

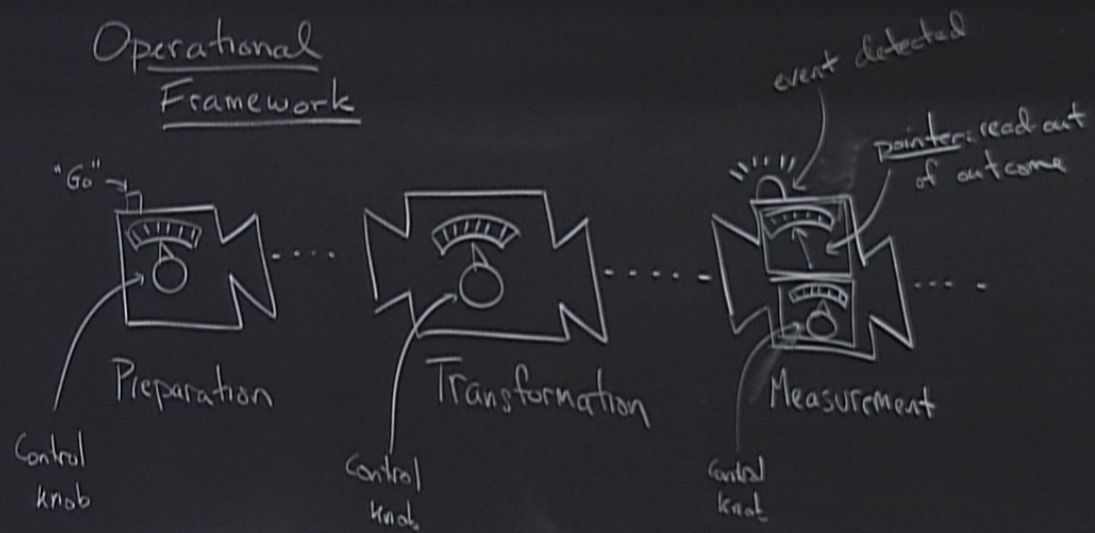
Title: PSI 2015/2016 Quantum Theory - Lecture 2

Date: Sep 09, 2015 10:45 AM

URL: <http://pirsa.org/15090043>

Abstract:

## Operational Framework





Postulate 1: An ideal  
preparation procedure  
is described by a  
a Hilbert space vector,  
called a pure state.

Q Why is it best to  
avoid the term  
state?

is confusing



Postulate 1: An ideal  
preparation procedure  
is described by a  
a Hilbert space vector,  
called a pure state.

Q Why is it best to  
avoid the term  
q. state?

A Because it is confusing,  
it evokes the wrong connotation  
with the uninitiated.



Follow the lady (queen)

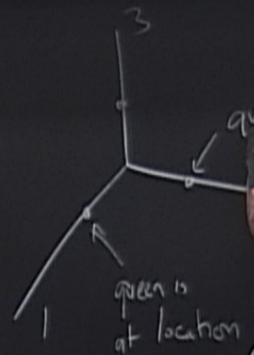


-  $X$  is the position (physical state)  
of the queen

$$X \in \{1, 2, 3\}$$

I have complete  
uncertainty about  
the physical state  
(i.e. location of queen)

Prob vector  $\vec{p} \in \mathbb{R}^3$





Follow the lady (queen)

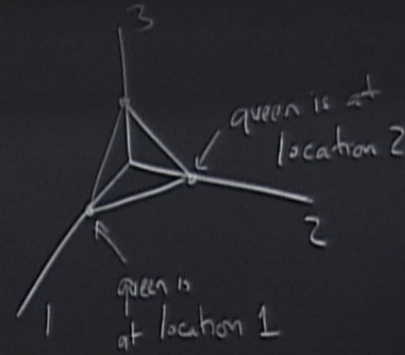


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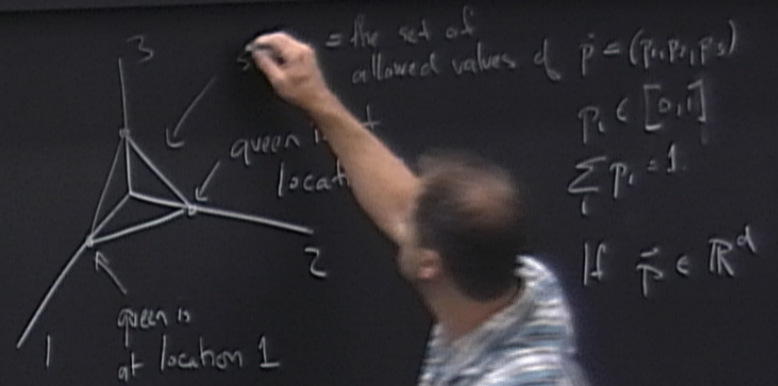


