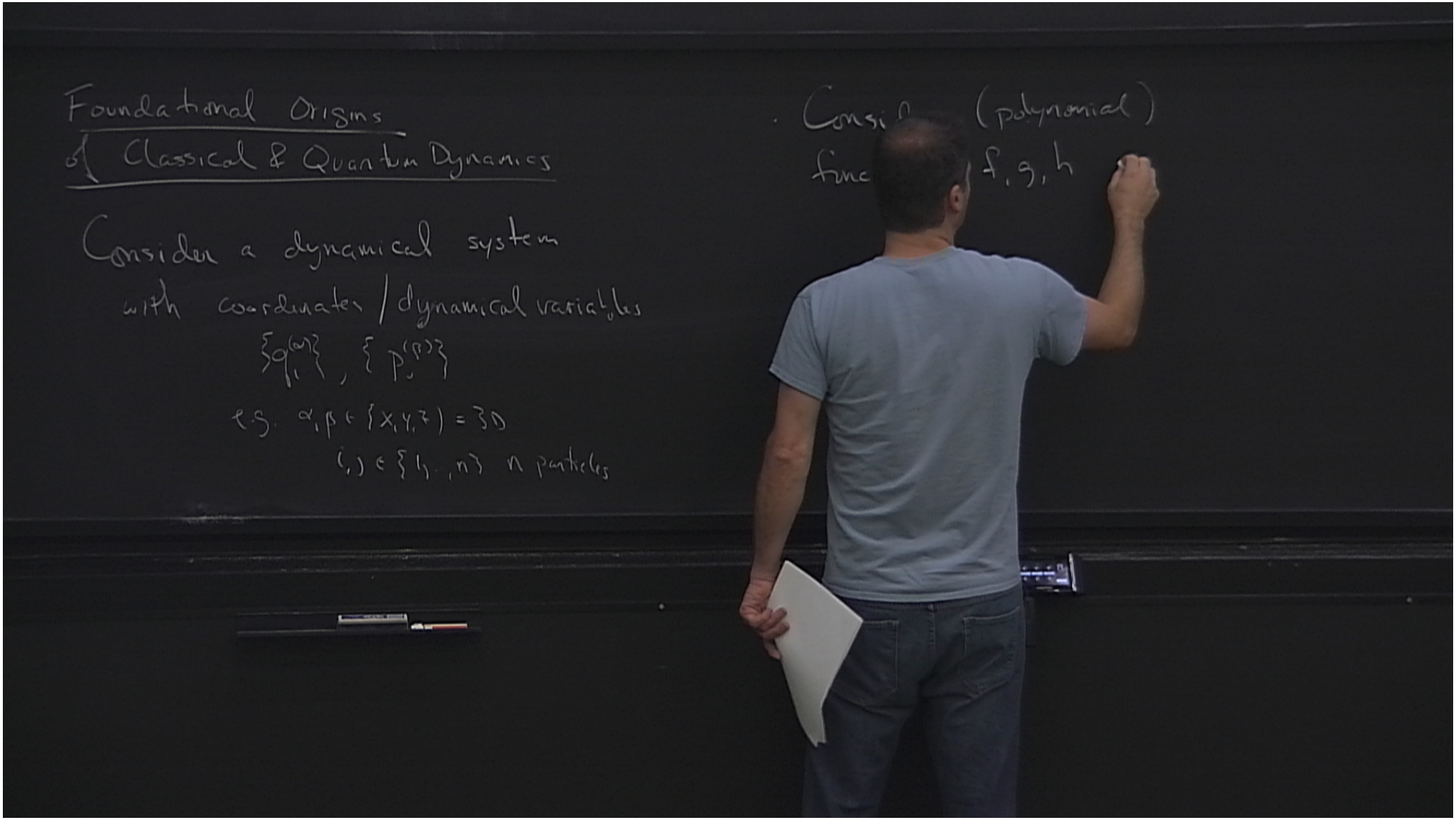
Title: PSI 2015/2016 Quantum Theory - Lecture 14

Date: Sep 25, 2015 10:45 AM

URL: http://pirsa.org/15090055

Abstract:


Foundational Origins
of Classical & Quantum Dynamics
Consider a dynamical system
with coordinates / dynamical variables
$\{q_i^{(\alpha)}\}, \{p_j^{(\beta)}\}$
e.g. $\alpha, \beta \in \{x, y, z\} = 3D$
$i, j \in \{1, \ldots, n\}$ n particles
Consi (polynomial)
func f, g, h

Foundational Origins of Classical & Quantum Dynamics

Consider a dynamical system with coordinates / dynamical variables

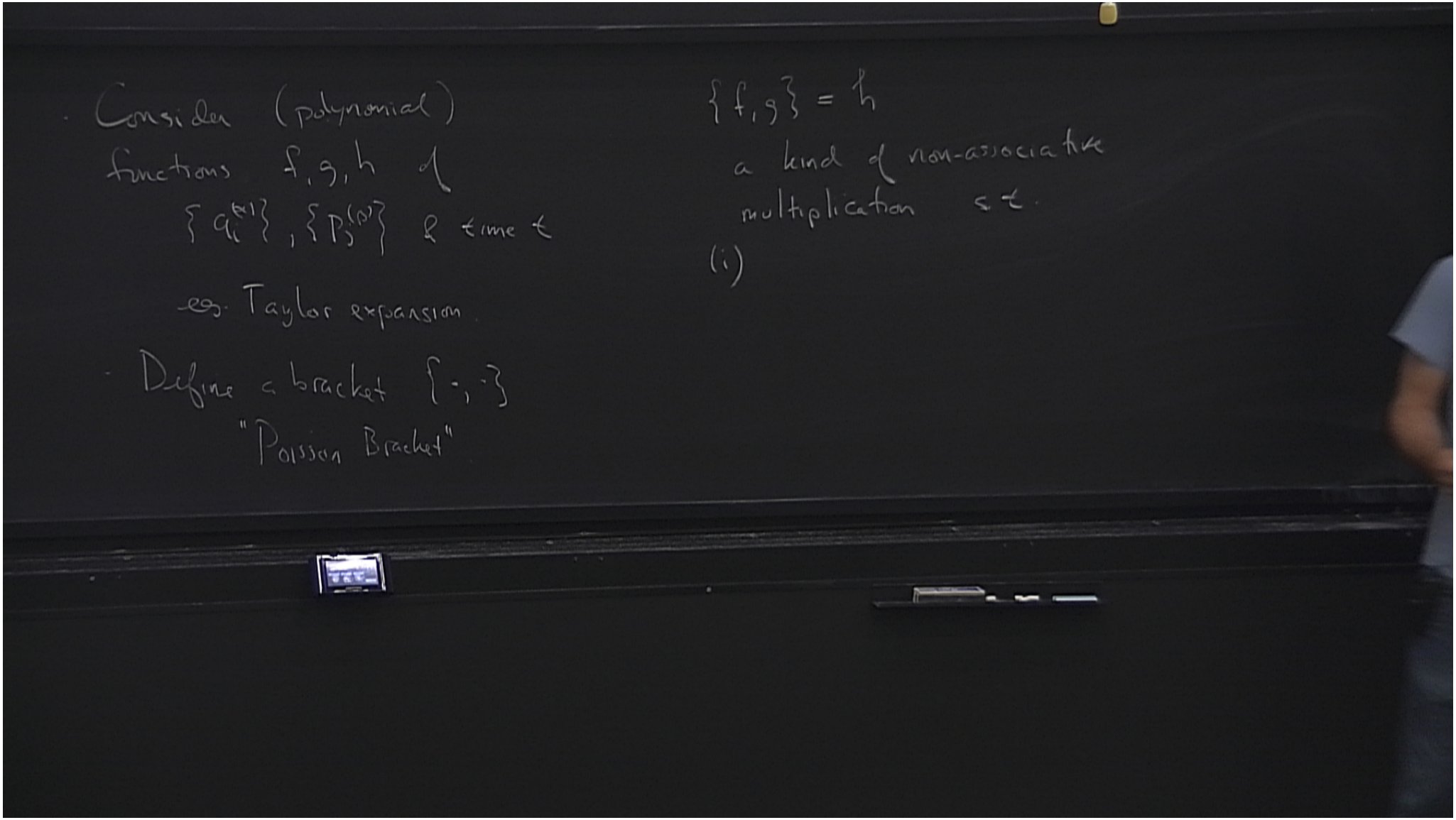
$\{q_i^{(\alpha)}\}, \{p_j^{(\beta)}\}$

e.g. $\alpha, \beta \in \{x, y, z\} = 3D$

$i, j \in \{1, \ldots, n\}$ n particles

Consider (polynomial) functions f, g, h of

$\{q_i^{(\alpha)}\}, \{p_j^{(\beta)}\}$ & time t


· Consider (polynomial)
functions f, g, h of
$\{q_i^{(n)}\}, \{p_j^{(n)}\}$ & time t
eg. Taylor expansion
· Define a bracket $\{\cdot,\cdot\}$
"Poisson Bracket"
$\{f,g\} = h$
a kind of non-associative
multiplication s.t.
(i)

- Consider (polynomial) functions f, g, h of $\{q_i^{(n)}\}, \{p_j^{(n)}\}$ & time t

  e.g. Taylor expansion

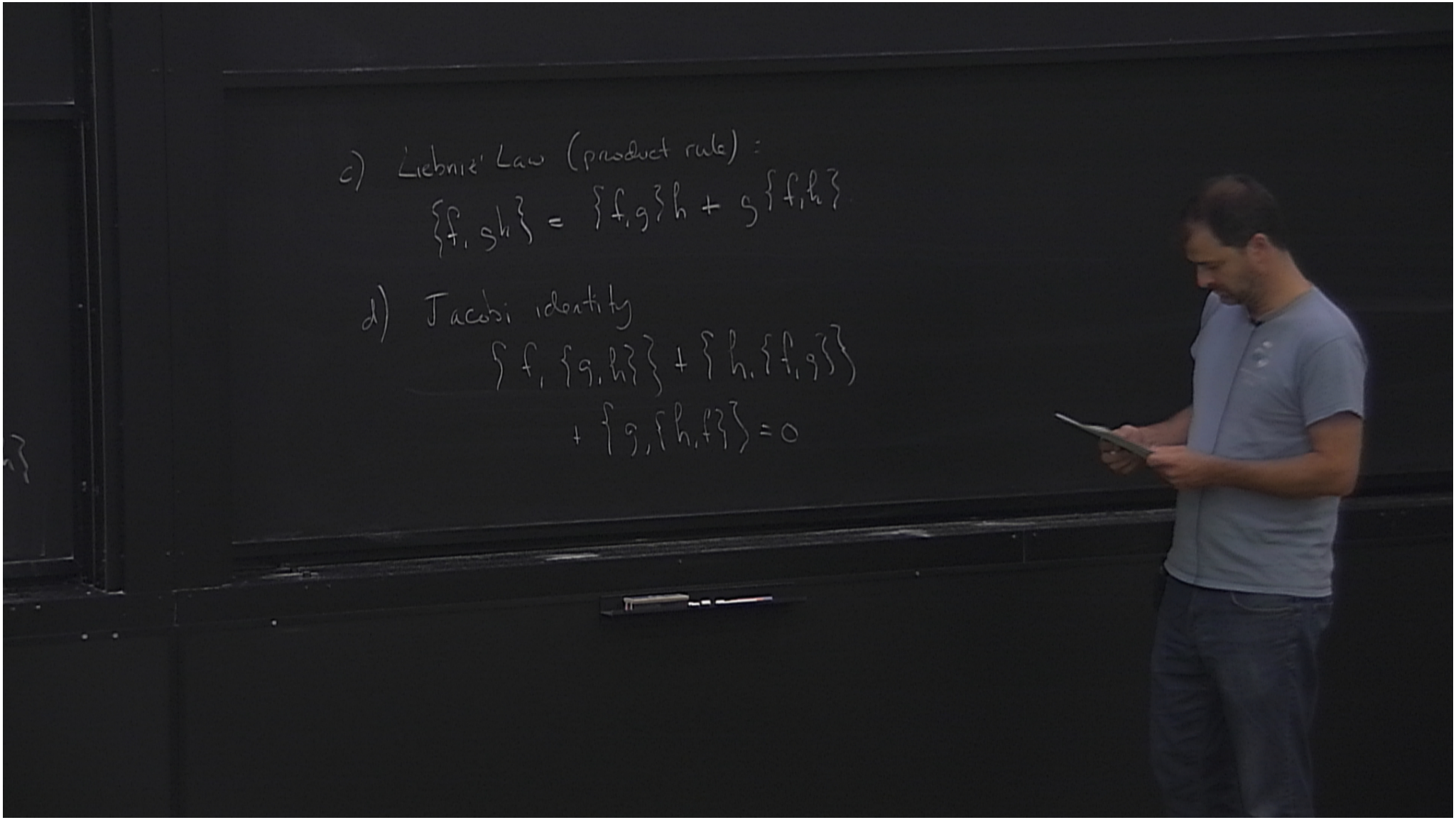
- Define a bracket $\{\cdot,\cdot\}$ "Poisson Bracket"

$\{f,g\} = h$

a kind of non-associative multiplication s.t.

