

Title: PSI 2015/2016 Quantum Theory - Lecture 6

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Abstract:

Realistic Preparation

Any preparation procedure for a q. system can be represented by a density operator $\hat{\rho}$.

A density operator is a non-negative operator st. $\text{tr}(\hat{\rho}) = 1$

Realistic Measurement

Two scenarios:

(i) Probabilistic mixture
of measurements.

Consider a P.V.M. $\{\hat{P}_i\}$, $i \in \{1, \dots, D\}$
where $\hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_i$, $\hat{P}_i = \hat{P}_i^\dagger$

a second PVM $\{\pi_k\}$, $k \in \{1, \dots, D\}$

$$\text{and } \pi_k \pi_l = \delta_{kl} \pi_k$$

$$\& \pi_k = \pi_k^\dagger$$

The actual measurement
procedure is to apply
 $\{\hat{P}_i\}$ with prob p ,
& $\{\pi_k\}$ with prob $1-p$.

$\{E_v\}$ satisfies

(a) $E_v \geq 0$ (non-negative)

(b) $\sum_v E_v = \mathbb{1}$.

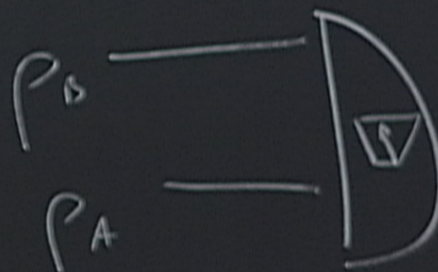
(ii) $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$\hat{\rho}_A \in \mathcal{L}(\mathcal{H}_A)$

$\hat{\rho}_B \in \mathcal{L}(\mathcal{H}_B)$.

We have a PVM $\{P_v\}$, $v \in$

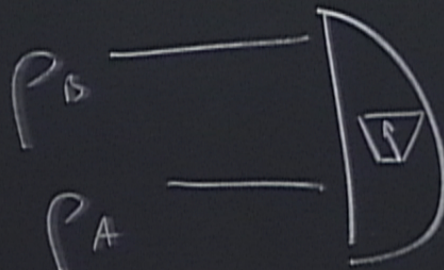
$P_v \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$



We have a PVM $\{P_v\}$, $v \in \{1, \dots, D\}$, where $D \leq \dim(\mathcal{H}_A) \dim(\mathcal{H}_B)$.

$$P_v \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

$$P_r(v) = \text{Tr}(\overbrace{P_A \otimes P_B}^{P_{AB}} \hat{P}_v)$$



We want an effective description
using operators on system A alone.

$$P_r(v) = \sum_{i,j=1}^M \sum_{k,l=1}^N (P_v)_{ik,jl} (p_A)_{ji} (p_B)_{lk}$$

$$M = \dim(H_A)$$

$$N = \dim(H_B)$$

$$\text{where } (P_v)_{ik,jl} \equiv \langle i | \otimes \langle k | \hat{P}_v | j \rangle \otimes | l \rangle$$

We can write this as

$$P_r(v) = \text{Tr}[E_v \rho_A]$$

$$\text{where } (E_v)_{ij} = \sum_{k,l} (P_v)_{ik,jl} (p_B)_{lk}$$

$$P_r(v) = \sum_{i,j=1}^M \sum_{k,l=1}^N (P_v)_{ik,jl} (p_A)_{ji} (p_B)_{lk}$$

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where $(P_v)_{ik,jl} \equiv \langle i| \otimes \langle k| \hat{P}_v |j\rangle \otimes |l\rangle$

We can write this as

$$P_r(v) = \text{Tr}[E_v p_A]$$

where $(E_v)_{ij} = \sum_{k,l} (P_v)_{ik,jl} (p_B)_{lk}$

$$\text{i.e. } \hat{E}_v = \text{Tr}_B (\hat{P}_v \hat{1}_A \otimes \hat{P}_B) \in \mathcal{L}(\mathcal{H}_A)$$

Remark the measurement operators E_v

satisfy

$$(a) E_v \geq 0$$

$$(b) \sum_v E_v = \mathbb{1}$$

$$\text{i.e. } \hat{E}_v = \text{Tr}_B (\hat{P}_v \hat{1}_A \otimes \hat{P}_B) \in \mathcal{L}(\mathcal{H}_A)$$

