

Title: PSI 2015/2016 Quantum Theory - Lecture 7

Date: Sep 16, 2015 10:45 AM

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Abstract:

Measurement

Example of POVM
(continuous outcomes)

Measurement

Example of POVM (continuous outcomes)

- $SU(2)$ coherent states
- (Regular) coh. states are minimum unc. states in $L^2(\mathbb{R}^n)$

• $SU(2)$ coh states
are an analog
for angular momentum

$\Rightarrow$ They are minimum unc.
states

$$\Delta J_x \Delta J_y \geq \frac{\hbar}{2} |\langle J_z \rangle|$$

$$[J_x, J_y] = i\hbar J_z$$

& cyclic permutations

$$J_z |j, m\rangle = \hbar m |j, m\rangle$$

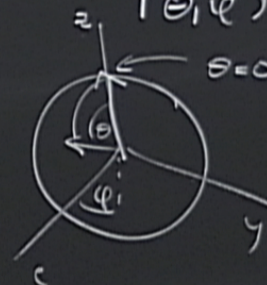
$$m \in \{-j, \dots, +j\}$$

Define rotation operator $U(\theta, \varphi) \in \text{SU}(2) \subset U(D)$
 $D = 2j + 1$

$$\hat{U}(\theta, \varphi) = e^{i\varphi \frac{\hat{J}_z}{\hbar}} e^{i\theta \frac{\hat{J}_x}{\hbar}}$$

$$|\theta, \varphi\rangle \equiv \hat{U}(\theta, \varphi) |j, j\rangle$$

↑
Maximum
Polarization
along $\hat{z}$.



(?)

The collection of states $\{|\theta, \varphi\rangle\}$
are $su(2)$ coh states.

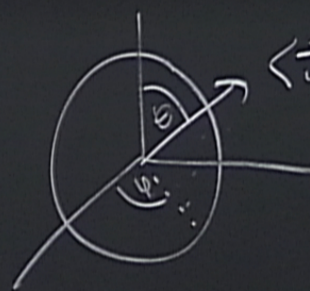
- Obviously not O.N

$$|\langle \theta', \varphi' | \theta, \varphi \rangle| = \left(\cos \frac{\Phi}{2} \right)^{2j}$$

$$\text{where } \cos \Phi := \cos \theta \cos \theta' - \sin \theta \sin \theta' \cos(\varphi - \varphi')$$

$$\{|\theta, \varphi\rangle\}$$

$$\langle \hat{\vec{J}} \rangle = \langle \hat{J}_x \rangle \hat{e}_x + \langle \hat{J}_y \rangle \hat{e}_y + \langle \hat{J}_z \rangle \hat{e}_z$$

$$\langle \hat{\vec{J}} \rangle = \langle \theta, \varphi | \hat{\vec{J}} | \theta, \varphi \rangle$$


$$\cos \frac{\theta}{2}$$

$$\cos \theta \cos \theta' - \sin \theta \sin \theta' \cos(\phi - \phi')$$

Perform a measurement w.r.t
this continuous set $\{|\theta, \varphi\rangle \in [0, 2\pi) \times [0, 2\pi)\}$.

Suppose we can only read-out (θ, φ) with
some finite precision $B \subset [0, \pi) \times [0, 2\pi)$

Define

$$\hat{E}(B) = \frac{2j+1}{4\pi} \int_B \overbrace{\sin\theta d\theta d\varphi}^{\text{d solid angle}} |\theta, \varphi\rangle \langle \theta, \varphi|$$

$$\Rightarrow \hat{E}(\Omega) = \hat{\mathbb{I}}$$

$$\Rightarrow E(B) \geq 0$$

$$P_r(B) = \text{Tr}[\hat{E}(B) \hat{\rho}]$$

$\forall$ input state $\hat{\rho}$

where $\{\hat{E}(B)\}$ is a POVM

Define

$$\hat{E}(B) = \frac{2j+1}{4\pi} \int_B \overbrace{\sin\theta d\theta d\varphi}^{d\omega_{S^2}} |\psi, \varphi\rangle \langle \psi, \varphi|$$

