2 eigenvalues of Dirac operators can be written in terms of the n eigenvalues μ_i of the matrix H or the eigenvalues $i\mu$ of L . For the (1, 0) case one has eigenvalues

$$\lambda_{ij} = \mu_i + \mu_j$$

while for the (0, 1) case

$$\lambda_{ij} = \mu_i - \mu_j$$

This follows from the fact that eigenvectors of D are of the form $u_i \otimes \bar{u}_j$, with u_i the eigenvectors of H or L .

For the $(1, 0)$ case, using the simplified action (13) one can then transform the integral over the Dirac operator into an integral over the Hermitian matrix H .

$$S^{(1,0)}(D) = \text{Tr} (D^2) = 2n \text{Tr} (H^2) + 2(\text{Tr} H)^2 \quad (3)$$

$$= 2n \sum_i \mu_i^2 + 2 \sum_i \sum_j \mu_i \mu_j \quad (4)$$

(5)

The $(0, 1)$ case is similar, but one has to take into account the fact that the integration over Dirac operators is an integration over traceless matrices L . Using (17) gives

$$S^{(0,1)}(D) = \text{Tr} (D^2) = -2n \text{Tr} (L^2) \quad (6)$$

$$= 2n \sum_i \mu_i^2 , \quad (7)$$

These random matrix models are close to the Gaussian Her-

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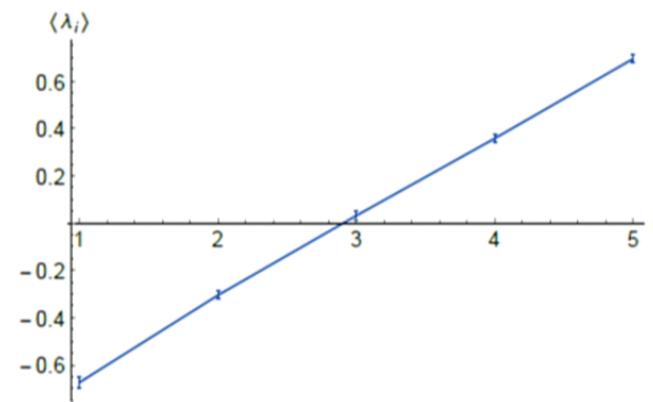
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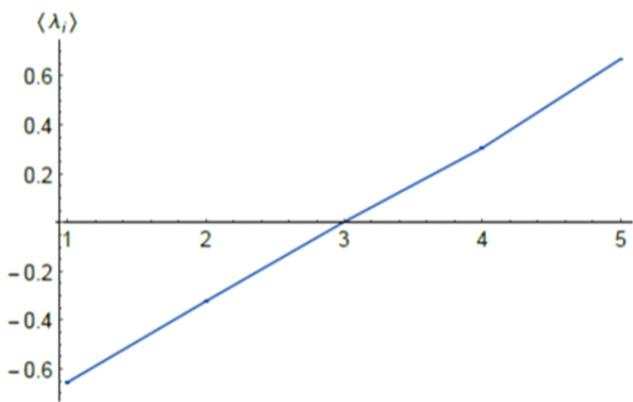
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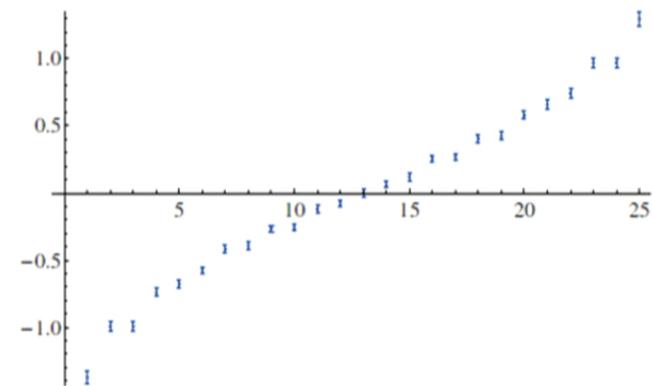
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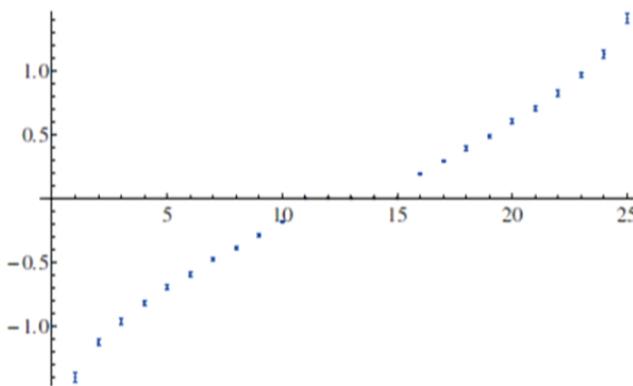
(a) Type $(1, 0)$ average eigenvalues of H



(b) Type $(0, 1)$ average eigenvalues of $-iL$



(c) Type $(1, 0)$ average eigenvalues of D



(d) Type $(0, 1)$ average eigenvalues of D

mitian matrix model [11, 3]. which has the similar action

$$S'(M) = 2n \text{Tr} (M^2) = 2n \sum_i \mu_i^2 ,$$

with integration over all Hermitian matrices.