Remark the measurement operators $\{\hat{E}_v\}$
satisfy

$$(a) \hat{E}_v \geq 0$$

$$(b) \sum_v \hat{E}_v = \hat{1}$$

Any set $\{\hat{E}_v\}$ satisfying (a) & (b)
is called a POVM (positive operator-
valued measure)

i.e. $\hat{E}_v = \text{Tr}_B (\hat{P}_v \hat{1}_A \otimes \hat{P}_B) \in \mathcal{L}(\mathcal{H}_A)$

Remark the measurement operators $\{\hat{E}_v\}$ satisfy

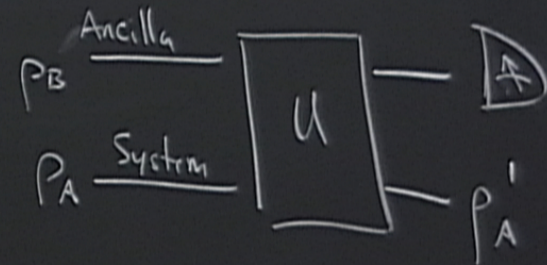
|| (a) $\hat{E}_v \geq 0$
 (b) $\sum_v \hat{E}_v = \hat{1}$

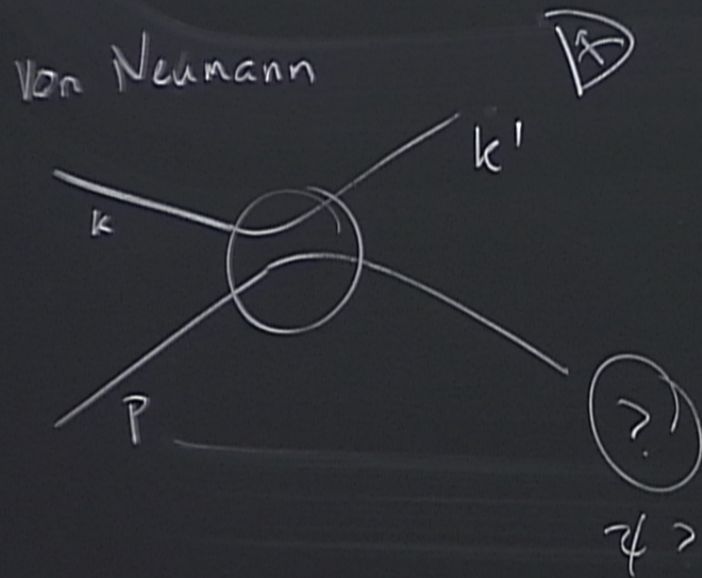
Any set $\{\hat{E}_v\}$ satisfying (a) & (b) is called a POVM (positive operator-valued measure)

Von Neumann's

model of

indirect measurement





System $\mathcal{H}_A$

$$A = \sum_x a_x |x\rangle\langle x|$$

Input state

$$\rho_A = |\psi\rangle\langle\psi|$$

$$|\psi\rangle = \sum_x c_x |x\rangle$$

$$T_{\mathcal{B}}(\hat{P}_V, \hat{1}_A \otimes \hat{P}_B) \in \mathcal{L}(\mathcal{H}_A)$$

the measurement operators $\{E_v\}$

satisfy

$$E_v \geq 0$$

$$\sum_v E_v = \mathbb{1}$$

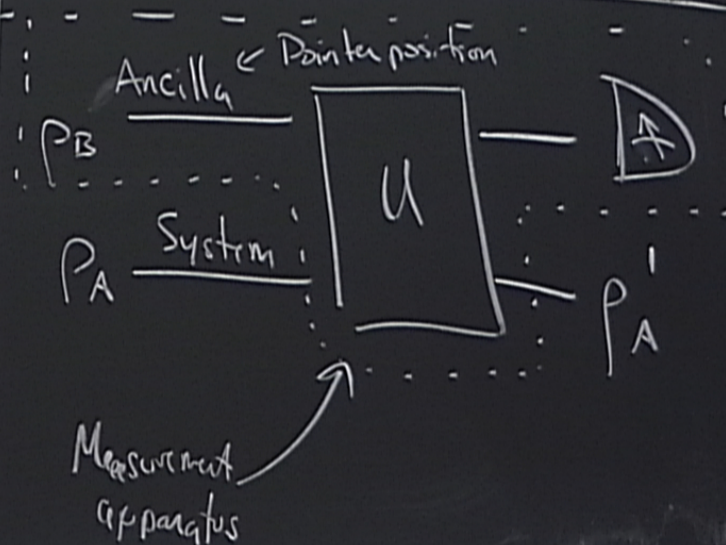
$\{E_v\}$ satisfying (a) & (b)