$$\Rightarrow \hat{E}(\Omega) = \hat{1} \quad \langle \psi, \varphi | \rho | \psi, \varphi \rangle$$

$$\Rightarrow E(B) \geq 0$$

Identity:

$$\hat{1} = \frac{2j+1}{4\pi} \int_{\Omega} |\psi, \varphi\rangle \langle \psi, \varphi| \sin\theta d\theta d\varphi$$

Naimark's Dilation Theorem

For any POVM $\{E\}$ and any state $\rho \in \mathcal{L}(\mathcal{H}_A)$

$\exists$ a PVM $\{\hat{P}\}$ acting $\mathcal{H}_A \otimes \mathcal{H}_B$

and $|\phi\rangle \in \mathcal{H}_B$ st

$$P_r(x) = \text{Tr}(E(x)\rho) = \text{Tr}(\rho \otimes |\phi\rangle\langle\phi| \hat{P}(x))$$

$$\forall x \in \mathcal{F}$$

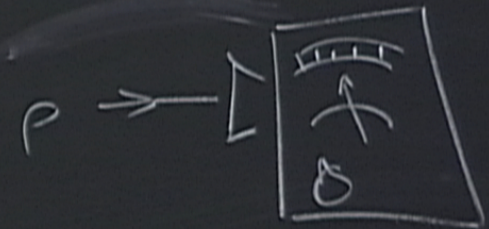
↑
Naimark's
projective-valued
measure

Moreover, $P(x)$ can always be realized via von Neumann's indirect measurement

$$P(x) = U^\dagger (\mathbb{I}_A \otimes \pi(x)) U$$

$$\pi \in \mathcal{L}(\mathcal{H}_B)$$

Start with



$\{E_k\}$

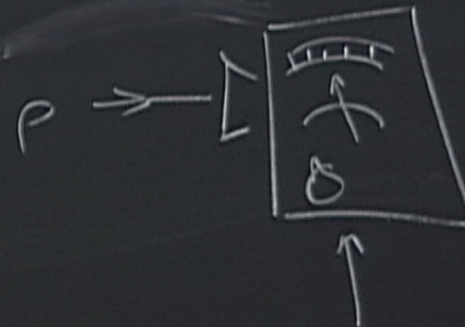
$$P(k) = \text{Tr}(\hat{E}_k \hat{\rho})$$

Moreover, $P(x)$ can always be realized via von Neumann's indirect measurement

$$P(x) = U^\dagger (\mathbb{1}_A \otimes \Pi(x)) U$$

$$\Pi \in \mathcal{L}(\mathcal{H}_B)$$

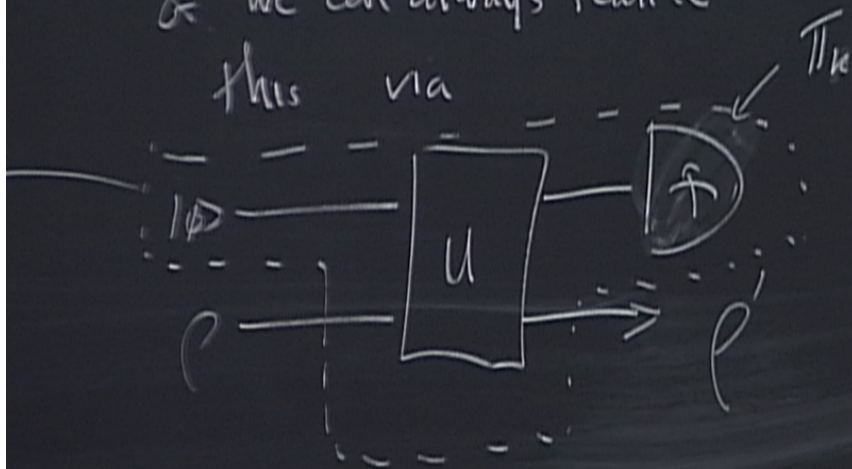
Start with



$$\{E_k\}$$

$$P(x) = \text{Tr}(\hat{E}_x \hat{\rho})$$

& we can always realize
this via



Realistic Provisional Postulate for Measurement (Discrete Case)

