Title: Nonassociative geometry, Hom-associative algebras, and cyclic homology - Mohammad Hassanzadeh

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Abstract:

Nonassociative Geometry, Hom-associative algebras, Cyclic homology

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Joint work with
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Nonassociatity

- Lie algebras
- Alternative algebras (Example: Octonion Algebra (Number theory, string theory))
- Alternative algebras

$$x(xy) = (xx)y \quad \text{(left alternative)}, \quad (yx)x = y(xx) \quad \text{(right alternative)}$$

- Jordan algebras
  Introduced by Pascual Jordan, mathematical physicist,
  to formalize the notion of an algebra of observable in quantum mechanics

$$xy = yx$$

$$(xy)(xx) = x(y(xx))$$

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Quaternion algebra $\mathbb{O}$

The octonions were discovered in 1843 by John T. Graves and independently by Arthur Cayley in 1845. The octonions algebra is also called Cayley algebra.

	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
u	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-u$	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	$-u$	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	$-u$	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	$-u$	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	$-u$	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	$-u$	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	$-u$

Non-Associative Geometry and the Spectral Action Principle,
Latham Boyle and Shane Farnsworth (2013-2014)

- Non-associative Spectral triple related to standard model of particle physics

$$(A, H, D) = (\mathbb{O}, \mathbb{O}, D)$$

Home-Lie algebra: Hartwig, Larsson, Silvestrove (2006)

A Home-Lie algebra is a triple $(V, [.,.], \alpha)$, consisting a vector space V with a bilinear map $[.,.] : V \times V \longrightarrow V$ and a linear map $\alpha : V \longrightarrow V$ satisfying

i) $[x, y] = -[y, x]$.

ii) $\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0$. (Twisted Jacobi Identity)

Example: Deformation of $sl(2)$

Example

Let V be a 3 dimensional k-vector space with basis $\{X_1, X_2, X_3\}$. We define

$$[X_1, X_2] = 2x_2, \quad [X_1, X_3] = -2X_3, \quad [X_2, X_3] = X_1,$$

by a map α defined, on the basis, by the matrix $M = \begin{bmatrix} a & d & c \\ 2c & b & f \\ 2d & e & b \end{bmatrix}$ where $a, b, c, d, e, f \in k$.

Note: If the matrix M is identity matrix then we get the classical Lie algebra $sl(2)$.

History

- Deformation of Lie algebras
  Ruggero Santilli (Italian-American nuclear physicist)
- In 1967, Santilli considered two-parametric deformations of Lie commutator bracket in an associative algebra $(A, B) = pAB - qBA$ where p and q are scalar parameters and A and B are elements in associative algebra. (Algebra of matrices or linear operators)
- In 1978, He extended this to operator deformations of Lie product $(A, B) = APB - BQA$, where P and Q are fixed elements in the associative algebra.
  Motivation: Resolving certain limitations of conventional formalism of classical and quantum mechanics. (Several books and papers)

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Hom-associative algebra: Silvestrov, Makhlouf (2008)

Hom-associative algebra (Hom-algebra) is a triple $(\mathcal{A}, \mu, \alpha)$ consisting of a k-vector space $\mathcal{A}$ over a field k, and k-linear maps $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ that we denote by $\mu(a, b) =: ab$, and a k-linear map $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying the Hom-associativity condition

$$\alpha(a)(bc) = (ab)\alpha(c),$$

for any $a, b, c \in \mathcal{A}$.

Hom-algebras

- A Hom-algebra $\mathcal{A}$ is unital if there exist $1 \in \mathcal{A}$ where $1a = a1 = a$.
- A Hom-algebra is multiplicative if $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in \mathcal{A}$.

Example

Let $\mathcal{A}$ be any associative algebra with multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, and let $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ be an algebra map. Then for $\mu_\alpha = \alpha \circ \mu : \mathcal{A} \longrightarrow \mathcal{A}$, the triple $(\mathcal{A}, \mu_\alpha, \alpha)$ is a multiplicative Hom-algebra.

Unitality condition is a restrictive

- If Hom-algebra $\mathcal{A}$ is unital, then for any $a, b \in \mathcal{A}$,

$$\alpha(a)b = a\alpha(b) = \alpha(ab).$$

- Therefore Unital Hom-algebras are restricted.
- There is no natural embedding of non-unital Hom-algebras to unital Hom-algebras.

Lemma (I. Shapiro, S. Sutlu, M. H)

Let $(\mathcal{A}, \mu, \alpha, 1)$ be a multiplicative unital Hom-associative algebra. Then $\mathcal{A} \cong A_1 \oplus A_2$ as algebras, where A_1 is a unital associative algebra, and A_2 is a unital (not necessarily associative) algebra. Furthermore, $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ is given by $\alpha(a_1 + a_2) = a_1$. Conversely, for any unital associative algebra A_1 and a unital (not necessarily associative) algebra A_2, $A_1 \oplus A_2$ is a multiplicative unital Hom-associative algebra with $\alpha : A_1 \oplus A_2 \longrightarrow A_1 \oplus A_2$ being the projection onto A_1.

Example

Let $\mathcal{A}$ be a two dimensional vector space over a field k with a basis $\{e_1, e_2\}$. Let the multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ be given by

$$e_i e_j = \begin{cases} e_1, & \text{if } (i,j) = (1,1) \\ e_2 & \text{if } (i,j) \neq (1,1). \end{cases}$$

Then via the map

$$\alpha : \mathcal{A} \longrightarrow \mathcal{A}, \qquad \alpha(e_1) = e_1 - e_2, \quad \alpha(e_2) = 0,$$

the triple $(\mathcal{A}, \mu, \alpha)$ is a multiplicative Hom-associative algebra with the unit $1 := e_1$. We have $\mathcal{A} = k(e_1 - e_2) \oplus ke_2$.

