

Title: PSI 2015/2016 Quantum Theory - Lecture 10

Date: Sep 21, 2015 10:45 AM

URL: <http://pirsa.org/15090051>

Abstract:

The theory of Measurement (continued)

- When does Luder's rule apply?
- How do we model state-update for general (POVM) measurements?

Is state-update (collapse of wavefunction)
actually a mysterious/unexplained
aspect of QM?

The theory of Measurement (continued)

- When does Luder's rule apply?
- How do we model state-update for general (POVM) measurements?
- Is state-update (collapse of wavefunction) actually a mysterious/unexplained aspect of QM?

Luder's Rule

Measurement of $\hat{A} = \sum_e a_e \hat{P}_e$

where P

Measurement

(red)

us' rule apply?

model state-update

(POVM) measurements?

(collapse of wavefunction)

a mysterious/unexplained
of QM?

Lüders' Rule

- Measurement of $\hat{A} = \sum_e a_e \hat{P}_e$
where $\hat{P}_e \hat{P}_u = \delta_{eu} \hat{P}_e$, $\hat{P}_e^\dagger = \hat{P}_e \forall e$

Input state $\hat{\rho}$

- We obtain outcome a_k with prob.
 $\Pr(a_k) = \text{Tr}(\hat{\rho} \hat{P}_k)$ Born rule

Give

$$\hat{A} = \sum_e a_e \hat{P}_e$$

$$\hat{P}_e^\dagger = \hat{P}_e \quad \forall e$$

a_k with prob.
 $(\hat{P}_k)$ Born rule

Given outcome a_k

$$\hat{\rho} \longrightarrow \frac{\hat{P}_k \hat{\rho} \hat{P}_k}{\text{Tr}(\hat{P}_k \hat{\rho})} =: \hat{\rho}_k$$

We call $\hat{\rho}_k$ the post-selected or conditional state.

Last class, if each $\hat{P}_k$ is rank-one

$$\hat{P}_k = |k\rangle\langle k|$$

called this maximally distinguishing.

the measurement is repeatable,

Luders' rule is required

or empirical consistency.

• Now suppose we
have some P_k with
rank > 1

Last class, if each $\hat{P}_k$ is rank-one

i.e. $\hat{P}_k = |k\rangle\langle k|$

we called this maximally distinguishing.

2. If the measurement is repeatable,

then Luder's rule is required
for empirical consistency.

• Now suppose we
have some P_k with
 $\text{rank} > 1$

• Then repeatability does not
guarantee Luder's rule.

We can have
10x

if each $\hat{P}_k$ is rank-one
 $= |k\rangle\langle k|$.

this maximally distinguishing.

Measurement is repeatable,

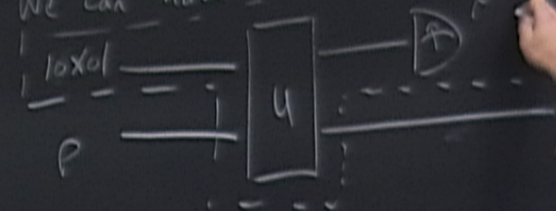
Lüder's rule is required

empirical consistency.

Now suppose we
 have some P_k with
 $\text{rank} > 1$

Then repeatability does not
 guarantee Lüder's rule.

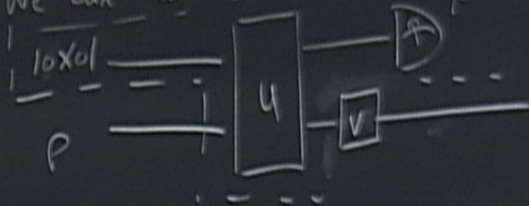
We can have



Now suppose we have some P_n with $\text{rank} > 1$

Then repeatability does not guarantee Luder's rule.