The most general measurement
on a system is represented by
a POVM $\{E_v\}$ where

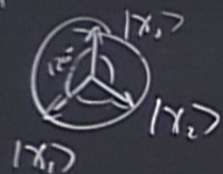
$$Pr(v) = \text{Tr}(\hat{\rho} \hat{E}_v).$$

Simple Perm "tree"



← See lecture notes

Simple POVM "trine"



← See lecture notes

$$E_v = |x_v\rangle\langle x_v| \quad \left(\frac{2}{3}\right)$$

Example of discrete POVM $\mathcal{H} = \mathbb{C}^D$

Mutually-unbiased bases (MUBs)

$$\{| \psi_a^b \rangle\}$$

$$b \in \{1, \dots, M\}$$

→ $| \psi_a^b \rangle$ a basis

$$a \in \{1, \dots, D\}$$

→ $| \psi_a^b \rangle$ a basis

$$| \langle \psi_{a'}^{b'} | \psi_a^b \rangle |$$

$$= \begin{cases} \frac{1}{\sqrt{D}} & \text{if } a=a' \text{ and } b=b' \\ 0 & \text{otherwise} \end{cases}$$

1 $H = \mathbb{C}^D$
bases (MUBs)

$b \in \{0, \dots, M\}$

→ index for a basis

$a \in \{0, \dots, D-1\}$

→ element of basis b .

$b \neq b'$

$b = b' \quad (a \neq a')$

The maximum # of such bases is $M \leq D+1$.

Maximum set is known to exist

when $D = p^k$ where p is prime.

1 $\mathcal{H} = \mathbb{C}^D$
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Maximum set is known to exist

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POVM:
$$E_a^b = \frac{1}{D+1} |\psi_a^b\rangle\langle\psi_a^b|$$

$\mathcal{H} = \mathbb{C}^D$
 bases (MUBs)
 $b \in \{0, \dots, M\}$
 $\rightarrow$ index for a basis
 $a \in \{0, \dots, D-1\}$
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 $b = b' \quad (a \neq a')$

The maximum # of such bases is $M \leq D+1$.

Maximum set is known to exist
 when $D = p^k$ where p is prime.

POVM: $E_a^b = \frac{1}{D+1} |\psi_a^b\rangle\langle\psi_a^b|$

This POVM is tomographically complete,

$\Rightarrow$ given $P_r(|\psi_a^b\rangle) = \text{Tr}(\hat{E}_a^b \hat{\rho}) \quad \forall a, b$

One can completely determine (parameterize)
the unknown input state $\hat{\rho}$

$$\rho \in \mathcal{L}(\mathbb{C}^D)$$

D diagonal entries
that are real and sum to 1.

$\Rightarrow D-1$ parameters

$\frac{D(D-1)}{2}$ complex numbers for off-diagonal
elements

$$D-1 + \frac{D(D-1)}{2} \times 2 = D^2$$

$$D-1 + \frac{D(D-1) \times 2}{2} = D-1 + D^2 - D$$

$$= \underline{\underline{D^2 - 1}}$$

real numbers
needed to
characterize
an arbitrary
propⁿ procedure.

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$\frac{D(D-1)}{2}$ complex numbers for off-diagonal
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$$D-1 + \frac{D(D-1)}{2} \times 2 = D-1 + D^2 - D = \underline{\underline{D^2 - 1}}$$

• For MUB POVM
we have $D+1$ prob vectors.

parameterize)

state $\hat{\rho}$

$$D-1 + \frac{D(D-1)}{2} \times 2 = D-1 + D^2 - D$$

$$= \underline{\underline{D^2 - 1}}$$

real numbers
needed to
characterize

an arbitrary
prepⁿ procedure.

