

Title: PSI 2015/2016 Quantum Theory - Lecture 8

Date: Sep 17, 2015 10:45 AM

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Abstract:

## Realistic Transformations

Last class Two kinds of  
generalized transformations

(i) mixtures of unitaries

$$\hat{\rho}' = \sum_x p_x U_x \hat{\rho} U_x^\dagger$$

$$\hat{\rho}, U \in \mathcal{L}(\mathcal{H})$$

$$0 \leq p_x \leq 1, \sum_x p_x = 1$$

(ii) reduced action of  $U \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$

$$\rho_{AB}' = U \rho_A \otimes \rho_B U^\dagger$$

In both cases we found

$$\rho_A' = \Lambda(\rho_A) = \sum_x A_x \rho_A A_x^\dagger$$

$$\text{where } A_x \in \mathcal{L}(\mathcal{H}_A)$$

$$\text{s.t. } \sum_x A_x^\dagger A_x = \mathbb{1}_A$$



$$(\in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B))$$

$$A \otimes P_B U^\dagger$$

2

$$\sum_x A_x \rho A_x^\dagger$$

$$(\mathcal{H}_A)$$

$$\mathbb{1}_A$$

• In case (ii) keep in mind

We assumed  $\rho_{AB} = \rho_A \otimes \rho_B$   
(uncorrelated input state)

• Given some superoperator  $\Lambda: \hat{\rho} \rightarrow \hat{\rho}$   
the  $\{A_x\}$  satisfying  $\Lambda(\rho) = \sum_x A_x \rho A_x^\dagger$   
is called an operator-sum decomposition.



Is the map  $\rho \rightarrow \rho' = \sum_e A_e \rho A_e^\dagger$

the most general possible  
transformation  $\rho \rightarrow \rho'$ ?

Assuming  $\rho$  is not correlated  
with anything else.

& if we demand  $\text{tr}(\rho') = \text{tr}(\rho)$   
 $\Rightarrow \sum_e A_e^\dagger A_e = \mathbb{1}$



## Complete positivity

Consider a linear map  $\Lambda: \rho \rightarrow \rho'$

Def<sup>n</sup> If  $\forall$  input  $\rho$  that is non-negative  
the output  $\rho'$  is also non-negative.  
then  $\Lambda$  is a positive map



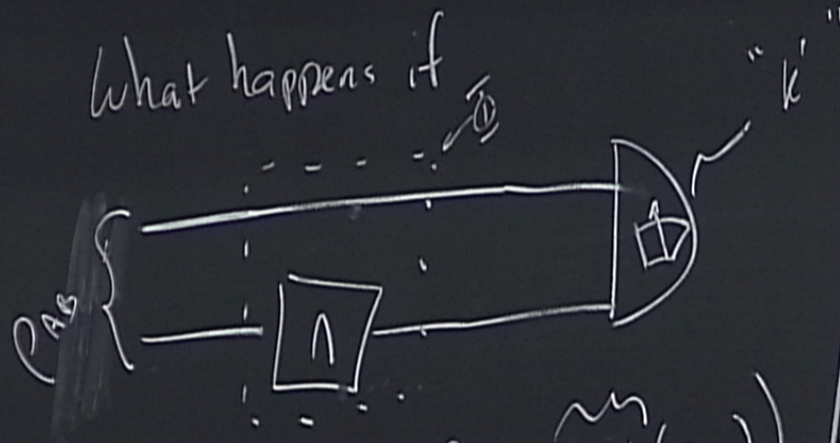
$$\Lambda: \rho \rightarrow \rho'$$

is non-negative

also non-negative

positive map

What happens if



$$P_k(k) = \text{Tr} \left( E_k \underbrace{\Lambda \otimes \mathbb{I}}_{\Phi} (\rho_{AB}) \right)$$

$$E_k \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

$$\rho_{AB}' = \underbrace{\Lambda \otimes \mathbb{I}}_{\Phi} (\rho_{AB})$$



Def<sup>n</sup>: A map  $\Lambda: \mathcal{P}_A \rightarrow \mathcal{P}_{A'}$   
is called completely-positive  
if  $\Lambda \otimes I_B$  is a positive map.

Def<sup>n</sup> A completely-positive (CP) map  
is also trace-preserving (CPTP)  
if  $\text{Tr}(\rho_{AB}') = \text{Tr}(\rho_{AB})$



Kraus' Theorem: Any linear map  $\Lambda$  is a CPTP map  
if and only if it admits an  
operator-sum decomposition (that  
preserves trace).