A standard technique in random matrix models is to calculate the joint probability density for the eigenvalues. The formula is [12]

$$P(\lambda_1, \dots, \lambda_n) = C \exp -S(\lambda_i) \prod_{i < j} (\lambda_i - \lambda_j)^2 . \quad (8)$$

The terms with the differences of eigenvalues result from the Jacobian for the change of variables from the matrix elements to the eigenvalues. Since this term is small when two eigenvalues are close, this results in the phenomenon of the repulsion of eigenvalues.

The matrix M can be split into traceless and trace parts, and these are statistically independent. It follows that expectation values in the $(0, 1)$ model can be calculated as expectation values

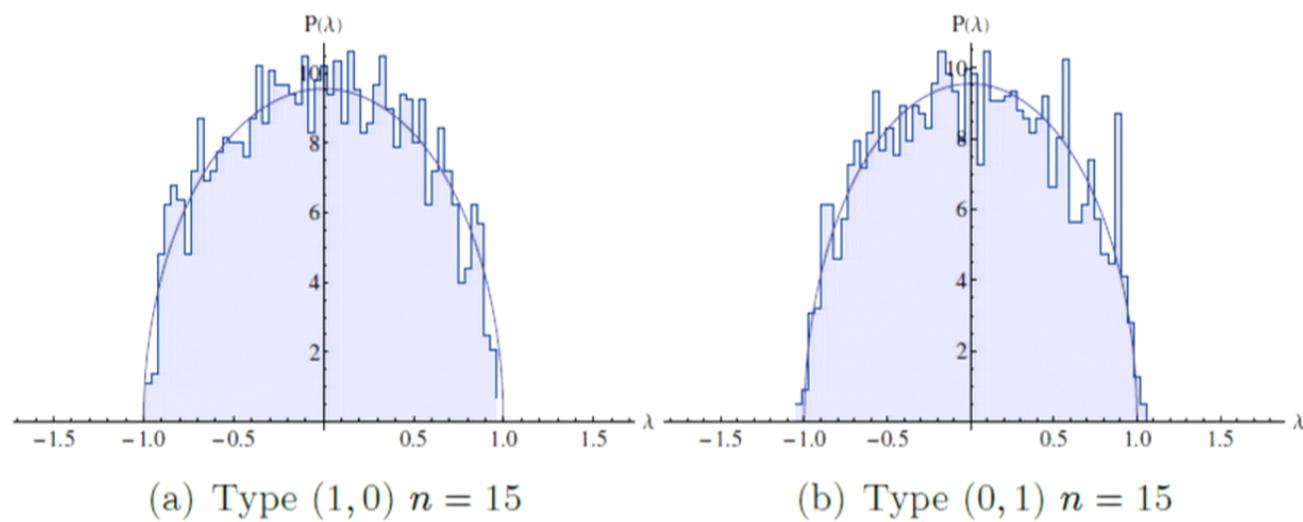
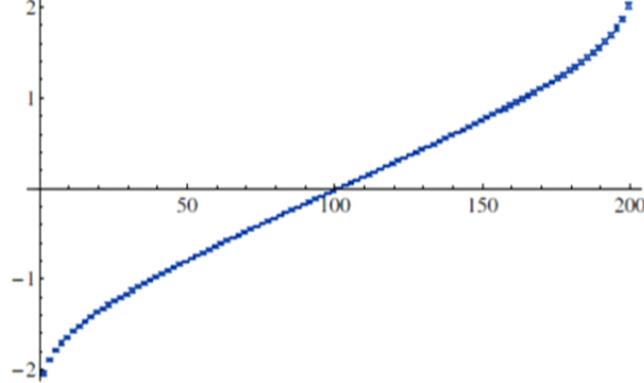
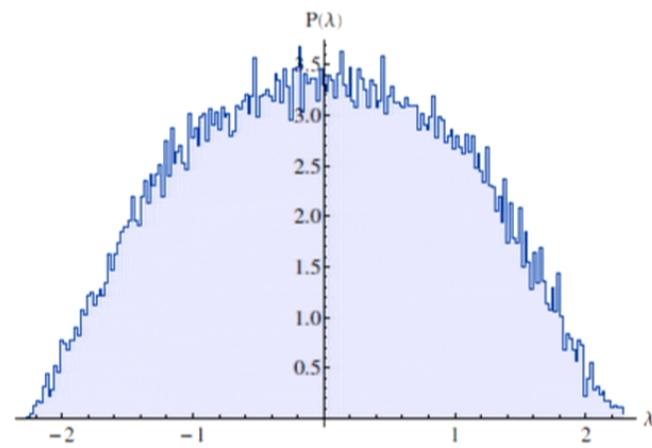