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Von Neumann's

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indirect measurement



$$T_{\mathcal{B}}(\hat{P}_V, \hat{1}_A \otimes \hat{P}_B) \in \mathcal{L}(\mathcal{H}_A)$$

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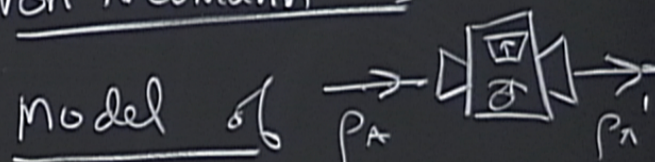
$$E_v \geq 0$$

$$\sum_v E_v = \mathbb{1}$$

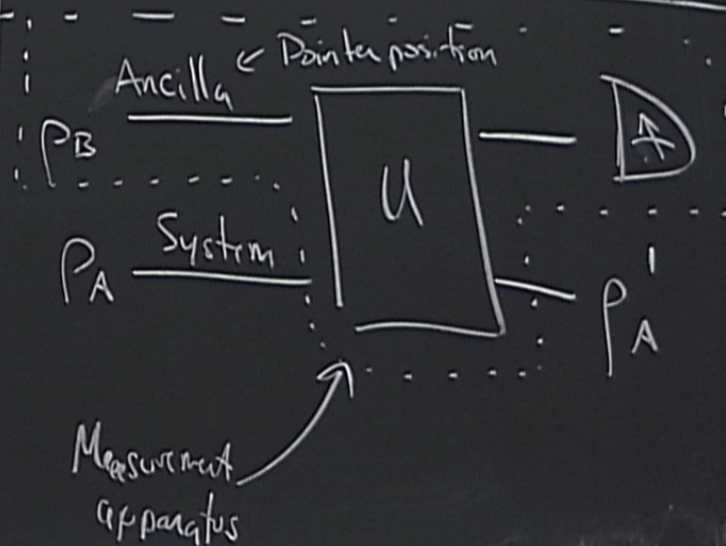
$\{E_v\}$ satisfying (a) & (b)

called a POVM (positive operator-valued measure)

Von Neumann's



in direct measurement



for $v \in \{D+1, \dots, 2L\}$

System H_A

$$A = \sum_k a_k |k\rangle\langle k|$$

Input state

$$\rho_A = |4\rangle\langle 4|$$

$$|4\rangle = \sum_k c_k |k\rangle$$

System B

Assume a fixed 'ready state'

$$\rho_B = |0\rangle\langle 0|$$

How do we design an interaction that correctly identifies/distinguishes the O.N. input states $\{|k\rangle\}$, $|k\rangle \in H_A$

System B

Assume a fixed 'ready state'

$$|p_B\rangle = |0\rangle \otimes |0\rangle$$

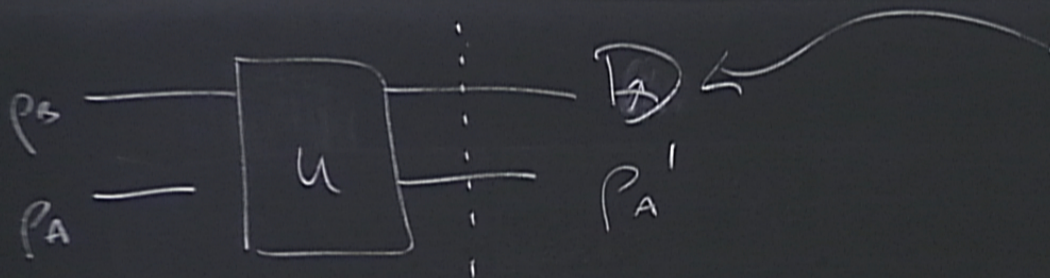
How do we design an interaction
that correctly identifies/distinguishes
the o.v. input states $\{|k\rangle\}$, $|k\rangle \in \mathcal{H}_A$?