(a) Anti-symmetry: $\{f,g\} = -\{g,f\}$

(b) Linearity (distributivity): $\{af+bg, h\}$ for scalars a,b $= a\{f,h\} + b\{g,h\}$


c) Leibniz Law (product rule):
{f, gh} = {f, g} h + g {f, h}
d) Jacobi identity
{f, {g, h}} + {h, {f, g}}
+ {g, {h, f}} = 0

c) Leibniz Law (product rule):

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

d) Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

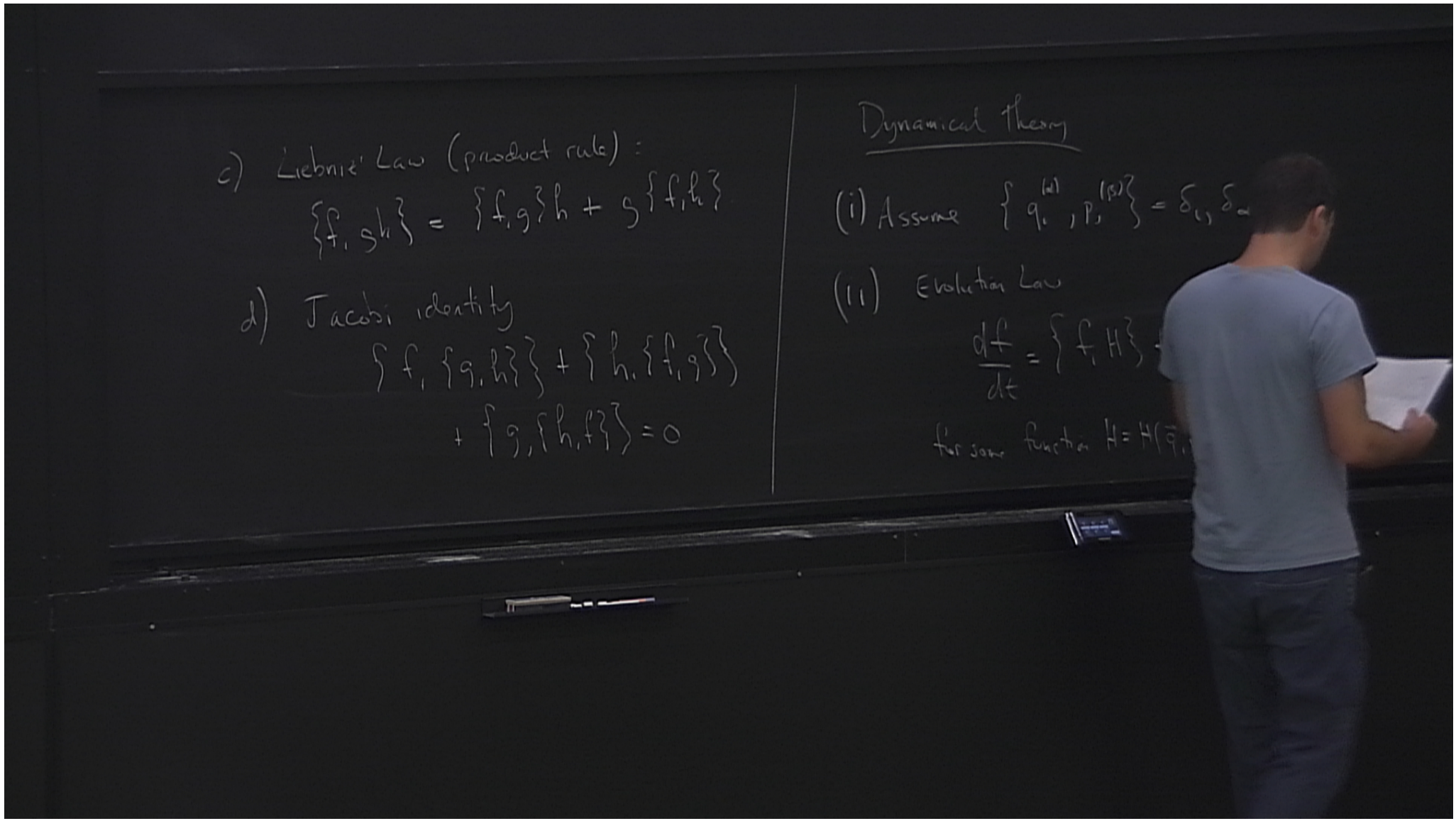
Dynamical theory

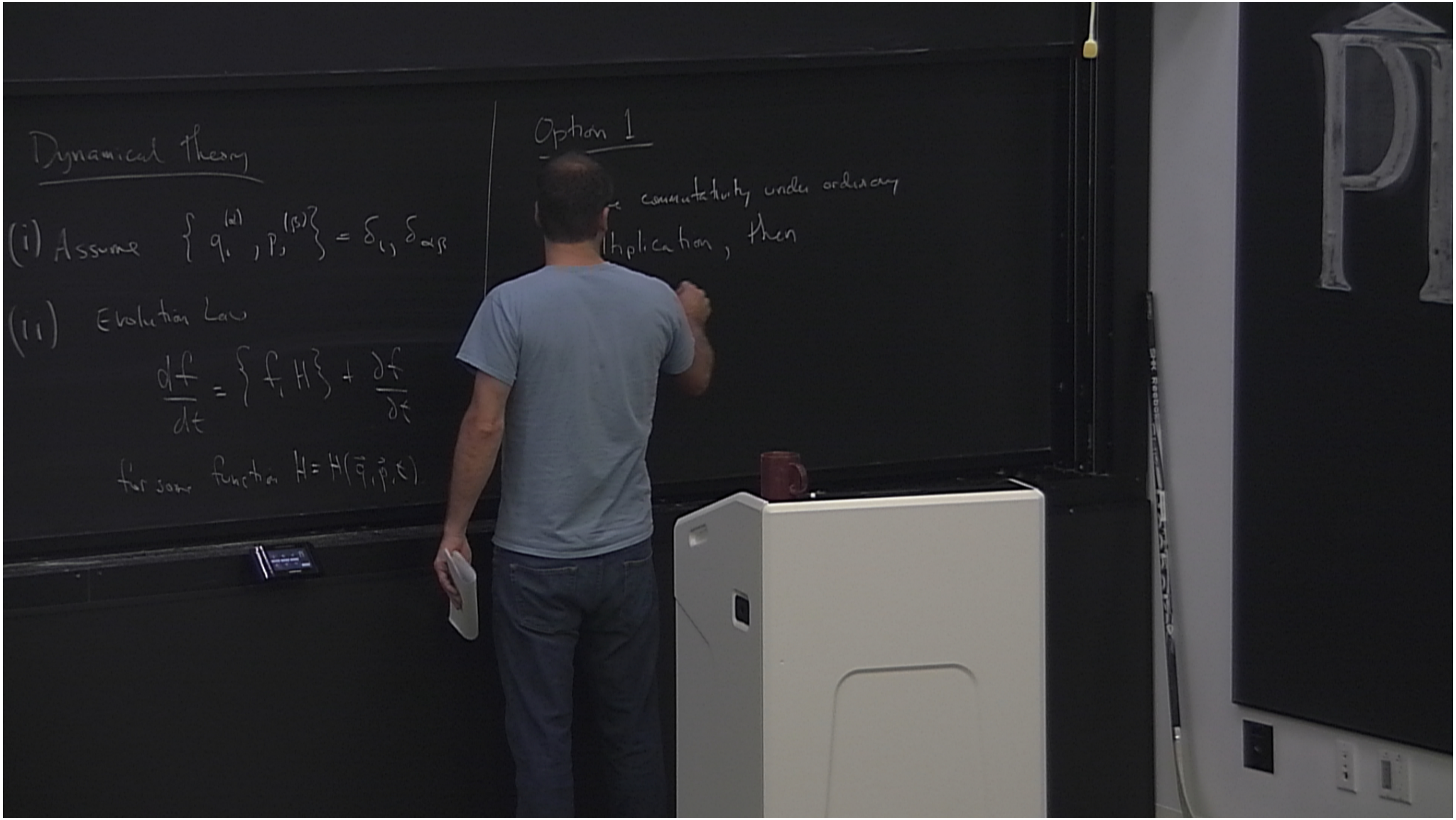
(i) Assume $\{q_\alpha^{(a)}, p_\beta^{(b)}\} = \delta_{ab}\delta_{\alpha\beta}$