Figure 3: The semicircle law is compared with the density of states for H or L .



(a) Average eigenvalues



(b) Eigenvalue distribution

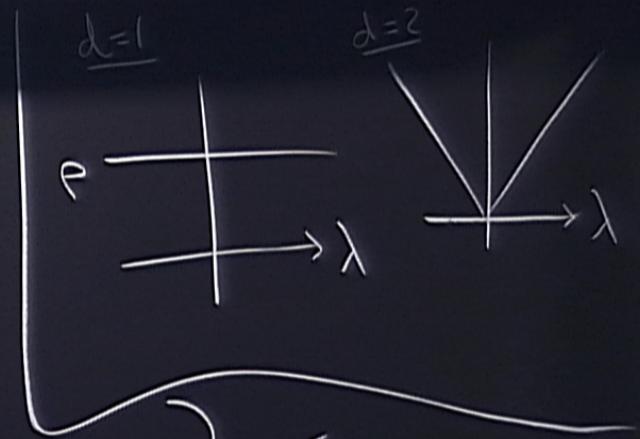
Figure 7: Type (0, 3). The action is $S(D) = \text{Tr}(D^2)$ and the matrix size is $n = 10$.

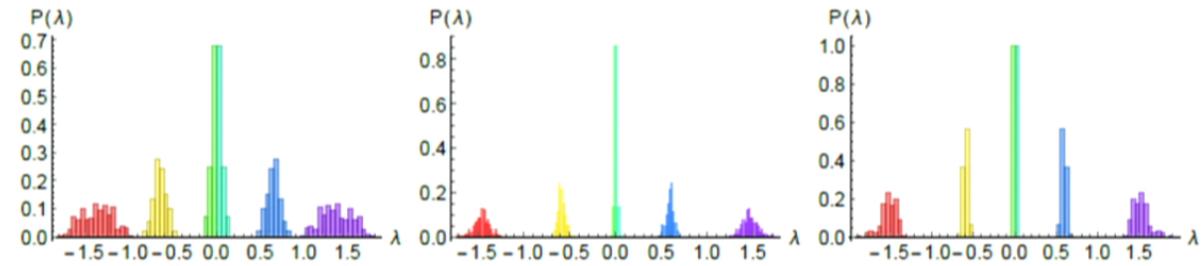
Application : particle physics : $M \times F$

$M : C$ spacetime

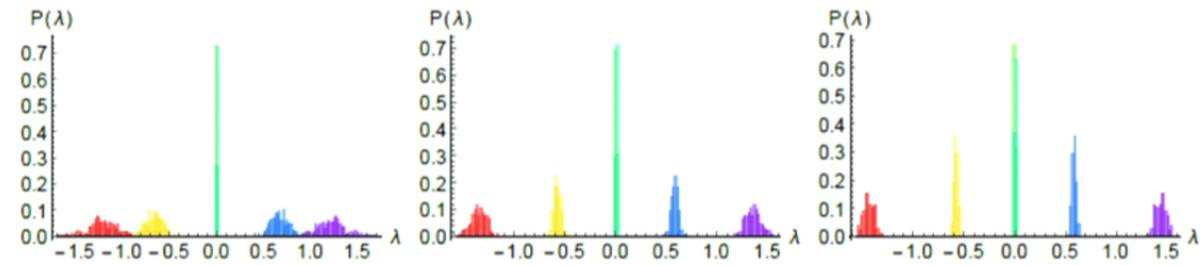
$F : NC$ internal space

- Project :
- Replace $M \sim NC$ geometry
 - Truncate to finite # nodes $A : \text{finite-dim}$
 - Compute (something) in random NCG *in progress*
 - Couple with F — future





(a) Type (2,0) $n = 5$ (b) Type (2,0) $n = 10$ (c) Type (2,0) $n = 15$



(d) Type (0,2) $n = 5$ (e) Type (0,2) $n = 10$ (f) Type (0,2) $n = 15$

Figure 9: The distribution of selected eigenvalues at different sizes.