We can have



Let $\mathcal{H}_k \subset \mathcal{H}$ be the eigenspace of a_k ,
we can have any unitary $V \in \mathcal{U}(\mathcal{H}_k)$

still meet the repeatability condition

Repeatability allows

Generalized rule

$$P \rightarrow p_k' = V p_k V^\dagger$$

or, more generally

$$p_k' = \Lambda(p_k)$$

where Λ is a CP map on $\mathcal{L}(\mathcal{H}_k)$.

Repeatability allows

Generalized rule

$$P \rightarrow P'_k = V P_k V^\dagger$$

or, more generally

$$P'_k = \Lambda(P_k)$$

where Λ is a CP map on $\mathcal{L}(\mathcal{H}_k)$.

To recover Ludus' rule
in this case we need
to assume that the
measurement procedure

• Maximize the overlap between
input & output states
(restricted to $\mathcal{H}_A$).

• Let $\rho = |4 \times 4|$,

then we want

$$\langle 4 | \rho' | 4 \rangle = \text{Tr}(|4 \times 4| \rho')$$

to be maximized.

Hilbert-Schmidt
inner-product
↓

Last class

i.e.

we call

2. If the m

then

for

recover Luder's rule
 this case we need
 assume that the
 measurement procedure for P_k
 "minimally disturbing" to input
 state.

• Maximize the overlap between
 input & output states
 (restricted to $\mathcal{H}_k$).

• Let $\rho = |\psi\rangle\langle\psi|$,

then we want $\langle\psi|P_k|\psi\rangle = \text{Tr}(|\psi\rangle\langle\psi|P_k)$
 to be maximized, $\forall |\psi\rangle$

Hilbert-Schmidt
 inner-product
 ↓

we measure a POVM?
update-rule.

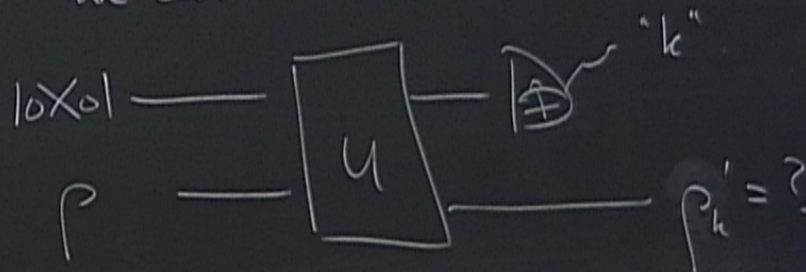
state $\hat{\rho}$

measure $\{E_k\} \Rightarrow E_k \geq 0$
s.t. $\sum_k \hat{E}_k = \mathbb{I}$

outcome "k" is obtained.

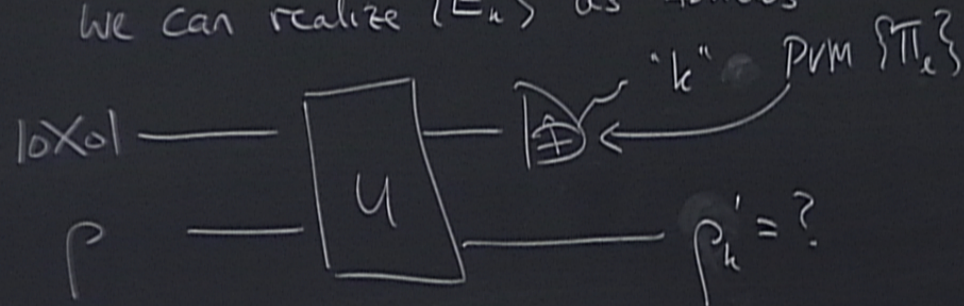
with $\Pr(k) = \text{Tr}(\hat{E}_k \rho)$