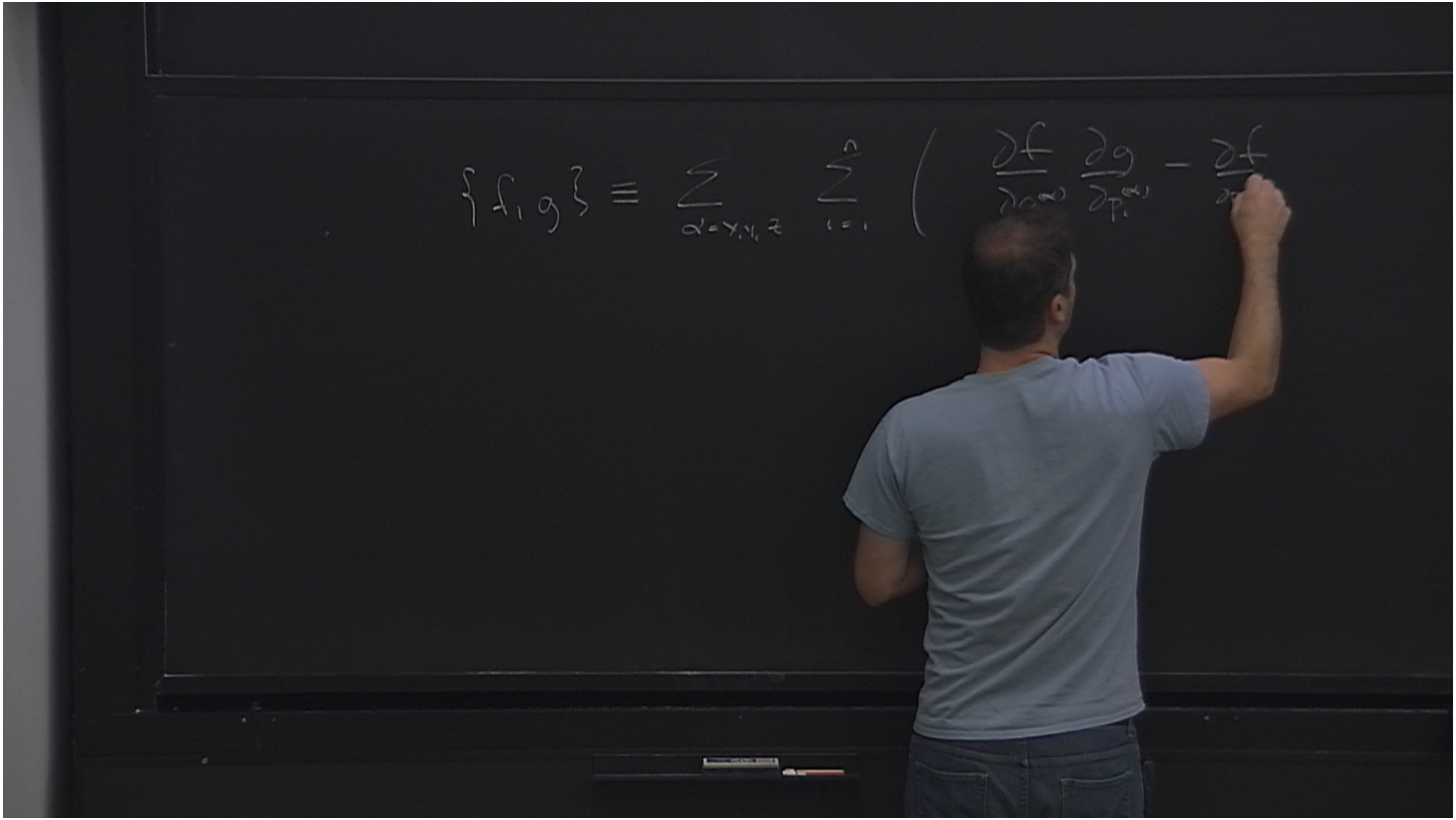
(ii) Evolution Law

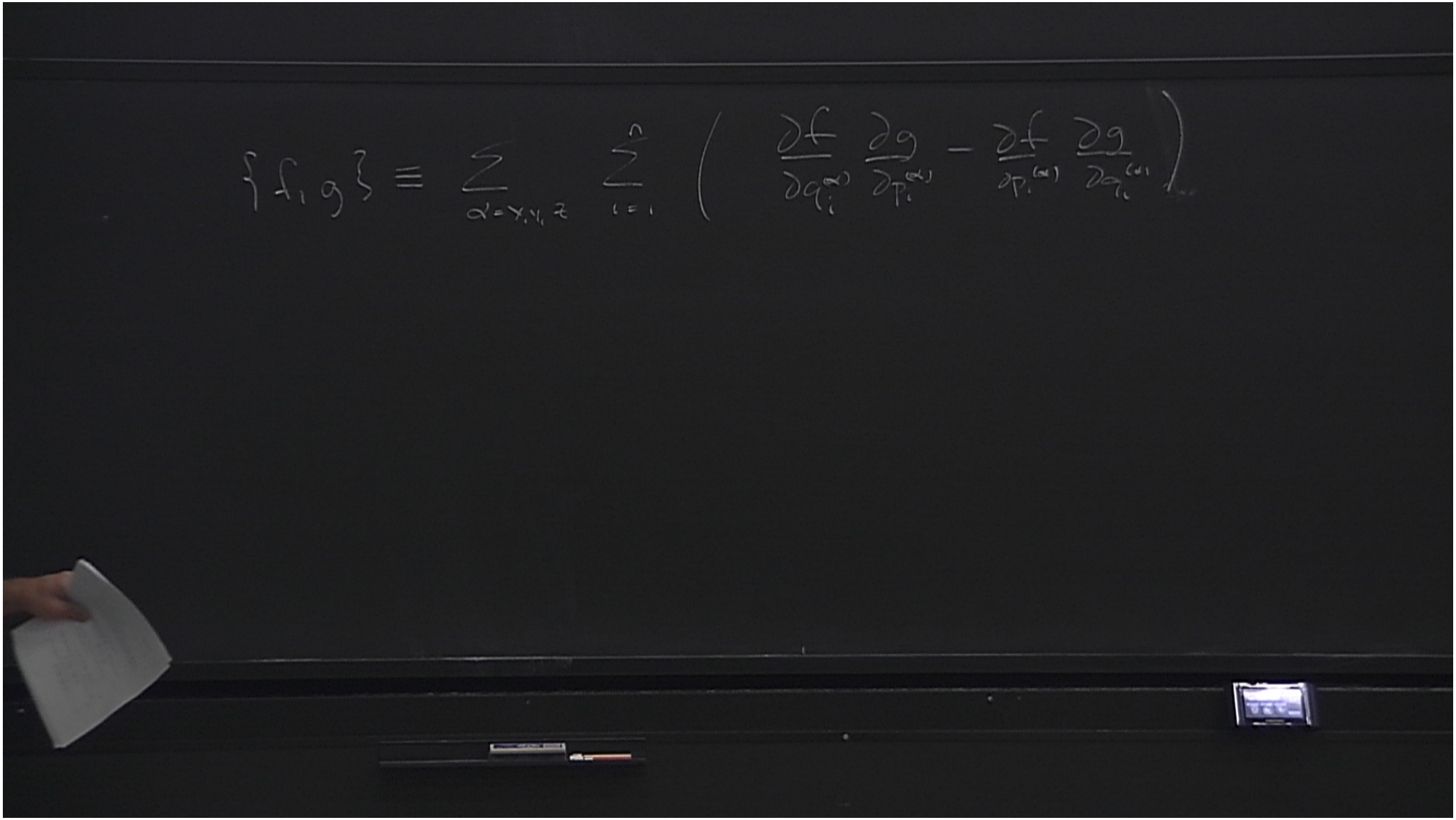
$$\frac{df}{dt} = \{f, H\}$$

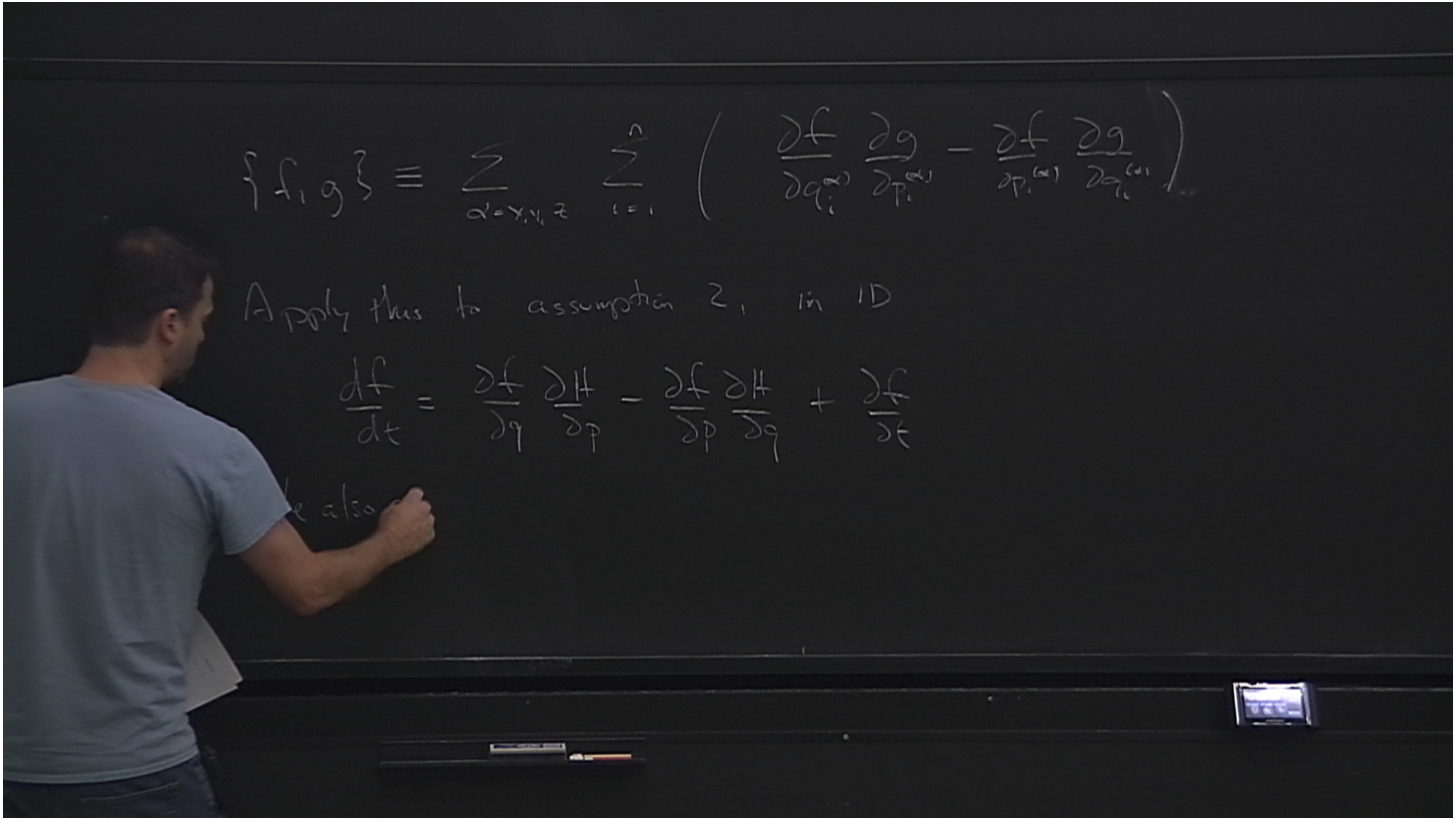
for some function $H = H(\vec{q}, \ldots)$




Dynamical Theory
(i) Assume $\{q_i^{(\alpha)}, p_j^{(\beta)}\} = \delta_{ij}\delta_{\alpha\beta}$
(ii) Evolution Law
$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$
for some function $H = H(\vec{q}, \vec{p}, t)$
Option 1
commutativity under ordinary
implication, then


$\{f, g\} \equiv \sum_{\alpha=x,y,z} \sum_{i=1}^{N} \left(\frac{\partial f}{\partial r_i^\alpha} \frac{\partial g}{\partial p_i^\alpha} - \frac{\partial f}{\partial}$


$\{f, g\} \equiv \sum_{\alpha=x,y,z} \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i^{(\alpha)}} \frac{\partial g}{\partial p_i^{(\alpha)}} - \frac{\partial f}{\partial p_i^{(\alpha)}} \frac{\partial g}{\partial q_i^{(\alpha)}} \right)$


$\{f, g\} \equiv \sum_{\alpha=x,y,z} \sum_{i=1}^{\hat{}} \left(\frac{\partial f}{\partial q_i^{(\alpha)}} \frac{\partial g}{\partial p_i^{(\alpha)}} - \frac{\partial f}{\partial p_i^{(\alpha)}} \frac{\partial g}{\partial q_i^{(\alpha)}} \right)$
Apply this to assumption 2, in 1D
$\frac{df}{dt} = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}$
e also

$$\{f, g\} \equiv \sum_{\alpha = x,y,z} \sum_{i=1}^{\hat{}} \left(\frac{\partial f}{\partial q_i^{(\alpha)}} \frac{\partial g}{\partial p_i^{(\alpha)}} - \frac{\partial f}{\partial p_i^{(\alpha)}} \frac{\partial g}{\partial q_i^{(\alpha)}} \right)$$

Apply this to assumption 2, in 1D

$$\frac{df}{dt} = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}$$

We also can apply chain rule to get

$$\frac{df}{dt} = \frac{\partial f}{\partial q} \frac{dq}{dt} - \frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial t}$$

$$\{f, g\} \equiv \sum_{\alpha = x,y,z} \sum_{i=1} \left(\frac{\partial f}{\partial q_i^{(\alpha)}} \frac{\partial g}{\partial p_i^{(\alpha)}} - \frac{\partial f}{\partial p_i^{(\alpha)}} \frac{\partial g}{\partial q_i^{(\alpha)}} \right)$$