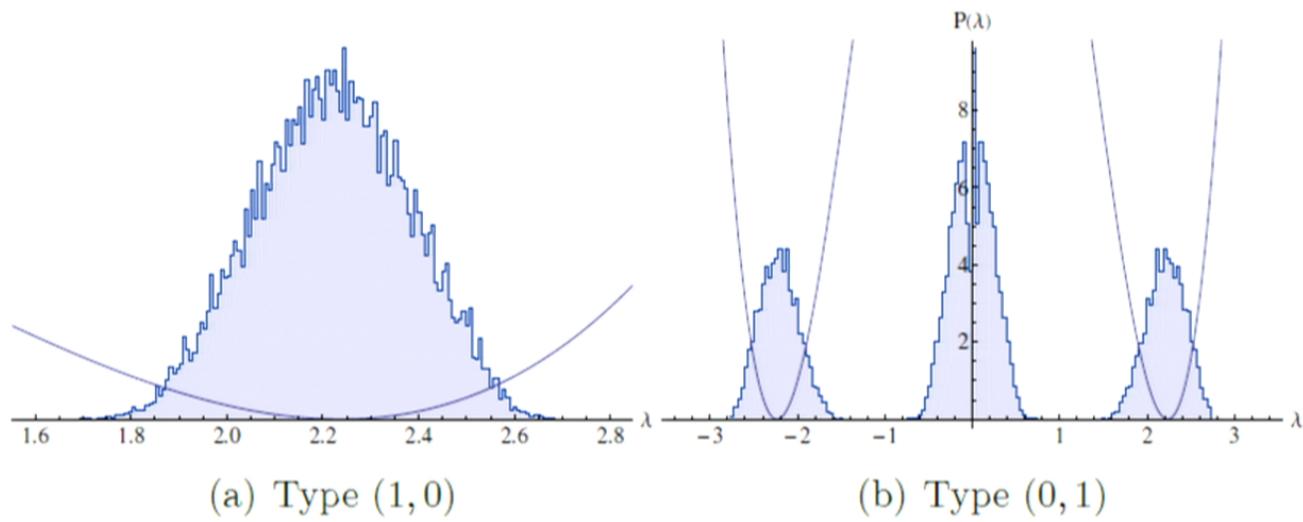


Figure 13: The eigenvalues of $S = \text{Tr} (D^4 - 10D^2)$ for $n = 10$.

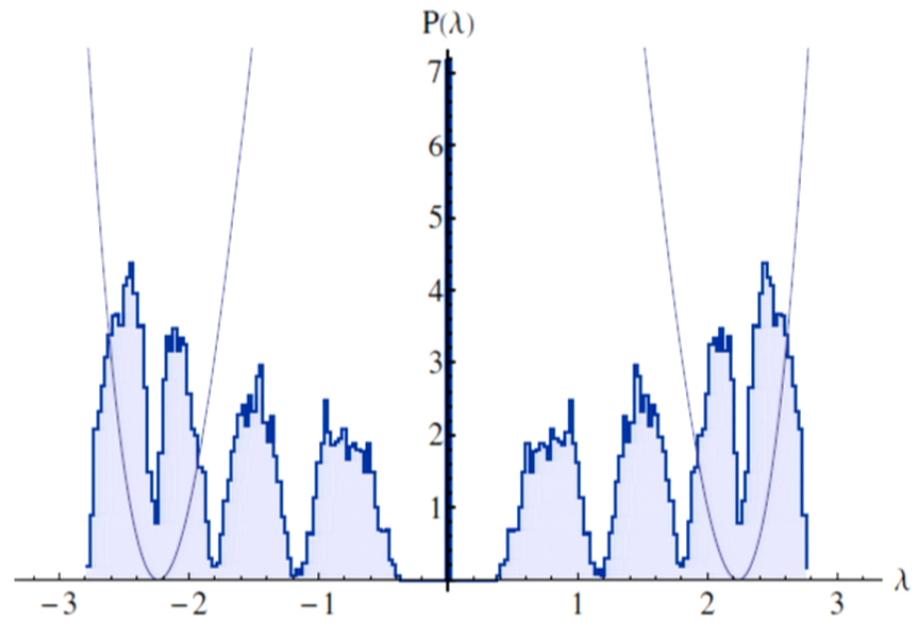


Figure 15: Type $(0, 2)$. The eigenvalues of $S = \text{Tr} (D^4 - 10D^2)$ for $n = 10$.