Demand:

$$U |k\rangle \otimes |0\rangle_B \rightarrow |k\rangle_A \otimes |k\rangle_B$$

By linearity

$$U \left(\sum_k c_k |k\rangle \right) \otimes |0\rangle \rightarrow \sum_k c_k |k\rangle \otimes |k\rangle$$



σ_{AB} = density operator
for A & B

$$\sigma_{AB} = |\phi\rangle\langle\phi|$$

$$|\phi\rangle = \sum_k c_k |k\rangle \otimes |k\rangle$$

$$B = \sum_k b_k \underline{|k\rangle} \in \mathcal{L}(\mathcal{H}_B)$$

For simplicity assume

$$\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$$

Pointer on $|k\rangle$ for B

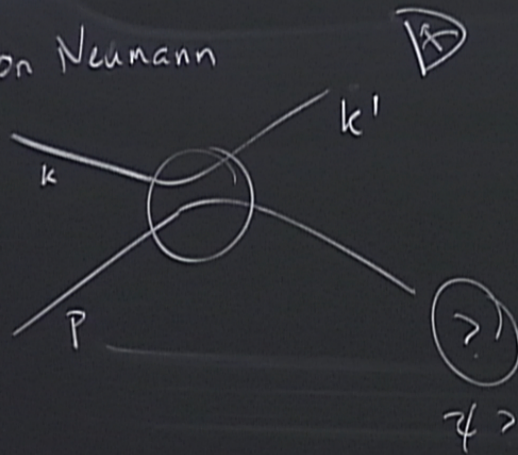
gives a random outcome "k"

$$\text{where } \text{Pr}(k_0) = \text{Tr}(\sigma_{AB} \mathbb{1}_A \otimes |k_0\rangle\langle k_0|)$$

$$\text{where } \sigma_{AB} = U \rho_A \otimes \rho_B U^\dagger$$

$$P_r(k_0) = |C_k|^2$$

Von Neumann



System H_A

$$A = \sum_k a_k |k\rangle\langle k|$$

Input state

$$\rho_A = |\psi\rangle\langle\psi|$$

$$|\psi\rangle = \sum_k c_k |k\rangle$$

→ $k=0$ spin up that correctly identifies the O.N. input state
 $k=1$ spin down

System B

Assume a fixed read

$$\rho_B = |0\rangle\langle 0|$$

How do we design

where $P_i P_j = \delta_{ij} P_i$, $P_i = P_i^\dagger$

& $\{\hat{\pi}_i\}$ with prob $1-p$

$$B = \sum_k b_k |k\rangle \in L(\mathcal{H}_B)$$

For simplicity assume

$$\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$$

Pointer on $|1\rangle$ for B

gives a random outcome " k "

$$\text{where } P_i(k_0) = \text{Tr}(\sigma_{AB} \mathbb{1}_A \otimes |k\rangle\langle k|)$$

$$\text{where } \sigma_{AB} = U P_A \otimes P_B U^\dagger$$

$$P_i(k_0) = |C_k|^2$$

Think $k=0 \rightarrow$ pointer to right

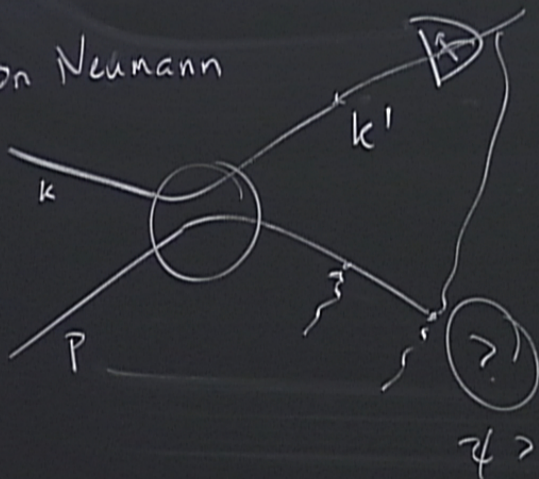
$k=1 \rightarrow$ pointer to left

$\Rightarrow$ Correlated final pointer position
with the input state spin-up or spin-down

$\Rightarrow$ Indirect measurement because the ~~"disturbance"~~ ^{"disruption"} of
measurement is only applied to system B

$$\tilde{E}_v = (1-p)\pi_{v-D} \text{ for } v \in \{D+1, \dots, 2D\}$$

Von Neumann



System $\mathcal{H}_A$

$$A = \sum_k a_k |k\rangle\langle k|$$

Input state

$$\rho_A = |4\rangle\langle 4|$$

$$|4\rangle = \sum_k c_k |k\rangle$$

→ $k=0$ spin up
 $k=1$ spin down
 that correctly
 the O.N.