Equat

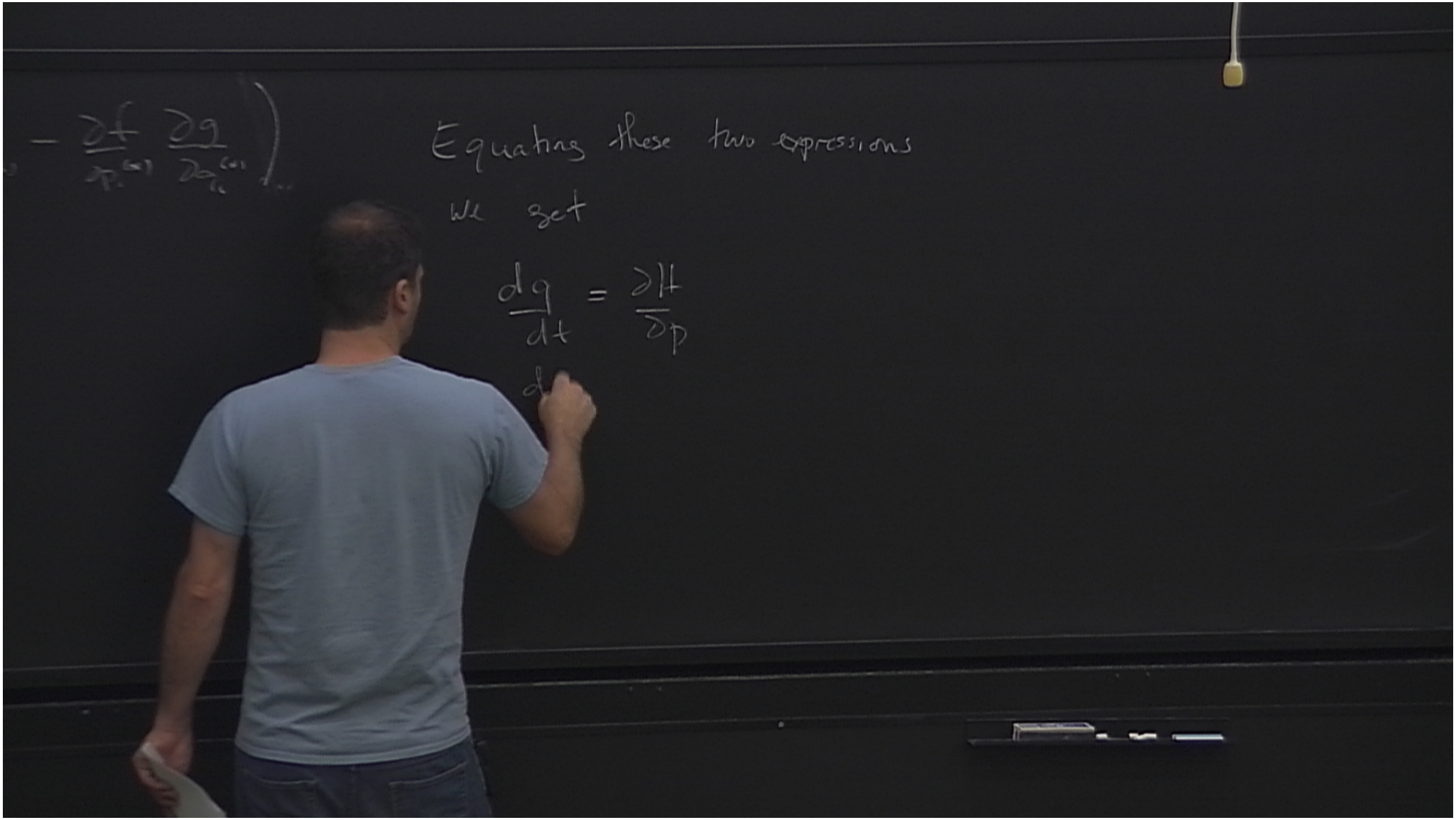
We

Apply this to assumption 2, in 1D

$$\frac{df}{dt} = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}$$

$f = f(q(t), p(t), t)$ We also can apply chain rule to get

$$\frac{df}{dt} = \frac{\partial f}{\partial q} \frac{dq}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial t}$$


$-\frac{\partial f}{\partial p_i^{(n)}}\frac{\partial g}{\partial q_i^{(n)}}\Big)$
Equating these two expressions
We get
$\frac{dq}{dt} = \frac{\partial H}{\partial p}$
$\dot{q}$

$- \frac{\partial f}{\partial p_i^{(a)}} \frac{\partial q}{\partial q_i^{(a)}} \Big)$

Equating these two expressions

We get

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

} Hamilton's eq^{n}s ⇒ Newton's Laws of Motion

$$-\frac{\partial f}{\partial p_i^{(a)}}\frac{\partial q}{\partial q_i^{(a)}}\Big)\Big)$$

Equating these two expressions

We get

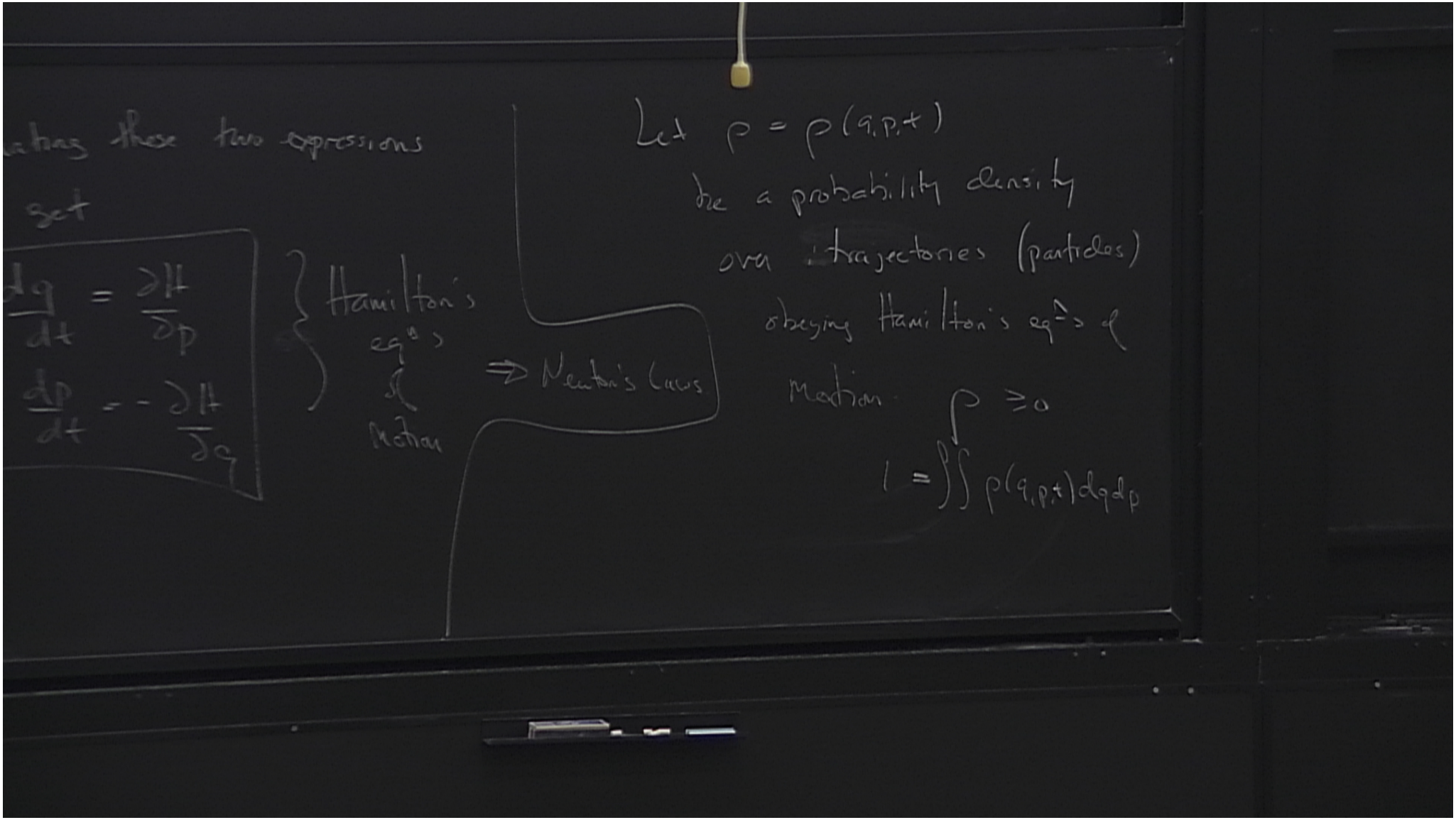
$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

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} Hamilton's eq^{n}s ⇒ Newton's Laws

↓

Motion


...ations these two expressions
get
$\frac{dq}{dt} = \frac{\partial H}{\partial p}$
$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$
Hamilton's eq^n's of Motion
⇒ Newton's Laws
Let $\rho = \rho(q,p,t)$
be a probability density over trajectories (particles) obeying Hamilton's eq^n's of Motion
$\rho \geq 0$
$1 = \int\int \rho(q,p,t)\,dq\,dp$