• For MU3 POVM

we have $D+1$ prob vectors.
 $\times D-1$ ind. numbers

$$(D+1)(\underline{D-1}) = D^2 - 1$$

sum to 1.

for off-diagonal
elements

Example of discrete POVM $\mathcal{H} = \mathbb{C}^D$

Mutually-unbiased bases (MUBs)

$$\{|\psi_a^b\rangle\} \quad b \in \{0, \dots, M\}$$

an angle in $\mathbb{C}^D$

$\rightarrow$ index for a basis

$$a \in \{0, \dots, D-1\}$$

$\rightarrow$ element of basis b .

$$|\langle \psi_{a'}^{b'} | \psi_a^b \rangle|$$

$$= \begin{cases} \frac{1}{\sqrt{D}} & \text{for } b \neq b' \\ 0 & \text{for } b = b' \quad (a \neq a') \end{cases}$$

Example of discrete POVM

$$\mathcal{H} = \mathbb{C}^D$$

Mutually-unbiased bases (MUBs)

$$\{|\psi_a^b\rangle\}$$

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→ element of basis b .

an angle in $\mathbb{C}^D$



$$|\langle \psi_{a'}^{b'} | \psi_a^b \rangle|$$

Analogs

$$|x\rangle, |k\rangle$$

$$= \begin{cases} \frac{1}{\sqrt{D}} & \text{for } b \neq b' \\ 0 & \text{for } b = b' \end{cases}$$

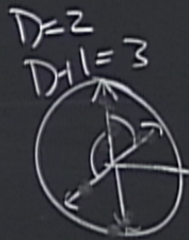
for $b \neq b'$

for $b = b'$ ($a \neq a'$)

$$\langle x | k \rangle = e^{ikx} \quad \forall k \& x$$

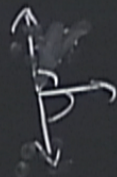
See lecture notes

$$\langle x | \psi \rangle \left(\frac{2}{3} \right)$$



Example of discrete PDM $\mathcal{H} = \mathbb{C}^D$

Mutually-unbiased bases (MUBs)



an angle in $\mathbb{C}^D$

$$\{ |\psi_a^b\rangle \}$$

$$b \in \{0, \dots, M\}$$

→ index for a basis

$$a \in \{0, \dots, D-1\}$$

→ element of basis b

$$|\langle \psi_{a'}^{b'} | \psi_a^b \rangle|$$

Analogs

$$|x\rangle, |k\rangle$$

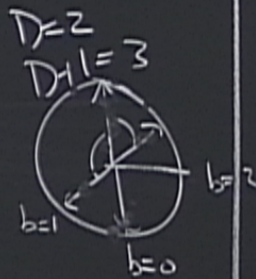
$$= \begin{cases} \frac{1}{\sqrt{D}} & \text{for } b \neq b' \\ 0 & \text{for } b = b' \end{cases}$$

for $b \neq b'$

for $b = b'$ ($a \neq a'$)

$$\langle x | k \rangle = e^{ikx} \quad \forall k \& x$$

See lecture notes



$$\sigma_x |+\rangle_x = |+\rangle_x$$

$$\sigma_z |+\rangle_z = |+\rangle_z$$

$$\langle + | + \rangle_z = \frac{1}{\sqrt{2}}$$

Example of discrete POVM $\mathcal{H} = \mathbb{C}^D$

Mutually-unbiased bases (MUBs)