System B

Assume a

$$\rho_B = I$$

How do we do

$$\text{where } \hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_i, \quad P_i = P_i^\dagger$$

$$B = \sum_k b_k \underline{|k\rangle} \in \mathcal{L}(\mathcal{H}_B)$$

For simplicity assume

$$\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$$

Pointer on $|x\rangle$ for B

Gives a random outcome " k "

$$\text{where } P_r(k') = \text{Tr}(\sigma_{AB} \mathbb{1}_A \otimes |k'Xk'|)$$

$$\text{where } \sigma_{AB} = U P_A \otimes P_B U^\dagger$$

$$P_r(k') = |C_k|^2$$

Think $k=0 \rightarrow$ pointer to
 $k=1 \rightarrow$ pointer to

$\Rightarrow$ Correlated final point
 with the input state

$\Rightarrow$ Indirect measurement
 measurement

$$P_r(k') = \text{Tr}(|\phi\rangle\langle\phi| \mathbb{1}_A \otimes |k'\rangle\langle k'|) = \langle\phi| \mathbb{1}_A \otimes |k'\rangle\langle k'| |\phi\rangle$$

$$|\phi\rangle = \sum_e c_e |e\rangle \otimes |e\rangle$$

$$= \sum_{e, e'} \left(\overline{c_{e'}} \langle e| \otimes \langle e| \right) \mathbb{1}_A \otimes |k'\rangle\langle k'|$$

$$\text{Tr}(\cdot) = \sum_\alpha \langle\alpha| \cdot |\alpha\rangle$$

Choose $\langle\alpha| = \langle\phi|$
 $\langle\alpha \neq \phi| \phi\rangle = 0$

$$c_v = (-1)^{v-D} \text{ for } v \in \{D+1, \dots, 2D\}$$

$$P_r(k') = \text{Tr} (|\phi\rangle\langle\phi| \mathbb{1}_A \otimes |k'\rangle\langle k'|) = \langle\phi| \mathbb{1}_A \otimes |k'\rangle\langle k'| |\phi\rangle$$

$$= \sum_{\alpha, \alpha'} \left(\bar{c}_\alpha \langle\alpha| \otimes \langle\alpha| \right) \mathbb{1}_A \otimes |k'\rangle\langle k'| (c_{\alpha'} |\alpha'\rangle \otimes |\alpha'\rangle)$$

$$|\phi\rangle = \sum_{\alpha} c_{\alpha} |\alpha\rangle \otimes |\alpha\rangle$$

$$\text{Tr}(\cdot) = \sum_{\alpha} \langle\alpha| \cdot |\alpha\rangle$$

Choose $\langle\alpha| = \langle\phi|$
 $\langle\alpha \neq \phi| \phi\rangle = 0$