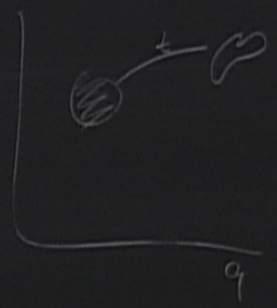
From assumption 2,

$$\frac{d\rho}{dt} = \{\rho, H\} + \frac{\partial \rho}{\partial t}$$

H = energy fn = Hamiltonian

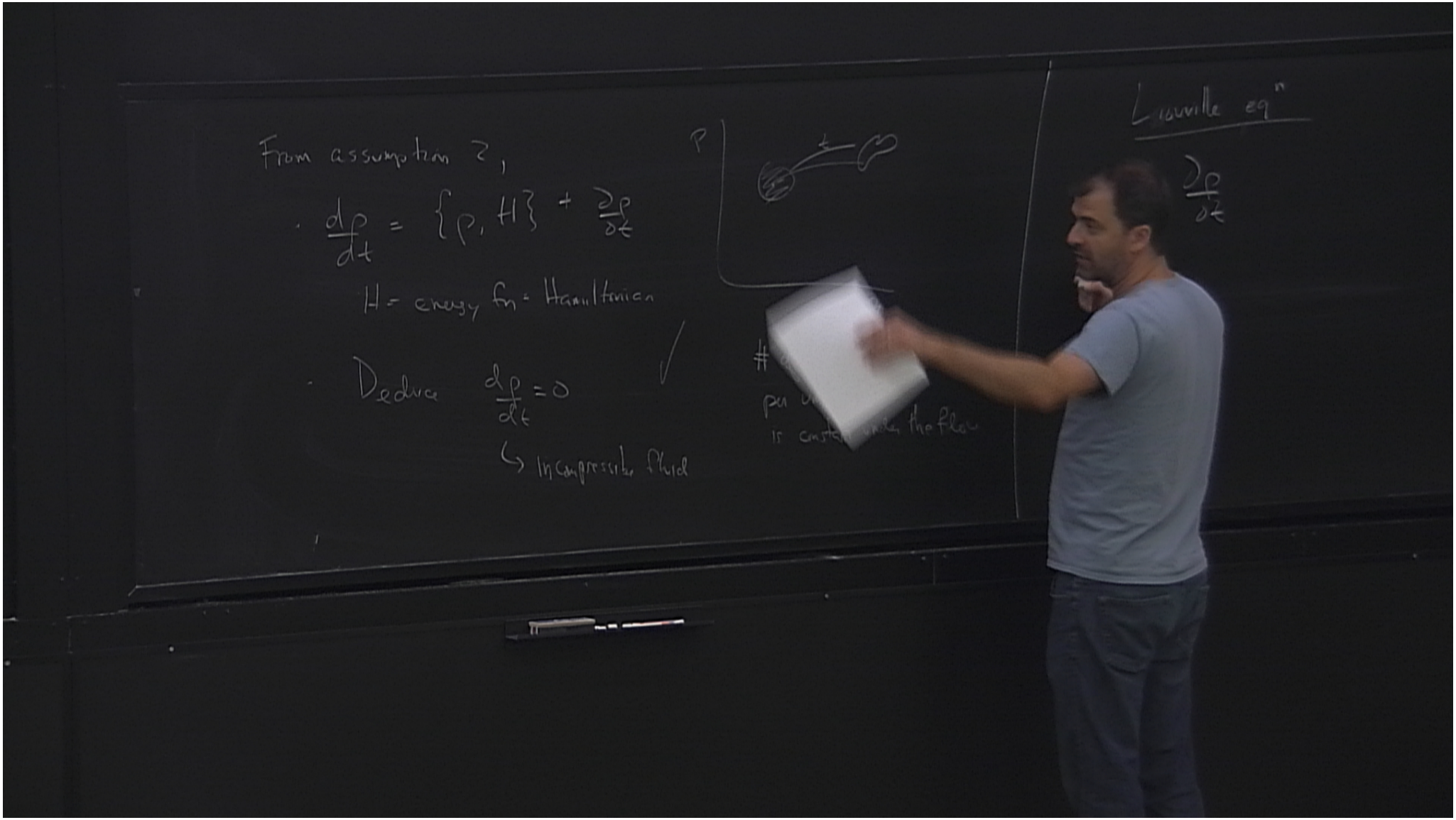
Deduce $\frac{d\rho}{dt} = 0$

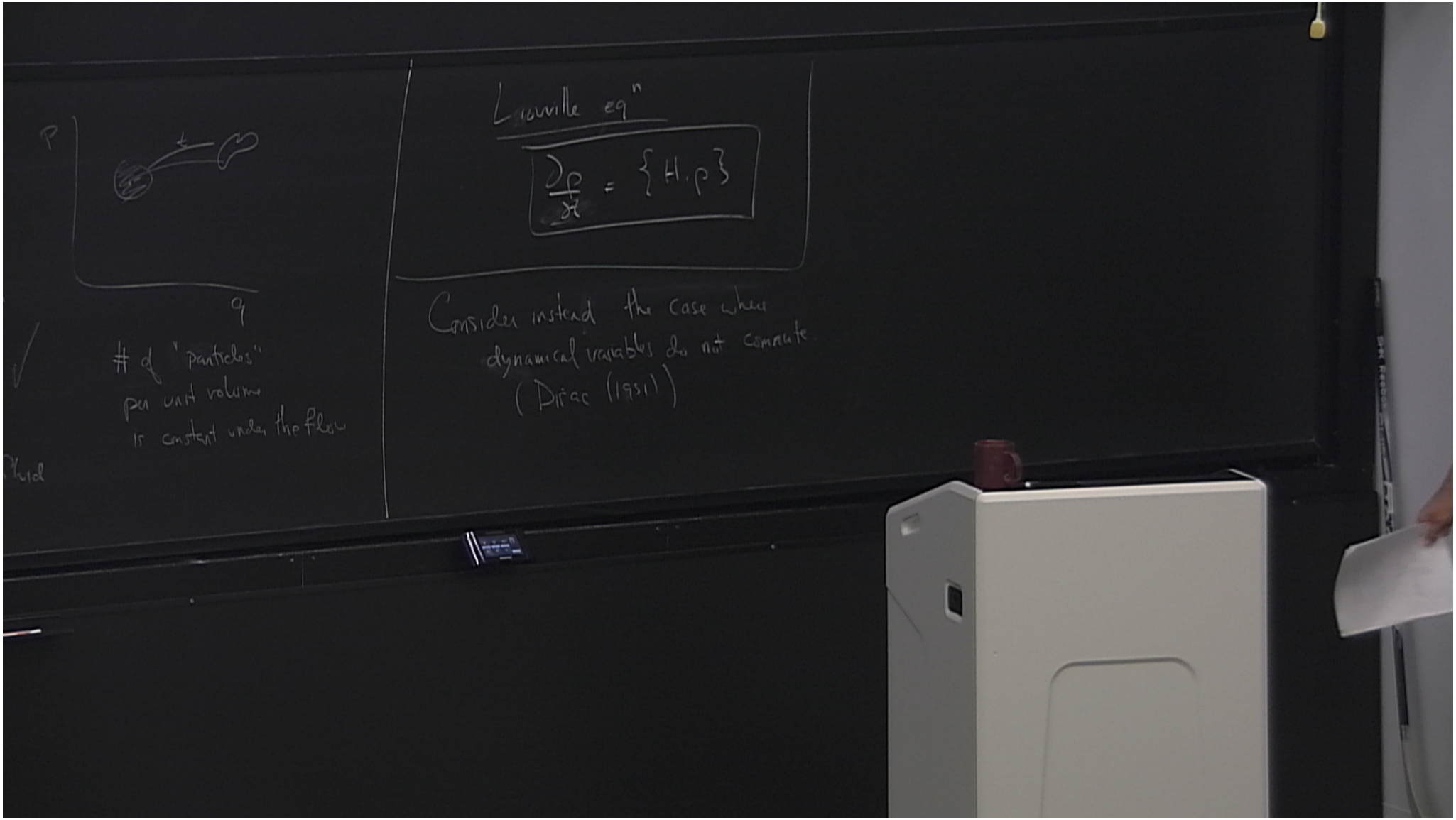
↳ Incompressible fluid

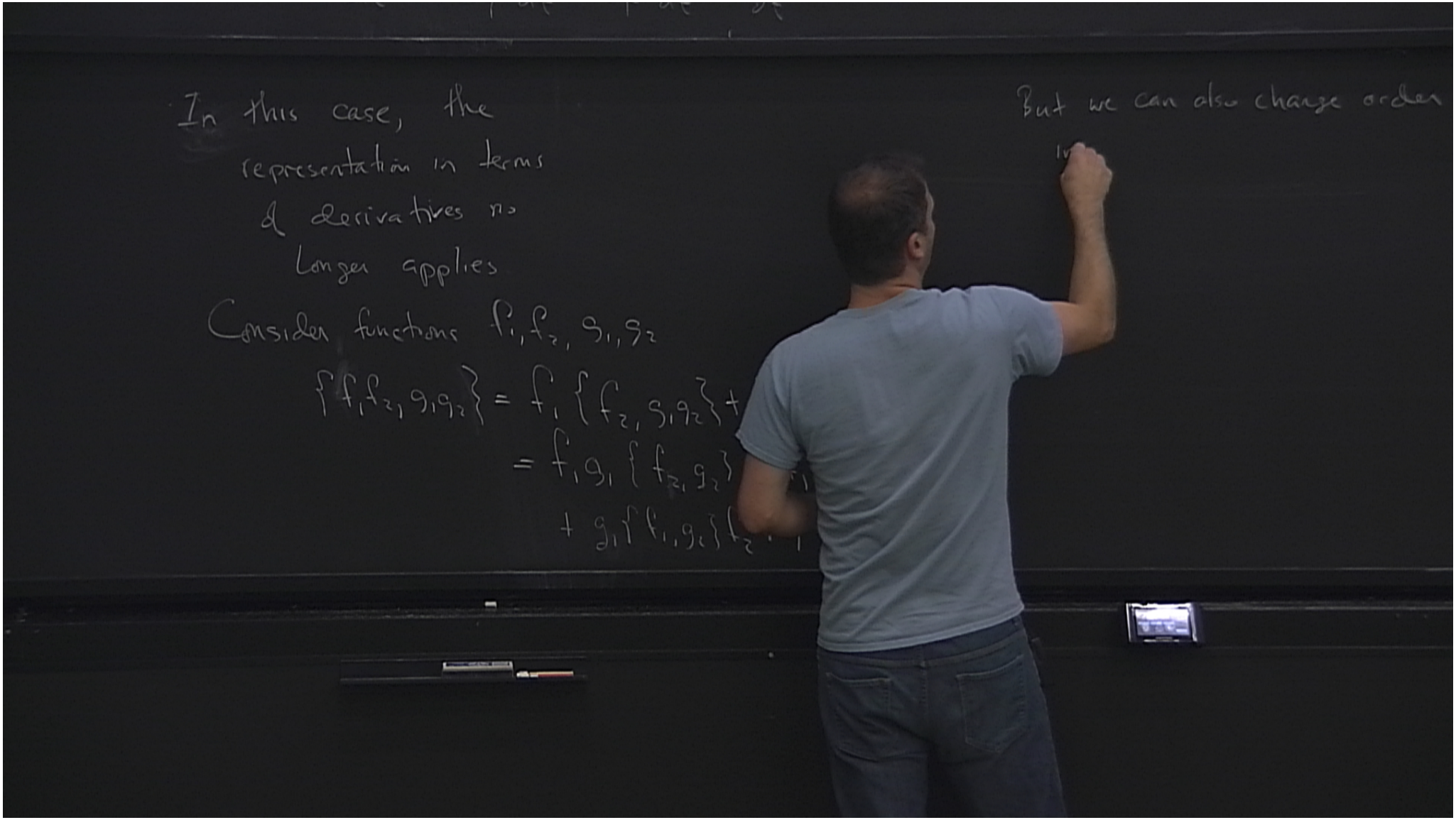


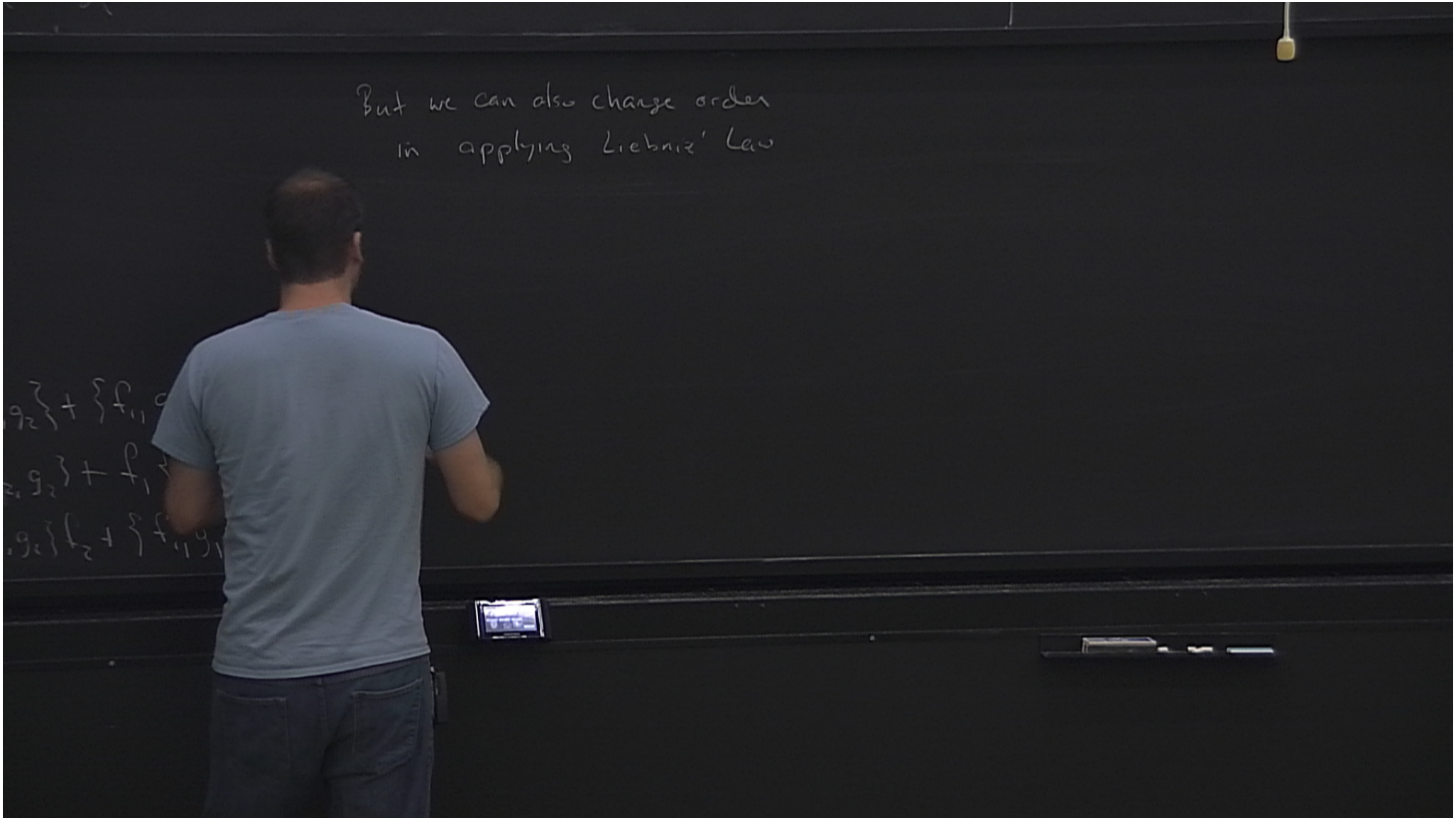
of "particles" per unit volume is constant under the flow