Modules

- $\mathcal{A}$-Right module

$$\beta(v) \cdot (ab) = (v \cdot a) \cdot \alpha(b)$$

- $\mathcal{A}$-bimodule

$$\alpha(a) \cdot (v \cdot b) = (a \cdot v) \cdot \alpha(b)$$

Modules

Example

Any Hom-associative algebra $(\mathcal{A}, \mu, \alpha)$ is a $\mathcal{A}$-bimodule over itself by multiplication and $\beta = \alpha$.

Remark

The algebraic dual $\mathcal{A}^*$ is NOT necessarily an $\mathcal{A}$-module via the coregular actions,

$$(a \cdot f)(b) = f(ba) \quad \text{or} \quad (f \cdot a)(b) = f(ab)$$

or their α-twisted versions

$$(a \cdot f)(b) = f(b\alpha(a)) \quad \text{or} \quad (f \cdot a)(b) = f(\alpha(a)b)$$

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Modules

Lemma

Given a Hom-associative algebra $(\mathcal{A}, \mu, \alpha)$, the pair $(\mathcal{A}^\circ, Id_{\mathcal{A}^*})$ where

$$\mathcal{A}^\circ = \{f \in \mathcal{A}^* \mid f(x\alpha(y)) = f(\alpha(xy)) = f(\alpha(x)y)\},$$

is a left $\mathcal{A}$-module via

$$(a \cdot f)(b) = f(b\alpha(a)),$$

for any $a, b \in \mathcal{A}$, and any $f \in \mathcal{A}^\circ$. (It is a bimodule)

Cohomology theory, Dual module, I.Shapiro, S. Sutlu, M. H (2015)

Definition

Let $(\mathcal{A}, \alpha)$ be a Hom-algebra. A vector space V is called a dual left $\mathcal{A}$-module if there are linear maps $\cdot : \mathcal{A} \otimes V \longrightarrow V$, and $\beta : V \longrightarrow V$ where

$$a \cdot (\alpha(b) \cdot v) = \beta((ab) \cdot v).$$

Example

Let $(\mathcal{A}, \alpha)$ be a Hom-algebra, and (V, β) an $\mathcal{A}$-bimodule. Then the algebraic dual V^* is a dual $\mathcal{A}$-bimodule.

Special case: for $V = \mathcal{A}$ the $\mathcal{A}^*$ is dual bimodule.

History: Between 1976-1978, Connes used Hochschild cohomology of A with <u>coefficients in A^*</u> to classify injective von Neumann algebra.

Cyclic cohomology, I.Shapiro, S. Sutlu, M. H (2015)

$$V = \mathcal{A}^*.$$

$$b\phi(a_0 \otimes \cdots \otimes a_{n+1}) = \phi(a_0 a_1 \otimes \alpha(a_2) \otimes \cdots \otimes \alpha(a_{n+1}))$$
$$+ \sum_{j=1}^{n} (-1)^j \phi(\alpha(a_0) \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes \alpha(a_{n+1}))$$
$$+ (-1)^{n+1} \phi(a_{n+1} a_0 \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_n)).$$

$$\tau_n \phi(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := (-1)^n \phi(a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}),$$

$$C^n_{\lambda, Hom}(\mathcal{A}, \mathcal{A}^*) = \ker(Id - \tau)$$
$$= \{\phi \in C^n_{Hom}(\mathcal{A}, \mathcal{A}^*) \mid \phi(a_0 \otimes a_1 \otimes \cdots \otimes a_n)$$
$$= (-1)^n \phi(a_n \otimes a_0 \otimes \cdots \otimes a_{n-1})\}.$$

$$b'(\varphi)(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n-1} (-1)^i \varphi(\alpha(a_0) \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes \alpha(a_n)),$$

$$N := Id + \tau + \ldots + \tau^n : C^n_{Hom}(\mathcal{A}) \longrightarrow C^n_{Hom}(\mathcal{A}).$$

New Results: non-associative differential calculus I.Shapiro, S. Sutlu, M. H (2015)

Definition

Let $(\mathcal{A}, \alpha)$ be a Hom-algebra and (M, β) a dual $\mathcal{A}$-bimodule. The k-linear function $f : \mathcal{A} \longrightarrow M$ is called a twisted α-derivation if

$$a_1 f(\alpha(a_2)) + f(\alpha(a_1))a_2 = \beta(f(a_1 a_2)). \tag{0.2}$$

The set of all twisted α-derivations is denoted by $Der_k(\mathcal{A}, M)$.

Lemma

Let $(\mathcal{A}, \alpha)$ be a Hom-algebra and (M, β) a dual $\mathcal{A}$-bimodule. The map $f_m : \mathcal{A} \longrightarrow M$ given by $f_m(a) = am - ma$ is a twisted α-derivation(called principal derivations).

I.Shapiro, S. Sutlu, M. H (2015)

Proposition

Let $(\mathcal{A}, \alpha)$ be a Hom-algebra and (M, β) a dual $\mathcal{A}$-bimodule. Then

$$H^1_{Hom}(\mathcal{A}, M) = \frac{Der_k(\mathcal{A}, M)}{PDer_k(\mathcal{A}, M)}.$$