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$$b \in \{0, \dots, M\}$$

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→ element of basis

an angle in $\mathbb{C}^D$



$$|\langle \psi_{a'}^{b'} | \psi_a^b \rangle|$$

Analogs

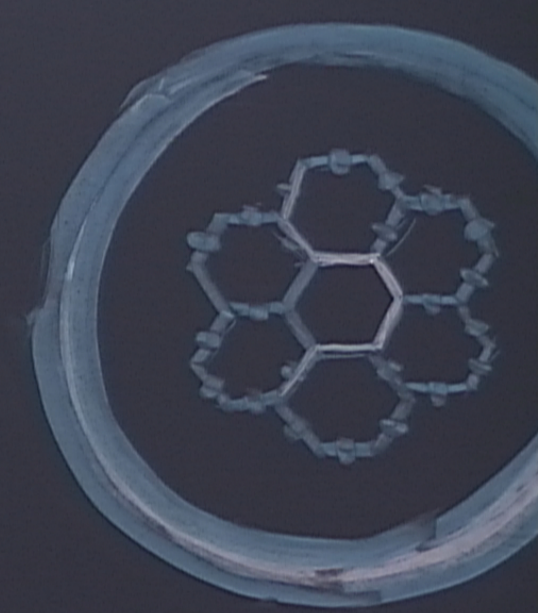
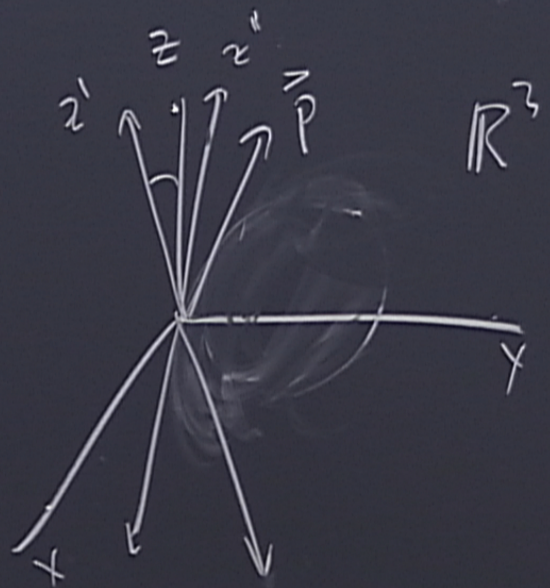
$$|x\rangle, |k\rangle$$

$$\langle x | k \rangle = e^{ikx} \quad \forall k \& x$$

$$= \begin{cases} \frac{1}{\sqrt{D}} & \text{for } b \neq b' \\ 0 & \text{for } b = b' \end{cases}$$

for $b \neq b'$

for $b = b'$ ($a \neq a'$)



Realistic Transformations

Same approach as before

(i) Probabilistic mixtures of unitaries

(ii) The reduced action on Hilbert space H_A given $U \in \mathcal{L}(H_A \otimes H_B)$.

$$(i) \quad \hat{\Lambda}(\hat{\rho}) = \sum_e p_e \hat{U}_e \hat{\rho} \hat{U}_e^\dagger$$

Introduce $\hat{A}_e = \sqrt{p_e} \hat{U}_e$

$$\rho' = \Lambda(\rho) = \sum_e \hat{A}_e \hat{\rho} \hat{A}_e^\dagger$$

Remark. $\text{Tr}(\rho') = \text{Tr}(\rho)$

$$\sum_e \hat{A}_e^\dagger \hat{A}_e = \mathbb{1}$$

$$(i) \quad \hat{\Lambda}(\hat{\rho}) = \sum_e p_e \hat{U}_e \hat{\rho} \hat{U}_e^\dagger$$

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Remark. $\text{Tr}(\rho') = \text{Tr}(\rho)$

$$\sum_e \hat{A}_e^\dagger \hat{A}_e = \mathbb{1}$$

$$(ii) \quad \rho_A' = \Lambda(\rho_A)$$

$$\rho_{AB}' = \mathcal{U} \rho_A \otimes \rho_B \mathcal{U}^\dagger$$

$$\rho_A' = \text{Tr}_B(\rho_{AB}')$$

$$\Lambda(\rho_A) = \text{Tr}_B(\mathcal{U} \rho_A \otimes \rho_B \mathcal{U}^\dagger)$$