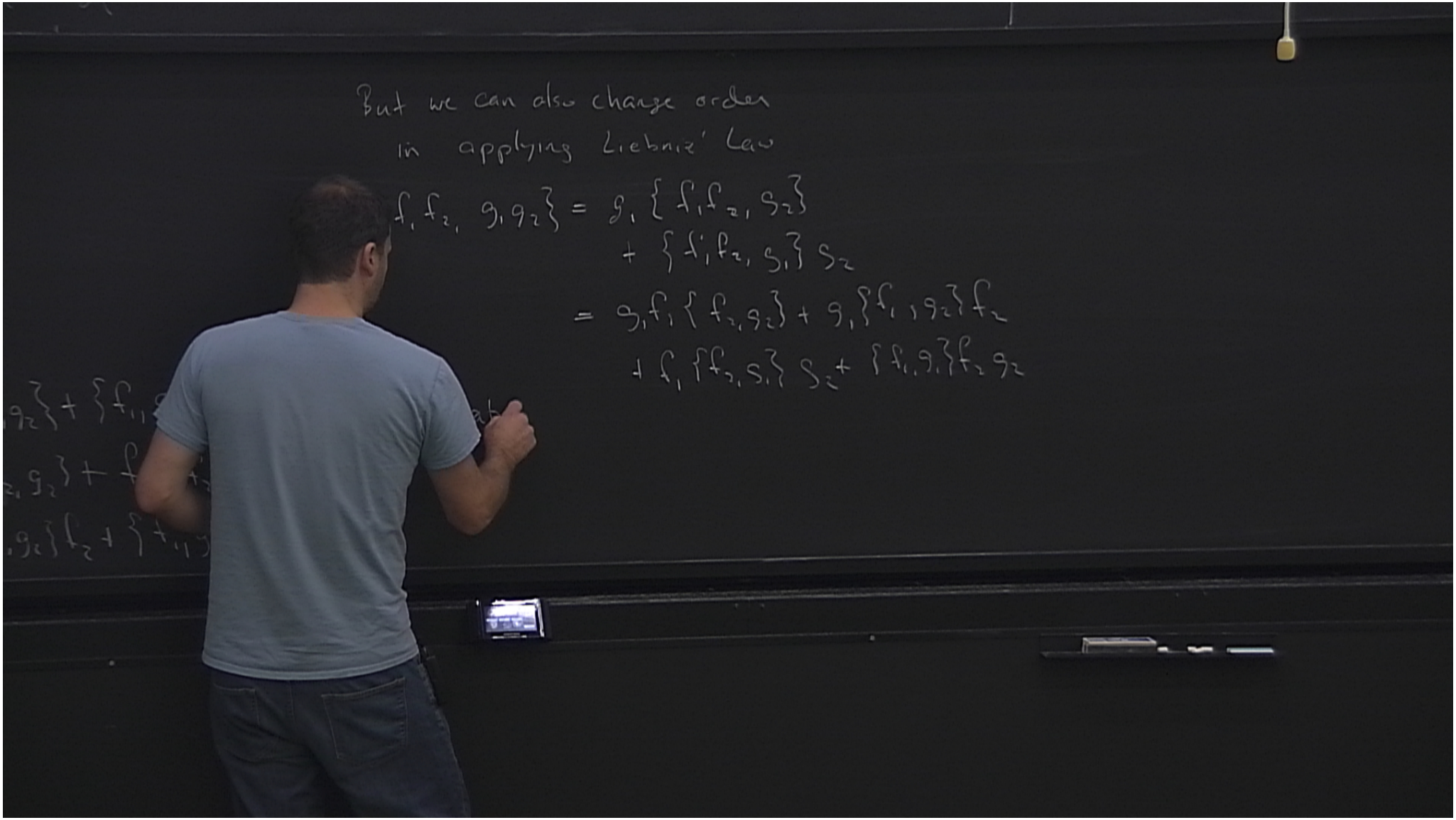
Liouville eq^{n}

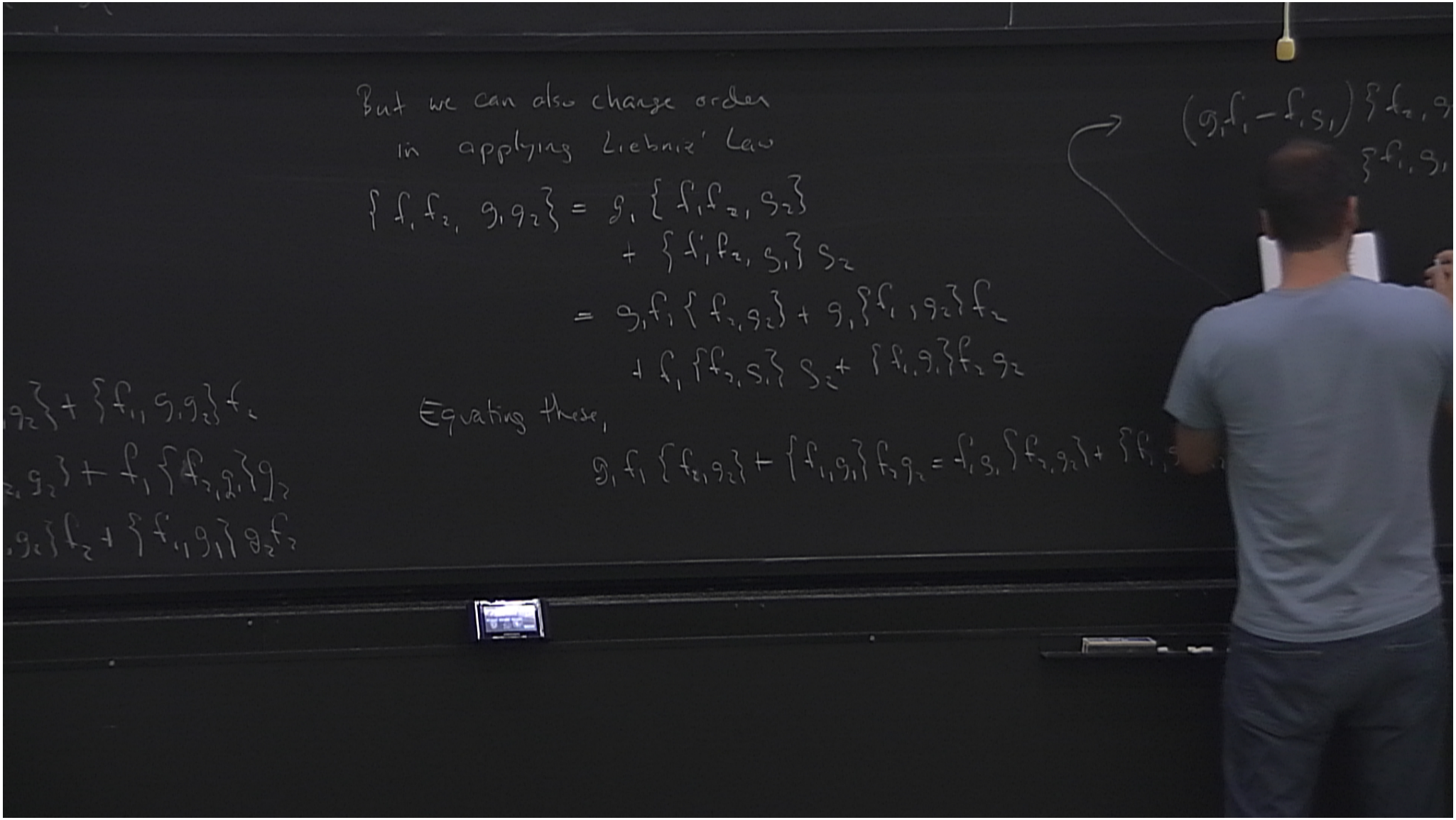

From assumption 2,
$\frac{dp}{dt} = \{p, H\} + \frac{\partial p}{\partial t}$
H = energy fn = Hamiltonian
Deduce $\frac{dp}{dt} = 0$
↳ Incompressible fluid
P
Liouville eqn
$\frac{\partial p}{\partial t}$

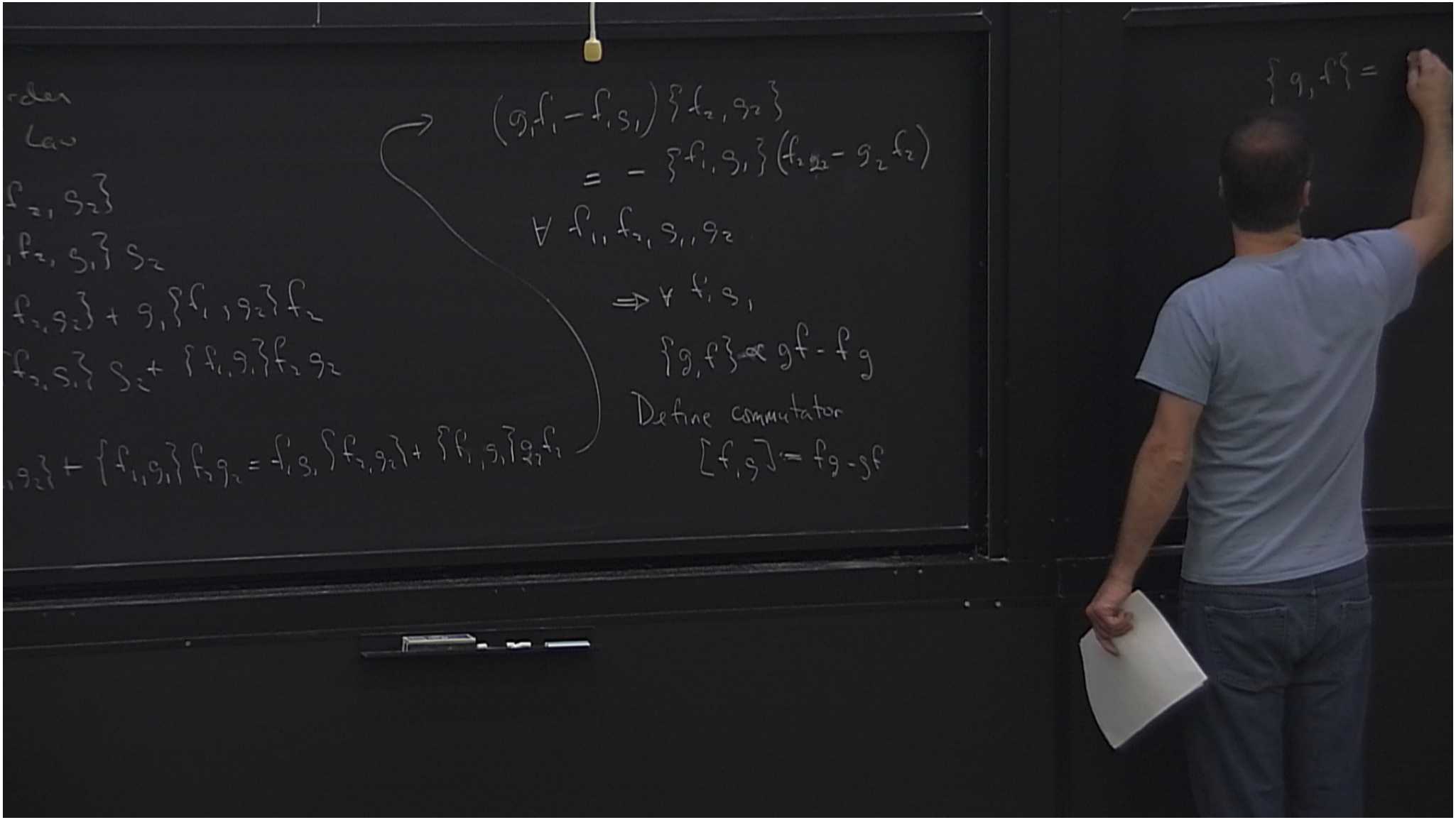

Liouville eq^n
$\frac{\partial \rho}{\partial t} = \{H, \rho\}$
Consider instead the case where dynamical variables do not commute.
(Dirac (1931))
of "particles" per unit volume is constant under the flow