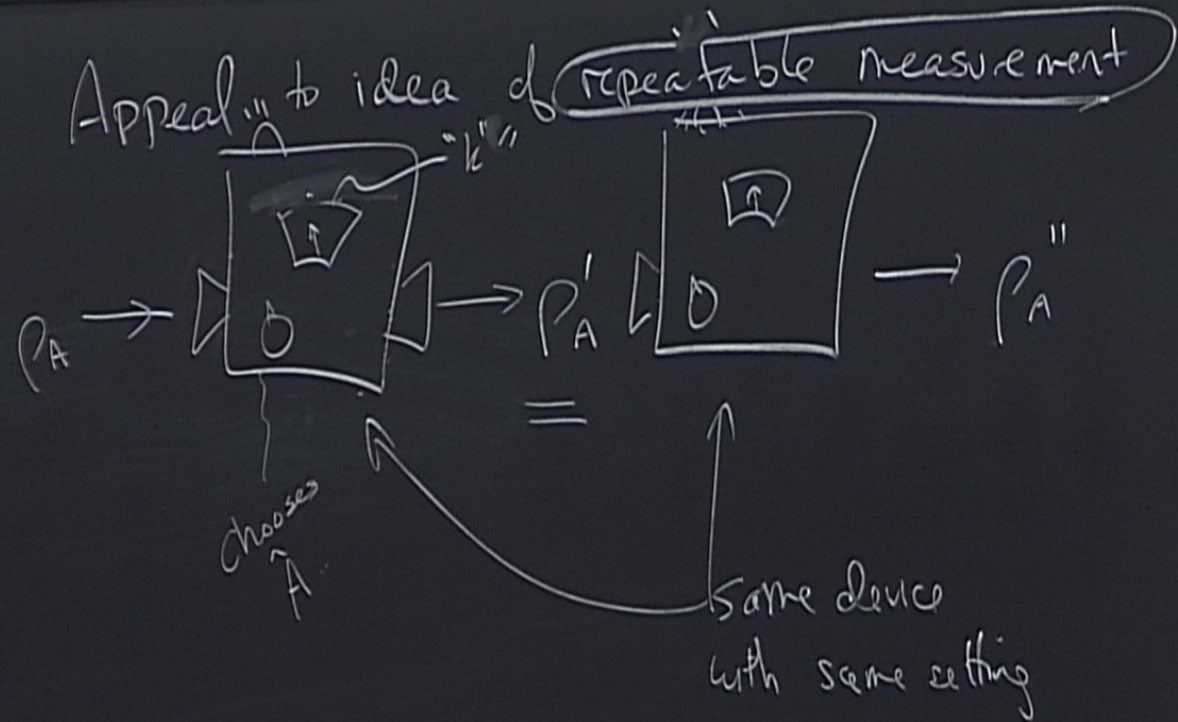
$$= \sum_{\alpha, \alpha'} \bar{c}_{\alpha} c_{\alpha'} \langle\alpha|\alpha'\rangle \langle\alpha|k'\rangle \langle k'|\alpha'\rangle$$

$$= |c_{k'}|^2$$

$$(c, |k'\rangle \otimes |k'\rangle)$$

$$\langle k' | k' \rangle$$

$$\delta_{k'k'}$$



Only way this ^{second} device

produces output k'

with certainty is if

$$P_A' = |k' \times k'|$$

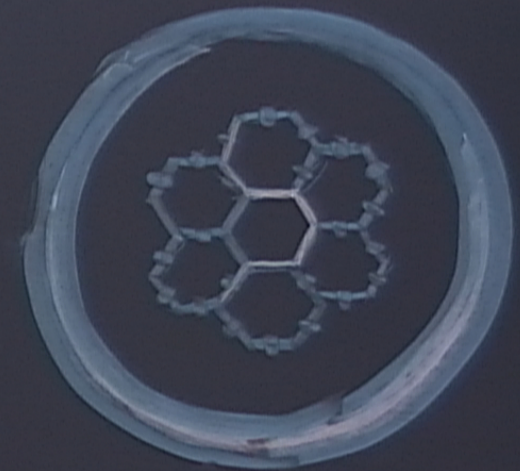
⇒ then we get the textbook
answer about the state
after measurement.

Formalism

Measurement consisted of

$$\mathbb{1} \otimes \pi_k \quad U \rho_A \otimes \rho_{TS} U^\dagger \quad \mathbb{1} \otimes \pi_k$$

$\uparrow$
 $|k \times k|$



Formalism

Measurement consisted of

$$P_r(l) = \text{Tr} \left(\underset{\substack{\uparrow \\ |k \times k|}}{\mathbb{1} \otimes \pi_k} U \rho_A \otimes \rho_B U^\dagger \mathbb{1} \otimes \pi_k \right)$$

Remark $\rightarrow$ Identity $P_U = U^\dagger \mathbb{1} \otimes \pi_k U$

$$\begin{aligned} P_r(l) &= \text{Tr} \left(P_U \rho_A \otimes \rho_B P_U^\dagger \right), \quad U^\dagger U = \mathbb{1} \\ &= \text{Tr} \left(P_U \rho_A \otimes \rho_B \right) = \text{Tr} \left(E_U \rho_A \right) \end{aligned}$$