For ease of notation, let $\rho_B = |0\rangle\langle 0|$

Introduce $A_k = \langle k | u | 0 \rangle$.

$$\Lambda(p_A) = \sum_k \langle k | U_{p_A \otimes p_B} u^\dagger | k \rangle$$

$$= \sum_k \langle k | u | 0 \rangle p_A \langle 0 | u^\dagger | k \rangle$$

$$= \sum_k A_k p_A A_k^\dagger$$

$$\rightarrow \sum_k A_k^\dagger A_k$$

if $\text{Tr}(p_A)$

$$\rightarrow \sum_k A_k^\dagger A_k = \mathbb{1}.$$

$$\text{if } \text{Tr}(\rho_{AB}') = \text{Tr}(\rho_{AB}).$$

Realistic Transformations

Same approach as before

(i) Probabilistic mixtures of unitaries

examples include
(Incoherence or
dephasing)

(ii) The reduced action on Hilbert space H_A

given $U \in \mathcal{L}(H_A \otimes H_B)$ (canonical
example
of
decoherence)

(i) $\hat{\Lambda}(\hat{\rho})$

$U \in \mathcal{L}(H)$ Introduce

$\rho \in \mathcal{L}(H)$

$\rho' = \Lambda(\rho)$

Remark

Defⁿ

Coherent state

$$|4_{cs}\rangle = |\alpha\rangle$$

$$\Delta x \Delta p = \frac{\hbar}{2}$$

Defⁿ a minimum unc. state

a state that is coherent

→ ρ is not diagonal

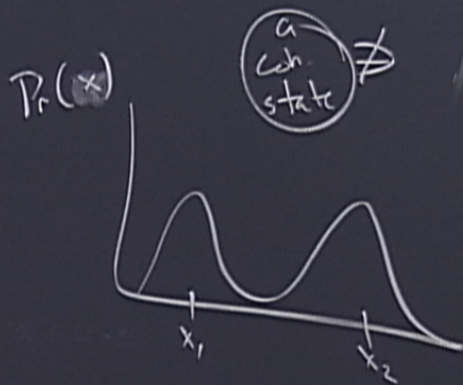
in a specific basis

→ not fully mixed.

$$|4\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\text{If } \alpha = \frac{1}{\sqrt{2}}, \beta = \frac{1}{\sqrt{2}}$$

$$|4\rangle = |1\rangle_x$$



coherent state $\neq$

$$|\psi\rangle = A e^{ipx} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

$$|\psi\rangle = A e^{ipx} e^{-\frac{(x-\bar{x}_1)^2}{2\sigma^2}} + B e^{ipx} e^{-\frac{(x-\bar{x}_2)^2}{2\sigma^2}}$$

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

$$\text{If } \alpha = \frac{1}{\sqrt{2}}, \beta = \frac{1}{\sqrt{2}}$$

$$|\psi\rangle = |1\rangle_x$$

Defⁿ

Coherent state

Defⁿ

a state that

→ p

in

→ not

Introduce $A_{k\alpha} = \langle k | U | \alpha \rangle$.

$$\begin{aligned} \Lambda(p_A) &= \sum_k \sum_{\alpha} \langle k | U | \alpha \rangle p_A \langle \alpha | U^\dagger | k \rangle \\ &= \sum_k \langle k | U | 0 \rangle p_A \langle 0 | U^\dagger | k \rangle \\ &= \sum_{k,\alpha} A_{k,\alpha} p_A A_{k,\alpha}^\dagger \end{aligned}$$

$$\rightarrow \sum_k A_k^\dagger A_k = \mathbb{1}$$

$$\text{if } \text{Tr}(p_{AB}) = 1$$

For both (i) & (ii)
in general $\text{Tr}(p_A)$
dec