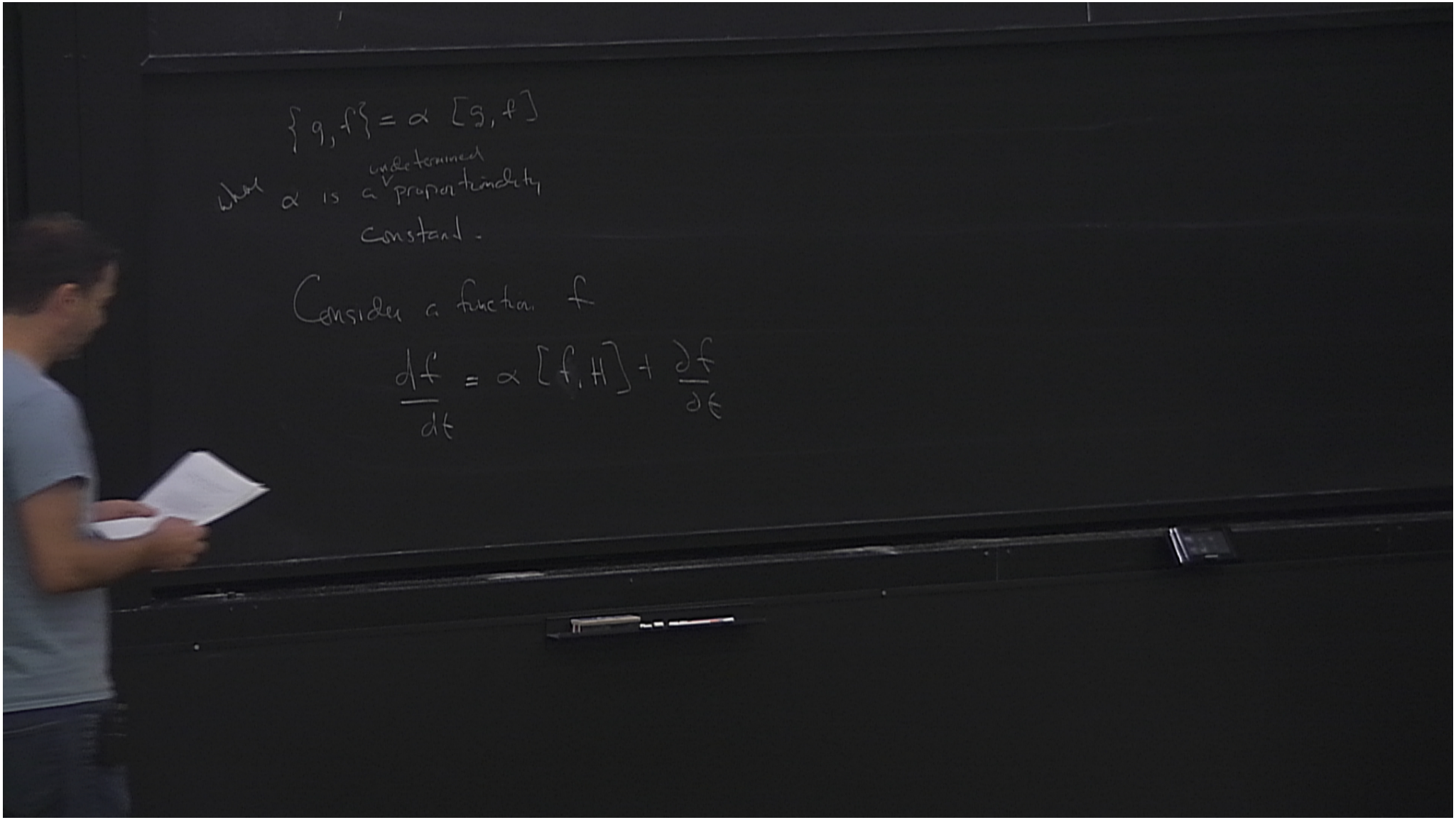

In this case, the representation in terms of derivatives no longer applies
But we can also change order
Consider functions f_1, f_2, g_1, g_2
$\{f_1 f_2, g_1 g_2\} = f_1 \{f_2, g_1 g_2\} +$
$= f_1 g_1 \{f_2, g_2\}$
$+ g_1 f \{f_1, g_1\} f_2$

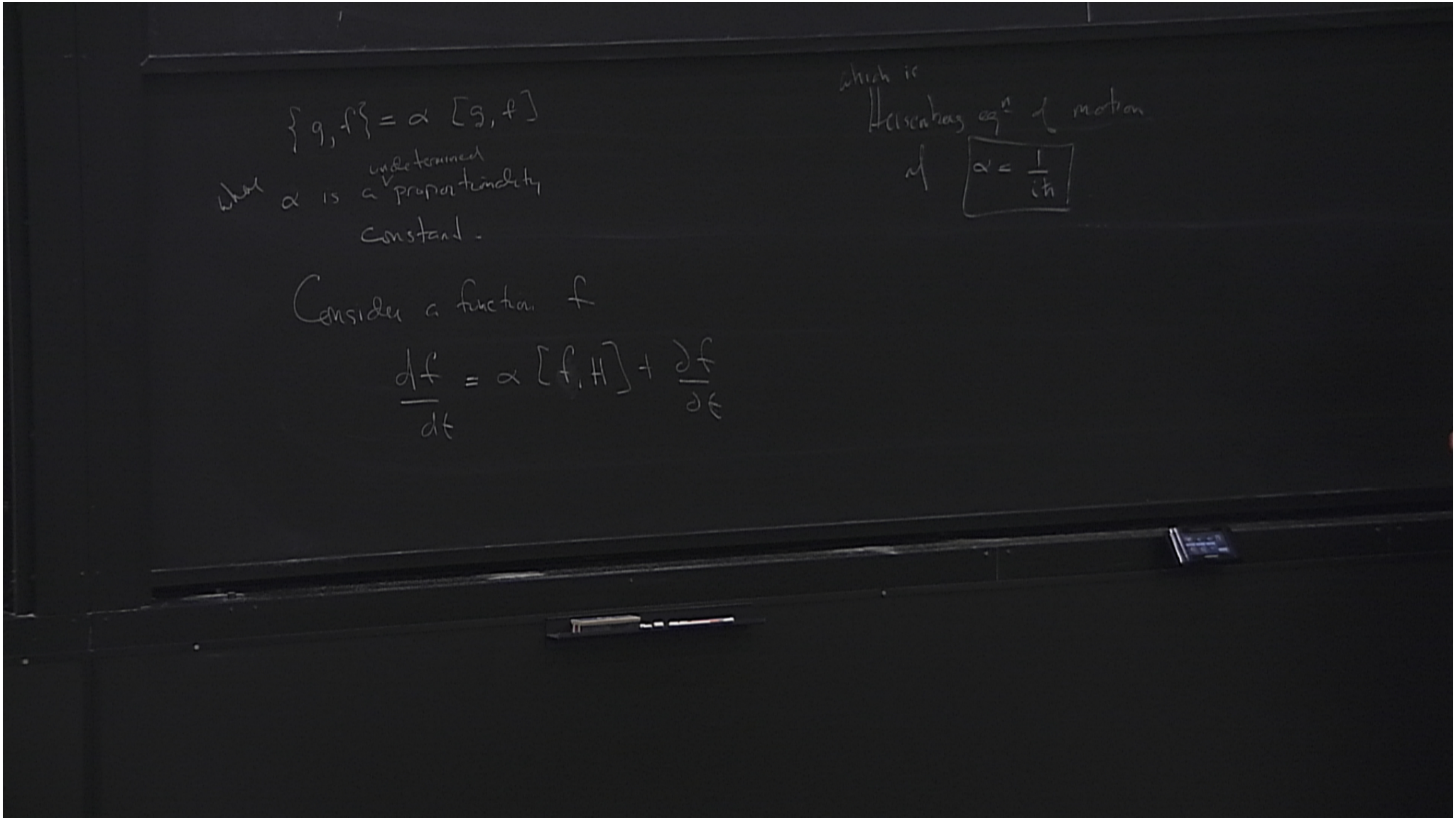

But we can also change order
in applying Leibniz' Law

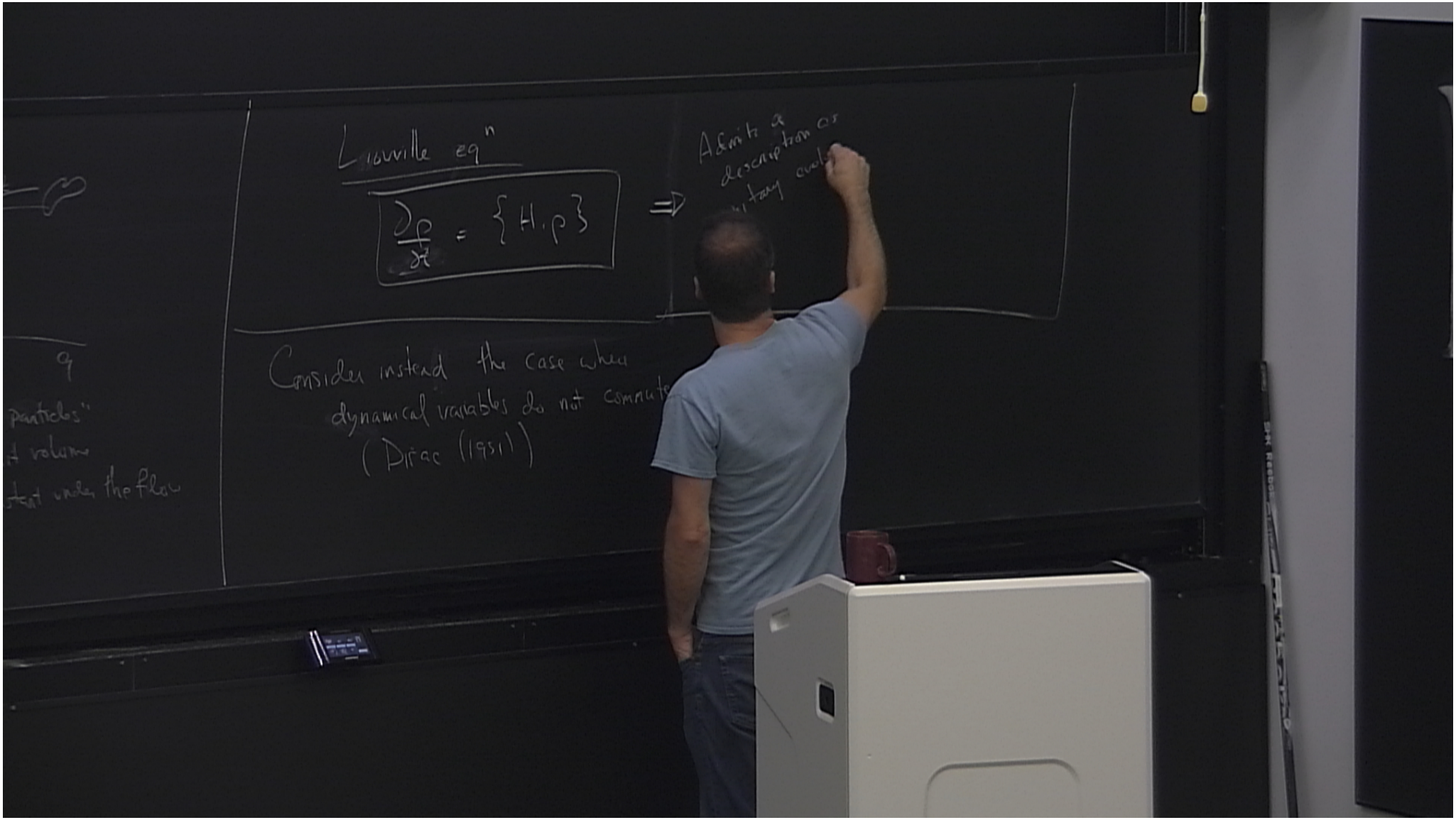

But we can also change order
in applying Leibniz' Law
$\{f_1 f_2, g_1 g_2\} = g_1 \{f_1 f_2, g_2\} + \{f_1 f_2, g_1\} g_2$
$= g_1 f_1 \{f_2, g_2\} + g_1 \{f_1, g_2\} f_2 + f_1 \{f_2, g_1\} g_2 + \{f_1, g_1\} f_2 g_2$