Appeal to Naimark's Theorem,
we can realize $\{E_k\}$ as follows



Appeal to Naimark's Theorem,

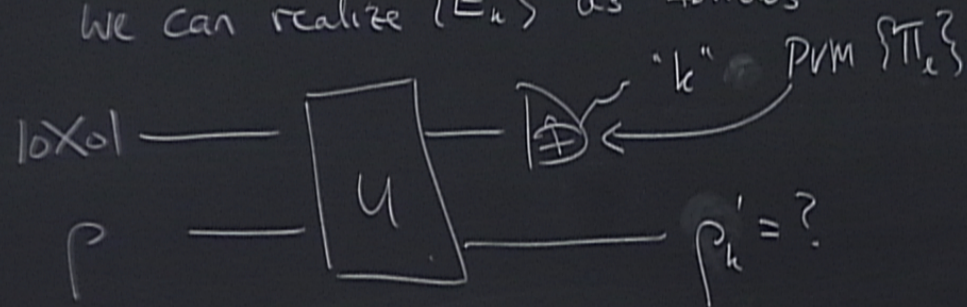
we can realize $\{E_k\}$ as follows



$$P_r(k) = \text{Tr} \left[\rho \otimes |0\rangle\langle 0| (U^\dagger I \otimes \Pi_k U) \right]$$

Appeal to Naimark's Theorem,

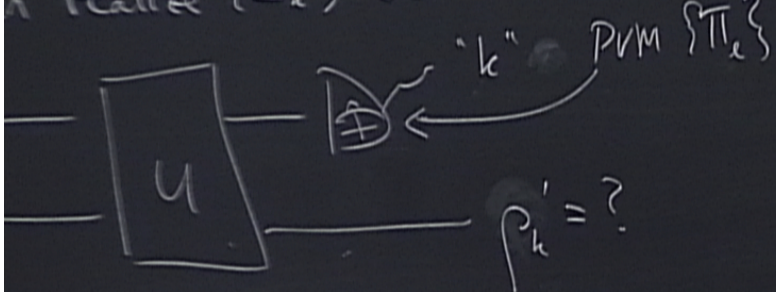
we can realize $\{E_k\}$ as follows



$$P_r(k) = \text{Tr} \left[\rho \otimes I \otimes \Pi_k (U^\dagger I \otimes \Pi_k U) \right]$$

to Naimark's Theorem,

realize $\{E_k\}$ as follows



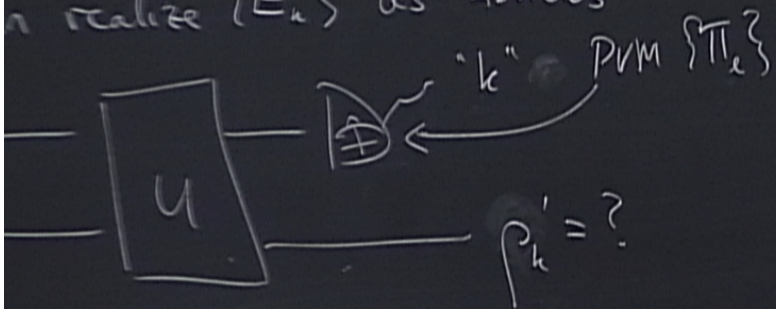
$$P_r(k) = \text{Tr} \left[\rho_{\infty} \otimes \chi_{\infty} (U^\dagger \underline{1 \otimes \Pi_k} U) \right] \quad P^2 = P$$

Apply Luders' rule (unjustifiably)

$$P \rightarrow P_k = \frac{\text{Tr}_B \left(1 \otimes \Pi_k U (\rho_{\infty} \otimes \chi_{\infty}) U^\dagger 1 \otimes \Pi_k \right)}{\text{Tr}(E_k \hat{P})} \leftarrow \text{normalization}$$

to Naimark's Theorem,

realize $\{E_k\}$ as follows



$$P(k) = \text{Tr} \left[\rho_{\infty} \otimes \chi_{\infty} (U^\dagger \underline{\mathbb{I} \otimes \Pi_k} U) \right] \quad P^2 = P$$

Apply Luders' rule (unjustifiably)

$$\hat{P} \rightarrow \hat{P}_k = \frac{\text{Tr}_B \left(\hat{\mathbb{I}} \otimes \hat{\Pi}_k \hat{U} (\hat{\rho}_{\infty} \otimes \chi_{\infty}) \hat{U}^\dagger \hat{\mathbb{I}} \otimes \hat{\Pi}_k \right)}{\text{Tr}(\hat{E}_k \hat{P})} \leftarrow \text{normalization}$$

where Λ is a CP map on $\mathcal{L}(\mathcal{H}_k)$.