Formalism

Measurement consisted of

$$P_r(k') = \text{Tr} \left(\underset{\substack{\uparrow \\ |k' \rangle \langle k'|}}{\mathbb{1} \otimes \Pi_{k'}} U \rho_A \otimes \rho_B U^\dagger \mathbb{1} \otimes \Pi_{k'} \right)$$

Remark $\rightarrow$ Identity $P_{k'} = U^\dagger \mathbb{1} \otimes \Pi_{k'} U$

$$\begin{aligned} P_r(k') &= \text{Tr} \left(P_{k'} \rho_A \otimes \rho_B P_{k'}^\dagger \right), \quad U^\dagger U = \mathbb{1} \\ &= \text{Tr} \left(P_{k'} \rho_A \otimes \rho_B \right) = \text{Tr} \left(\underline{E}_{k'}(\rho_A) \right) \end{aligned}$$

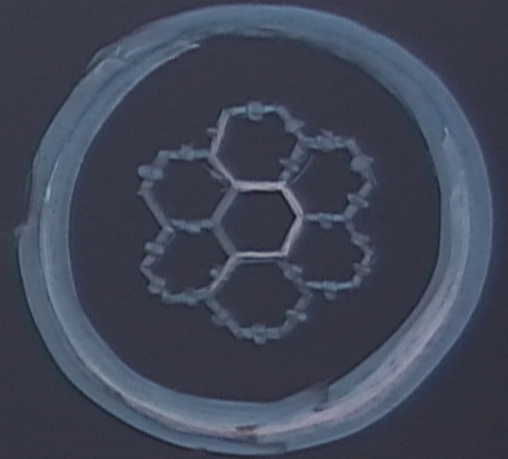
Formalism

Measurement consisted of

$$P_r(k') = \text{Tr} \left(\underbrace{1 \otimes \pi_{k'}}_{|k' \times k'|} U_{\rho_A \otimes \rho_B} U^\dagger 1 \otimes \pi_{k'} \right)$$

Remark $\rightarrow$ Identity $P_{k'} = U^\dagger 1_{\infty} \pi_{k'} U$

$$\begin{aligned} \text{Tr}(U) &= \text{Tr}(P_{k'} \rho_A \otimes P_B P_j^\dagger), \quad U^\dagger U = \mathbb{1} \\ &= \text{Tr}(P_{k'} \rho_A \otimes P_B) = \text{Tr}(E_{k'} \rho_A) \end{aligned}$$



← We've modelled Born rule statistics via PVM for VN's indirect measurement model.

Formalism

Measurement consisted of

$$P_r(k') = \text{Tr} \left(\underbrace{\mathbb{1} \otimes \pi_{k'}}_{|k' \times k'|} U \rho_A \otimes \rho_B U^\dagger \mathbb{1} \otimes \pi_{k'} \right)$$

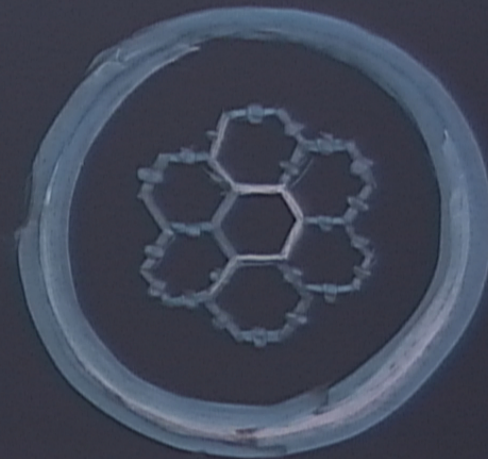
Remark

→ Identity

$$P_{k'} = U^\dagger \mathbb{1} \otimes \pi_{k'} U$$

$$P_r(k') = \text{Tr} \left(P_{k'} \rho_A \otimes \rho_B P_{k'}^\dagger \right), \quad U^\dagger U = \mathbb{1}$$

$$= \text{Tr} \left(P_{k'} \rho_A \otimes \rho_B \right) = \text{Tr} \left(E_{k'} \rho_A \right)$$



← We've modelled
Born rule statistics
via POVM for
VN's indirect
measurement model.