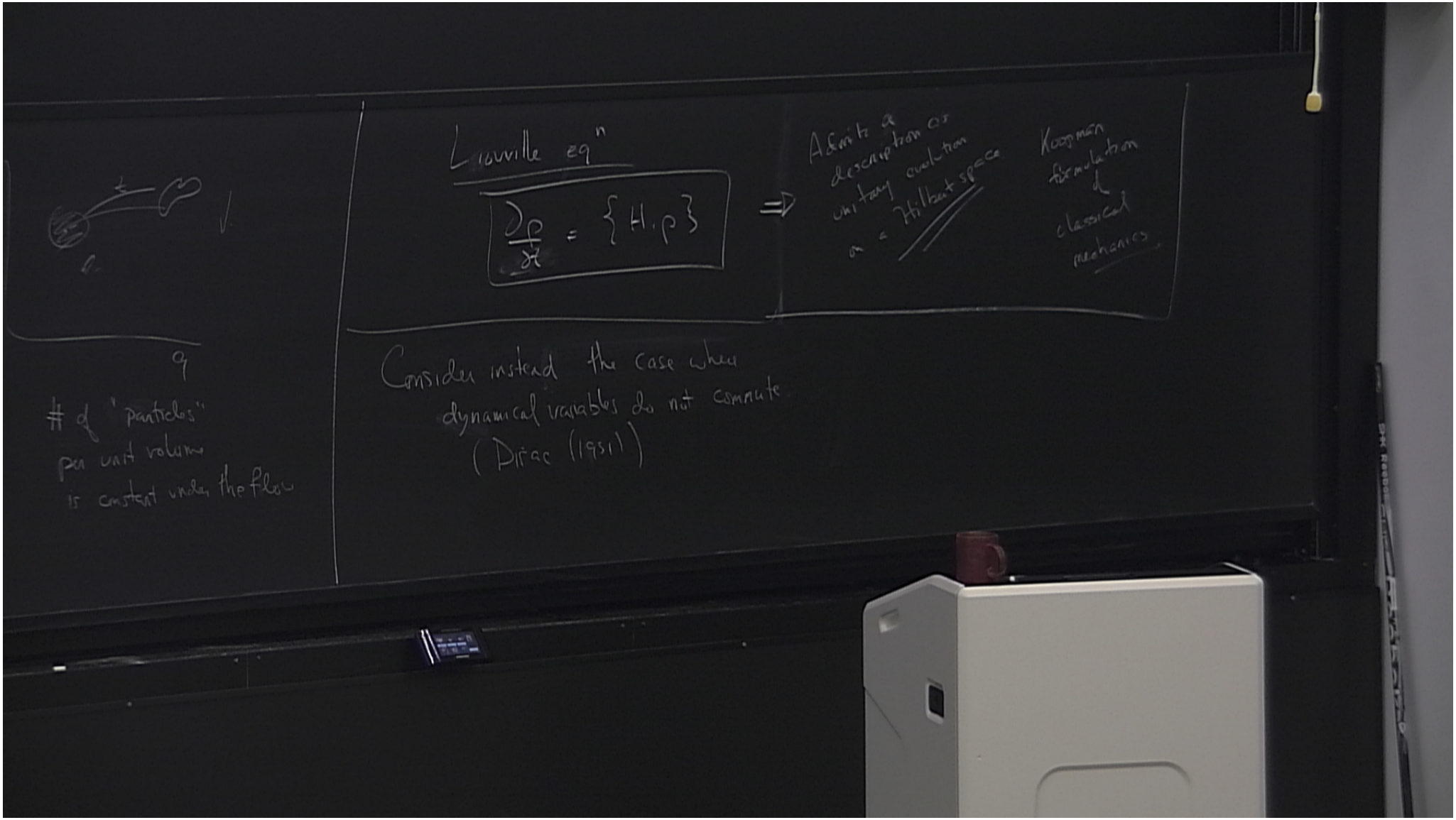

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$\{f_1 f_2, g_1 g_2\} = g_1 \{f_1 f_2, g_2\} + \{f_1 f_2, g_1\} g_2$
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Equating these,
$g_1 f_1 \{f_2, g_2\} + \{f_1, g_1\} f_2 g_2 = f_1 g_1 \{f_2, g_2\} + \{f_1$
$(g_1 f_1 - f_1 g_1) \{f_2, g$
$\{f_1, g_1$

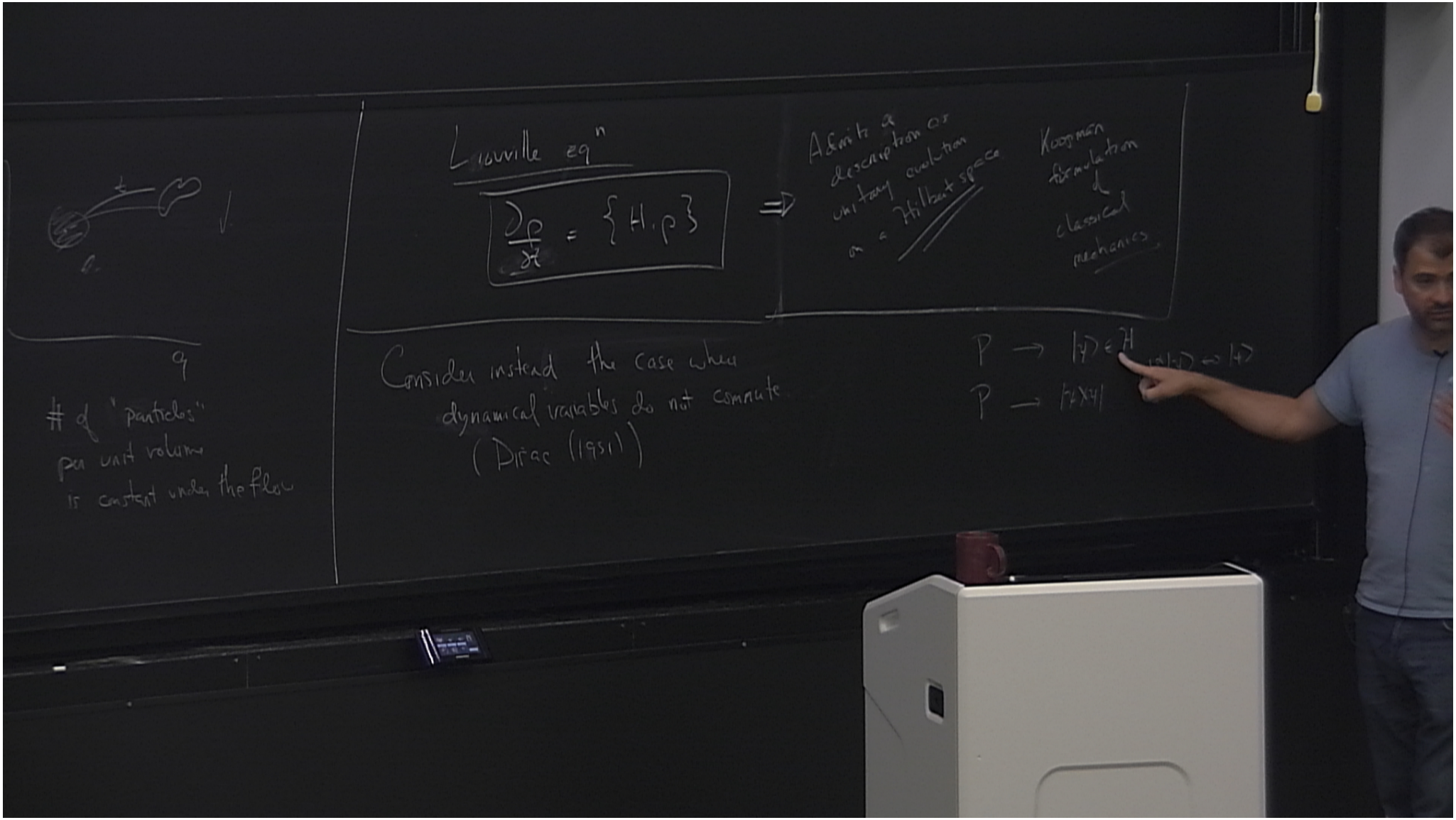

rder
Law
$\{f_2, g_2\}$
$, f_2, g_1\} g_2$
$f_2, g_2\} + g_1\{f_1, g_2\} f_2$
$f_2, g_1\} g_2 + \{f_1, g_1\} f_2 g_2$
$, g_2\} + \{f_1, g_1\} f_2 g_2 = f_1 g_1 \{f_2, g_2\} + \{f_1, g_1\} g_2 f_2$
$(g_1 f_1 - f_1 g_1)\{f_2, g_2\}$
$= -\{f_1, g_1\}(f_2 g_2 - g_2 f_2)$
$\forall f_1, f_2, g_1, g_2$
$\Rightarrow \forall f, g$
$\{g, f\} \propto gf - fg$
Define commutator
$[f, g] := fg - gf$
$\{g, f\} =$

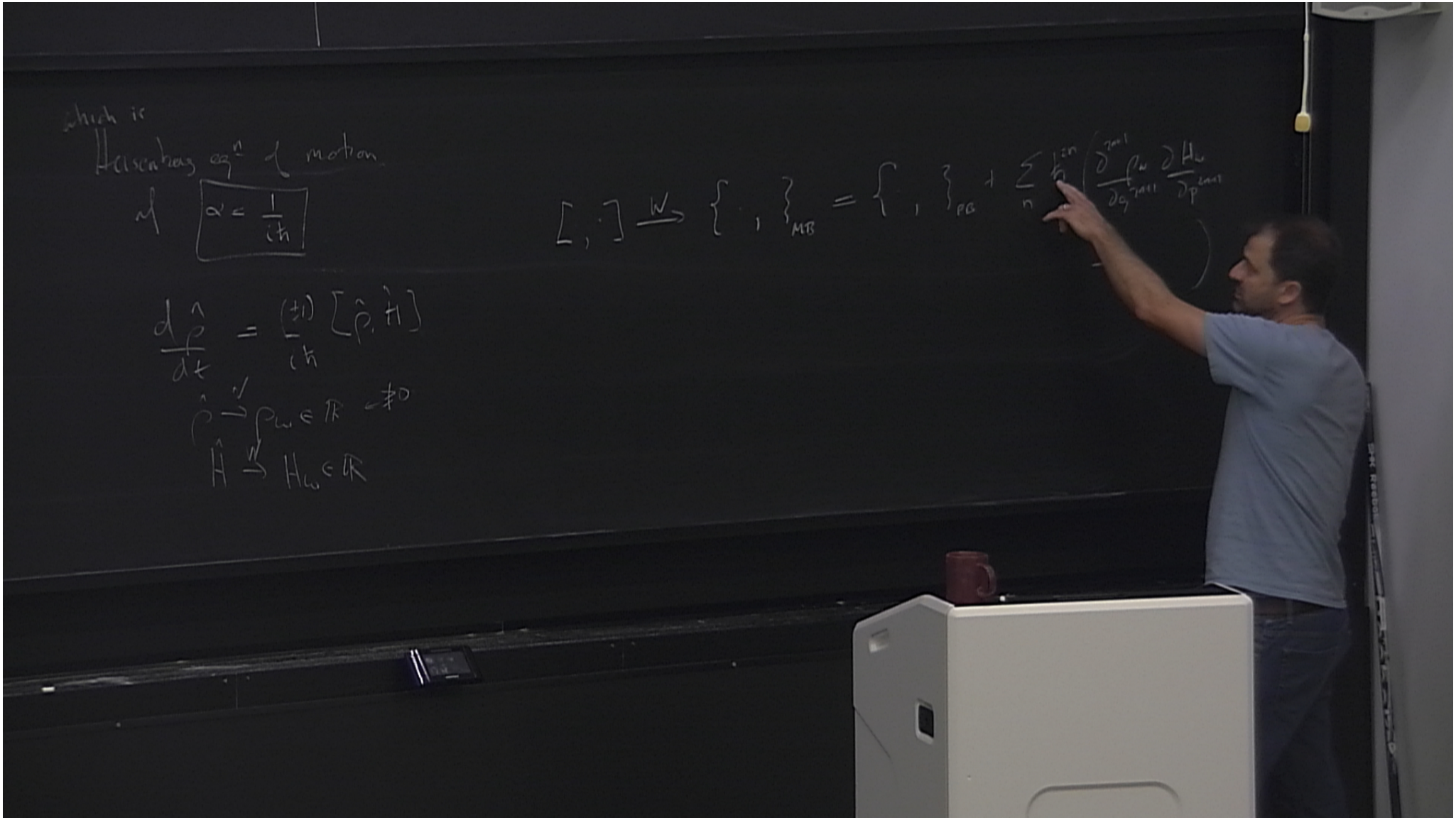

$\{g, f\} = \alpha [g, f]$
where α is a undetermined proportionality constant.
Consider a function f
$\frac{df}{dt} = \alpha [f, H] + \frac{\partial f}{\partial t}$


$\{g, f\} = \alpha [g, f]$
where α is a undetermined proportionality constant.
which is Heisenberg eqn of motion if $\alpha = \frac{1}{i\hbar}$
Consider a function f
$\frac{df}{dt} = \alpha [f, H] + \frac{\partial f}{\partial t}$


Liouville eq^n
$\frac{\partial \rho}{\partial t} = \{H, \rho\}$
Consider instead the case when
dynamical variables do not commute
(Dirac (1951))


Liouville eq^n
$\frac{\partial \rho}{\partial t} = \{H, \rho\}$
⇒
A finite d description of unitary evolution on a Hilbert space
Koopman formulation of classical mechanics
q
of "particles" per unit volume is constant under the flow
Consider instead the case when dynamical variables do not commute
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Liouville eq^n
∂ρ/∂t = {H, ρ}
⇒
A finite dimensional description of unitary evolution on a Hilbert space
Koopman formulation of classical mechanics
of "particles" per unit volume is constant under the flow
Consider instead the case when dynamical variables do not commute
(Dirac (1951))
P → |ψ⟩ ∈ ℋ
P → |x⟩⟨x|


which is
Heisenberg eq^n of motion
of
$\alpha = \frac{1}{ih}$
$[\,,\,] \xrightarrow{W} \{\,,\,\}_{MB} = \{\,,\,\}_{PB} + \sum_n \frac{i\hbar^n}{n!} \left(\frac{\partial^{n+1} \rho_W}{\partial q^{n+1} \partial p^{n+1}} \frac{\partial H_W}{\partial p^{n+1}} \right)$
$\frac{d\hat{\rho}}{dt} = \frac{(\pm 1)}{ih} [\hat{\rho}, \hat{H}]$
$\hat{\rho} \xrightarrow{W} \rho_W \in \mathbb{R} \neq 0$
$\hat{H} \xrightarrow{W} H_W \in \mathbb{R}$