Kraus' Theorem: Any linear map  $\Lambda$  is a CPTP map  
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$\Rightarrow \{A_k\}$  are called Kraus operators



$\Lambda$  is a CPTP map  
it admits an  
decomposition (that  
trace)

operator

### Stinespring dilation theorem

Any CPTP  $\Lambda: \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A)$   
can be realized via a  
unitary transf<sup>n</sup>  $U \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$   
via

$$\Lambda(\rho) = \text{Tr}_B(U \rho \otimes \hat{\sigma} U^\dagger)$$

where  $\hat{\sigma}$  is a density operator  
for system B



W.l.o.g. we can take

$$\sigma \rightarrow |0\rangle_{\text{A}}|0\rangle_{\text{B}}$$

(more of a theoretical  
consideration)



W.l.o.g. we can take

$$\sigma \rightarrow 10 \times 10^3$$

(more of a theoretical  
consideration)

Remark if  $\rho \otimes \sigma \rightarrow \sum_x p_x \hat{\rho}_x \otimes \hat{\sigma}_x$

$$\text{s.t. } \rho_A = \text{Tr}_B \left( \sum_x p_x \hat{\rho}_x \otimes \hat{\sigma}_x \right) \\ \equiv \rho$$



W.l.o.g. we can take

$$\sigma \rightarrow I_A \otimes \sigma|_B$$

(more of a theoretical consideration)

- Remark if  $\rho \otimes \sigma \rightarrow \sum_x p_x \hat{\rho}_x \otimes \hat{\sigma}_x$

$$\text{s.t. } \rho_A = \text{Tr}_B \left( \sum_x p_x \hat{\rho}_x \otimes \hat{\sigma}_x \right) \\ \equiv \rho$$

$$\text{the } \Lambda(\rho) = \text{Tr}_B \left( U \left( \sum_x p_x \hat{\rho}_x \otimes \sigma_x \right) U^\dagger \right)$$

may not be completely positive

& moreover it may not even be a linear map.



Examples: Depolarizing channel

$$\Lambda_{\text{dep}}(\hat{\rho}) = p \frac{\mathbb{1}}{D} + (1-p)\hat{\rho}$$

$p$  is a probability

Kraus' operators?

$$\text{Tr}(\hat{\rho}) \frac{\mathbb{1}}{D} = \frac{1}{D^2} \sum_{a,b=1}^D P_{(a,b)} \hat{\rho} P_{(a,b)}^\dagger$$



$$P_{AB} = P_{AB}$$

$$a, b \in \{1, \dots, D\}$$

Heisenberg-Weyl operators

$$\hat{P}_{(a,b)} = \hat{X}^a \hat{Z}^b$$

$$\mathcal{H} = \mathbb{C}^D, \quad j \in \{0, \dots, D-1\}$$

$$\hat{X} |j\rangle = |j+1 \pmod{D}\rangle$$

$$\hat{Z} |j\rangle = \exp\left(i \frac{2\pi j}{D}\right) |j\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{array}{l} |0\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |1\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X |0\rangle = |1\rangle$$

$$X |1\rangle = |0\rangle$$

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$$\begin{aligned} &\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Substituting

$$\begin{aligned} \Lambda_{dep}(p) = & \left(1 - p + \frac{p}{D^2}\right) p \\ & + \frac{p}{D^2} \sum_{a,b=2} \end{aligned}$$



Examples: Depolarizing channel

$$\Lambda_{\text{dep}}(\hat{\rho}) = p \frac{\mathbb{1}}{D} + (1-p)\hat{\rho}$$

$p$  is a probability

Kraus' operators?

$$\text{Tr}(\hat{\rho}) \frac{\mathbb{1}}{D} = \frac{1}{D^2} \sum_{a,b=0}^{D-1} P_{(a,b)} \hat{\rho} P_{(a,b)}^\dagger$$



$$\begin{aligned} |0\rangle &\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |1\rangle &\rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Substituting

$$\hat{P}_{(0,0)} = \frac{\hat{1}}{D}$$

$$\Lambda_{\text{dep}}(\rho) = \left(1 - p + \frac{p}{D^2}\right) \hat{1} \hat{\rho} \hat{1}$$