Substituting Let's define $M_{kij} := \langle j |_B (1_A \otimes \pi_k) U_{AB} | i \rangle_B \in \mathcal{L}(\mathcal{H}_k)$

$$\text{Then } \hat{\rho}_k = \frac{\sum_j \hat{M}_{kij} \hat{\rho} \hat{M}_{kij}^\dagger}{\text{Tr}(E_k \rho)}$$

$$\text{where } E_k = \sum_j M_{kij}$$

$$U_{AB} |0\rangle_B \in \mathcal{L}(\mathcal{H}_A)$$

where $\hat{E}_k = \sum_j \hat{M}_{k,j}^\dagger \hat{M}_{k,j}$

Assume $\{\Pi_k\}$ consists of
rank-1 projectors,

then

$$\rho \rightarrow \rho_k = \frac{\hat{M}_k \rho \hat{M}_k^\dagger}{\text{Tr}(\hat{E}_k \rho)}$$

where $E_k = M_k^\dagger M_k$

state.

to be maximized, $\forall |k\rangle$

$$|k\rangle \in \mathcal{L}(\mathcal{H}_A)$$

where $\hat{E}_k = \sum_j \hat{M}_{k,j}^\dagger \hat{M}_{k,j}$

Assume $\{\Pi_k\}$ consists of
rank-1 projectors,

then
$$\rho \rightarrow \rho_k = \frac{\hat{M}_k \rho \hat{M}_k^\dagger}{\text{Tr}(\hat{E}_k \rho)}$$

where $\hat{E}_k = \hat{M}_k^\dagger \hat{M}_k$

Let then define U :

$$U |k\rangle \otimes |0\rangle$$

$$\rightarrow |k\rangle \otimes |k\rangle$$

$$\forall k \perp U |k\rangle \otimes |0\rangle$$

$$\xrightarrow{U'} \left(\sum_{e=1}^{\text{deg}(\hat{a}_k)} c_e |e\rangle \right) \otimes |k\rangle$$

How do we remove the "unjustifiably"
claim?

⇒ Allow for the unitary freedom "V"
in case of projective measurement.

$$\Rightarrow E_k = \sum_i M_{k,i}^\dagger M_{k,i}$$

Remark $\hat{M}_{k,i} \rightarrow \tilde{V} \hat{M}_{k,i}$

$\tilde{V}$ is unitary

E_k is invariant under
freedom V

remove the "unjustified"

for the unitary freedom "V"
 of projective measurement.

$$= \sum_j M_{k,j}^\dagger M_{k,j}$$

ack $\hat{M}_{k,j} \rightarrow V \hat{M}_{k,j}$, V is unitary on $\mathcal{H}_A$

E_k is invariant under
 freedom V

$$\begin{aligned} E_k &= \sum_j \hat{M}_{k,j}^\dagger M_{k,j} = \sum_j M_{k,j} V^\dagger V M_{k,j} \\ &= \sum_j M_{k,j}^\dagger M_{k,j} \end{aligned}$$

$P_r(k)$ is invariant
 under V

where Λ is a CP map on $\mathcal{L}(\mathcal{H}_k)$.

Modelling measurement
as a non-post-selected transfⁿ

PVM case

$$\rho \rightarrow \rho_k = \frac{\hat{\pi}_k \rho \hat{\pi}_k}{\text{Tr}(\hat{\pi}_k \rho)}$$

post-selected
on outcome "k"

no post-selection

$$\rho \rightarrow \rho' = \sum_k \frac{\text{Tr}(\rho \hat{\Pi}_k)}{\text{Tr}(\rho)} \rho_k$$

$$= \sum_k \frac{\text{Tr}(\rho \hat{\Pi}_k)}{\text{Tr}(\rho)} \hat{\Pi}_k \rho \hat{\Pi}_k$$

$\sum_k \hat{\Pi}_k \rho \hat{\Pi}_k \rightarrow$ in general
we have "decohered"
the input state.