Define a POVM measurement
for continuous outcomes

- Set Ω
- $\underbrace{\sigma\text{-algebra}}_F$ on set Ω

Formalism

Measurement

$$Pr(k') = \text{Tr} \left(\underbrace{1 \otimes \pi_{k'}}_{|k' \times k'|} U \rho_A \otimes \right)$$

Remark $\rightarrow Id$

$$Pr(k') = \text{Tr}(\dots) = \text{Tr}(\dots)$$

$$E: \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H})$$

- (i) $E(X) \geq 0 \quad \forall X \in \mathcal{F}$
- (ii) $E(\Omega) = \mathbb{1}$

Define a POVM measurement
for continuous outcomes

- Set $\mathcal{L}$
- σ -algebra on set $\mathcal{L}$

$X \in \mathcal{F} \Rightarrow$ Think of X as
a subinterval $[a, b] \in \mathbb{R}$

$$P_r(k') = \text{Tr} \left(\mathbb{1} \otimes \pi_{k'} \quad \underbrace{U \rho_A \otimes \mathbb{1}}_{|k' \rangle \langle k'|} \right)$$

$\delta = |a-b|$
resolution

Formalism

Measurement

Remark $\rightarrow$ Id

$$P_r(k') = \text{Tr}(\dots) = \text{Tr}(\dots)$$

$$E: \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H})$$

$$(i) E(X) \geq 0 \quad \forall X \in \mathcal{F}$$

$$(ii) E(\Omega) = \mathbb{1}$$

$$(iii) E(U; X_i) = \sum_i E(X_i)$$

Generalized Born rule

$$Pr(X) = \text{Tr}(E(X) \rho)$$

Define a POVM measurement
for continuous outcomes

Set Ω

σ -algebra on set Ω

$X \in \mathcal{F} \Rightarrow$ Think of X as
a subinterval $[a, b] \subset \mathbb{R}$

$$Pr(k') = \text{Tr}(\mathbb{1} \otimes \Pi_{k'} U \rho U^\dagger)$$

$\delta = |a-b|$
resolution

Formalism

Measurement

Remark $\rightarrow$

$$Pr(k') = \text{Tr}(\dots)$$

$$E: \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H})$$

- (i) $E(X) \geq 0 \quad \forall X \in \mathcal{F}$
- (ii) $E(\Omega) = \mathbb{1}$
- (iii) $E(U_i X_i)$
 $= \sum_i E(X_i)$

Generalized Born rule

$$Pr(X) = \text{Tr}(E(X)\rho)$$

Define a POVM measurement
for continuous outcomes

Set Ω

σ -algebra on set Ω

$X \in \mathcal{F} \Rightarrow$ Think of X as
a subinterval $[a, b] \subset \mathbb{R}$

$$Pr(k') = \text{Tr}(\mathbb{1} \otimes \Pi_{k'} U \rho U^\dagger)$$

$\delta = |a-b|$
resolution

Formalism

Measurement

Remark $\rightarrow$

$$Pr(k') = \text{Tr}(\dots)$$

$$E: \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H})$$

- (i) $E(X) \geq 0 \quad \forall X \in \mathcal{F}$
- (ii) $E(\Omega) = \mathbb{1}$
- (iii) $E(U_i X_i)$
 $= \sum_i E(X_i)$

Generalized Born rule

$$Pr(x) = \text{Tr}(\hat{E}(x) \hat{\rho})$$

for input state $\hat{\rho}$.

Define a POVM measurement
for continuous outcomes

Set Ω

σ -algebra on set Ω
 $\mathcal{F}$

$X \in \mathcal{F} \Rightarrow$ Think of X as
a subinterval $[a, b] \subset \mathbb{R}$

$$Pr(k') = \text{Tr}(\mathbb{1} \otimes \Pi_{k'} U)$$

$\delta = |a-b|$
resolution

$|k' \times k'|$

Formalism

Measurement

Remark $\rightarrow$

$$Pr(k') = \text{Tr}(\dots)$$

output k'

certainty is if

$$|k' \times k'|$$

get the textbook
about the state
measurement.

A "positive operator"

is a non-negative operator.

$$\langle \psi | E | \psi \rangle \geq 0$$

$$\forall |\psi\rangle \in \mathcal{H}.$$