Check that  
 $\sum_{a,b} \hat{A}_{(a,b)}^\dagger \hat{A}_{(a,b)} = \hat{1}$

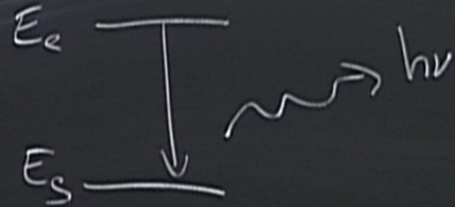
$$+ \frac{p}{D^2} \sum_{(a,b) \neq (0,0)} \hat{P}_{(a,b)} \hat{\rho} \hat{P}_{(a,b)}$$

$$\hat{A}_{(a,b)} = \frac{\sqrt{p}}{D} \hat{P}_{(a,b)} \quad \text{for } (a,b) \neq (0,0)$$

$$\hat{A}_{(0,0)} = \sqrt{1 - p + \frac{p}{D^2}} \hat{P}_{(0,0)}$$



Spontaneous emission



$$H = \omega (|e\rangle\langle e| - |g\rangle\langle g|)$$

$$= \omega (\hat{\sigma}_z)$$

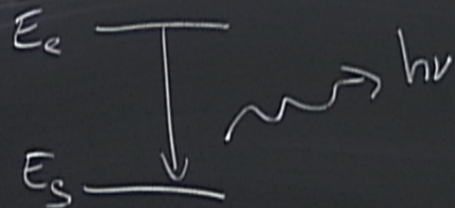
$$\rho(0) = |e\rangle\langle e|$$

$$U = \exp\left(-i\frac{Ht}{\hbar}\right)$$

$$U |e\rangle\langle e| U^\dagger = |e\rangle\langle e|$$



Spontaneous emission



$$H = \omega (|e\rangle\langle e| - |g\rangle\langle g|)$$

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$$U |e\rangle\langle e| U^\dagger = |e\rangle\langle e|$$

$$H = H_{\text{atom}} \otimes H_{\text{EM}}$$

$$H \rightarrow H + H_{\text{int}}$$

To describe dynamics of atom  
we trace out  $H_{\text{EM}}$



$$\rho_{AB}(0) \in \mathcal{L}(\mathcal{H}_{\text{system}} \otimes \mathcal{H}_{\text{env}})$$

$$U \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

We're interested in

$$\rho_A(t) = \text{Tr}_B \left( U(t) \rho_{AB}(0) U^\dagger(t) \right)$$

$$\rho_{AB}(0)$$



$$\rho_{AB}(0) \in \mathcal{L}(\mathcal{H}_{\text{system}} \otimes \mathcal{H}_{\text{env}})$$

$$U \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

We're interested in

$$\rho_A(t) = \text{Tr}_B \left( U(t) \rho_{AB}(0) U^\dagger(t) \right) = \mathcal{P}(\rho_{AB}(0))$$

$$\rho_{AB}(0) = \rho_A(0) \otimes \rho_B(0)$$

is a CPTP map



$$|p_{AB}\rangle = |1\rangle |p_{AB}\rangle$$

Example. Amplitude damping channel  
(simple model of spontaneous emission)

$$\mathcal{H} = \mathbb{C}^2$$

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$$\rho' = \Lambda(\rho)$$

$$\Lambda(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

$$\hat{\rho} \Rightarrow \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

$$\begin{aligned} \rho_{00} + \rho_{11} &= 1 \\ \rho_{01} &= \sqrt{\gamma} \rho_{10} \end{aligned}$$



$$\rho' \rightarrow \begin{pmatrix} \rho_{00} + \gamma \rho_{11} & \sqrt{1-\gamma} \rho_{01} \\ \sqrt{1-\gamma} \rho_{10} & (1-\gamma) \rho_{11} \end{pmatrix}$$

$$\rho_{11} = 1$$



$$\rho' \rightarrow \begin{pmatrix} \rho_{00} + \gamma \rho_{11} & \sqrt{1-\gamma} \rho_{01} \\ \sqrt{1-\gamma} \rho_{10} & (1-\gamma) \rho_{11} \end{pmatrix}$$

$$0 \leq \gamma \leq 1$$

$$\text{If } \rho(x) = |e^{\chi} e| \leftrightarrow |x|$$

$$\begin{aligned} \rho_{00} &= 0 \\ \rho_{11} &= 1 \end{aligned}$$

$$\rho_{00} + \rho_{11} = 1$$

$$\rho_{01} = \overline{\rho_{10}}$$

$$A_0^+ + A_1 \rho A_1^+$$