Example

Let Each Π_k be rank-1.

$$\rho = |\psi\rangle\langle\psi|$$

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$$

$$\rightarrow \begin{bmatrix} |c_0|^2 & c_0 \bar{c}_1 & c_0 \bar{c}_2 \\ \bar{c}_0 c_1 & |c_1|^2 & c_1 \bar{c}_2 \\ \bar{c}_0 c_2 & \bar{c}_1 c_2 & |c_2|^2 \end{bmatrix}$$

$$\sum_k \Pi_k \rho \Pi_k \rightarrow \begin{bmatrix} |c_0|^2 & 0 & 0 \\ 0 & |c_1|^2 & 0 \\ 0 & 0 & |c_2|^2 \end{bmatrix}$$

no post-selection

$$\rho \rightarrow \rho' = \sum_k \text{Pr}(k) \rho_k$$

$$= \sum_k \frac{\text{Tr}(\hat{\rho} \hat{\pi}_k)}{\text{Tr}(\rho \hat{\pi}_k)} \pi_k \rho \pi_k$$

$$= \sum_k \hat{\pi}_k \hat{\rho} \hat{\pi}_k \rightarrow \text{in general we have "decohered" the input state.}$$

Example

Let Each π_k be rank-1.

$$\rho = |4\rangle\langle 4|$$

$$|4\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$$

$$\rightarrow \begin{bmatrix} |c_0|^2 & c_0 \bar{c}_1 & c_0 \bar{c}_2 \\ \bar{c}_0 c_1 & |c_1|^2 & c_1 \bar{c}_2 \\ & & |c_2|^2 \end{bmatrix}$$

$$\sum_k \pi_k \rho \pi_k \rightarrow \begin{bmatrix} |c_0|^2 & 0 & 0 \\ 0 & |c_1|^2 & 0 \\ 0 & 0 & |c_2|^2 \end{bmatrix}$$

where Λ is a CP map on $\mathcal{L}(\mathcal{H}_k)$.

Modelling measurement
as a non-post-selected transfⁿ

w/o post-select
 $\rho \rightarrow \rho'$

PVM case

$$\rho \rightarrow \rho_k = \frac{\hat{\pi}_k \rho \hat{\pi}_k}{\text{Tr}(\hat{\pi}_k \rho)}$$

post-selected
on outcome "k"

POVM case

$$\rho \rightarrow \rho' = \sum_k P_r(k) \rho_k$$

$$= \sum_{k,j} M_{k,j} \rho M_{k,j}^\dagger$$

← w/o
post-
selection

$$\sum_k E_k = \mathbb{1} = \sum_k \underbrace{\sum_j M_{k,j}^\dagger M_{k,j}}_{\text{Kraus operator decomposition}} \Rightarrow \text{Kraus operator decomposition} \Rightarrow \text{CPTP}$$

Measurement problem

$$|y\rangle = \sum_e c_e |e\rangle \xrightarrow{\text{"k"}} |k\rangle$$

This conflicts with unitary evolⁿ

- (i) discontinuous
- (ii) non-deterministic
- (iii) non-linear

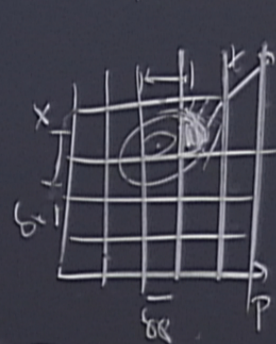
Classically

$$\rho(x, p, t)$$



$$\rho \cap \Delta_i$$

$$\rho \rightarrow \rho_i$$



Measurement problem

$$|\psi\rangle = \sum_e c_e |e\rangle$$

This conflicts with

- (i) discontinuous
- (ii) non-deterministic
- (iii) non-linear