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## Examples of Hilbert space

$$\mathcal{H} = L^2(\mathbb{R})$$

$\Rightarrow$  the space of square-integrable functions

$$\text{Let } \varphi, \psi \in L^2(\mathbb{R})$$

$$\text{Inner-product } \langle \varphi | \psi \rangle = (\varphi, \psi) = \int_{-\infty}^{\infty} \varphi(x) \overline{\psi(x)} dx$$

Square-integrable  
implies



$\psi$  is square-integrable if

$$(\psi, \psi) \equiv \int_{-\infty}^{\infty} dx \, \overline{\psi(x)} \psi(x)$$

$$= \int_{-\infty}^{\infty} dx \, |\psi(x)|^2 < \infty$$

integrable

$$(\psi, \phi) = \int_{-\infty}^{\infty} dx \, \overline{\psi(x)} \phi(x)$$



ble if

All of wavemechanics

occurs in  $\mathcal{H} = L^2(\mathbb{R}^n)$



1 particle in  $n$  dimensions

or  $n$  particles in 1 dimension

or  $n/3$  particles in 3 dimensions

etc...

$$\int_{-\infty}^{\infty} dx \, \overline{\psi(x)} \psi(x)$$

$$\int_{-\infty}^{\infty} dx \, |\psi(x)|^2 < \infty$$

$\psi$  is Square-integrable if

$$\begin{aligned}(\psi, \psi) &\equiv \int_{-\infty}^{\infty} dx \, \overline{\psi(x)} \psi(x) \\ &= \int_{-\infty}^{\infty} dx \, |\psi(x)|^2 < \infty\end{aligned}$$

$$\psi(x) < \infty$$

All of wavemechanics

occurs in  $\mathcal{H} = L^2(\mathbb{R}^n)$

↑  
1 particle in  $n$   
or  $n$  particles in  
or  $n/3$  particles  
etc...



ble if

$$\int_{-\infty}^{\infty} \overline{\psi(x)} \psi(x) dx < \infty$$

$$\int_{-\infty}^{\infty} dx |\psi(x)|^2 < \infty$$

All of wavemechanics

occurs in  $\mathcal{H} = L^2(\mathbb{R}^n)$



- 1 particle in n dimensions  $\propto L^2(\mathbb{R}^n)$
- or n particles in 1 dimension  $\propto L^2(\mathbb{R})$
- or  $n/3$  particles in 3 dimensions  $\propto L^2(\mathbb{R}^3)$
- etc...

2 particles in 3 D

$$\omega = (x_A, y_A, z_A, x_B, y_B, z_B)$$

Natural  $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$   
 $= L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$

Example  $\mathcal{H} = \mathbb{C}^2$

- This is two-dimensional complex vector space.

- Describes, eg, spin- $\frac{1}{2}$  particle

$$\chi \in \mathbb{C}^2$$

$$\chi = (\alpha, \beta)$$

$$\|\chi\|^2 = 1 \Rightarrow |\alpha|^2 + |\beta|^2 = 1$$



$$\psi = (\psi, 0) = \int_{-\infty}^{\infty} dx \, \tilde{\psi}(x) \varphi(x) < \infty$$

$$\psi = \sum_{k=1}^{\infty} c_k |\varphi_k\rangle, \quad |\varphi_k\rangle \text{ is an ON. basis}$$

etc.

where  $\{a_k\}$  are eigenvalues  
of  $\hat{A}$

&  $\{P_k\}$  are orthogonal

projection operators

$$\hat{P}_k \hat{P}_h = \delta_{kh} \hat{P}_k$$

$$\& \quad \hat{P}_k = \hat{P}_k^\dagger$$



$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \psi(x) < \infty$$

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etc...

where  $\{a_{\ell}\}$  are eigenvalues  
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projection operators

$$\hat{P}_{\ell} \hat{P}_k = \delta_{\ell k} \hat{P}_{\ell}$$

$$\& \quad \hat{P}_{\ell} = \hat{P}_{\ell}^{\dagger}$$

Each  $\hat{P}_{\ell}$  projects onto  
the (possibly degenerate)

eigenspace associated with  $a_{\ell} \in \mathbb{R}$ .

if  $a_{\ell}$  is  $M$ -fold degenerate

then we can choose an O.N. set  $\{|\ell, m_{\ell}\rangle\}$

$$\& \quad P_{\ell} = \sum_{m_{\ell}=1}^M |\ell, m_{\ell}\rangle \langle \ell, m_{\ell}|$$



$$\psi = \int_{-\infty}^{\infty} dx \, \psi(x) \phi(x) < \infty$$

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$$\text{s.t. } \hat{P}_{\ell} = \sum_{m_{\ell}=1}^M |\ell, m_{\ell}\rangle \langle \ell, m_{\ell}|$$

$$\|x\|^2 = 1 \Rightarrow |\alpha|^2 + |\beta|^2 = 1$$

Postulate 2: An ideal measurement

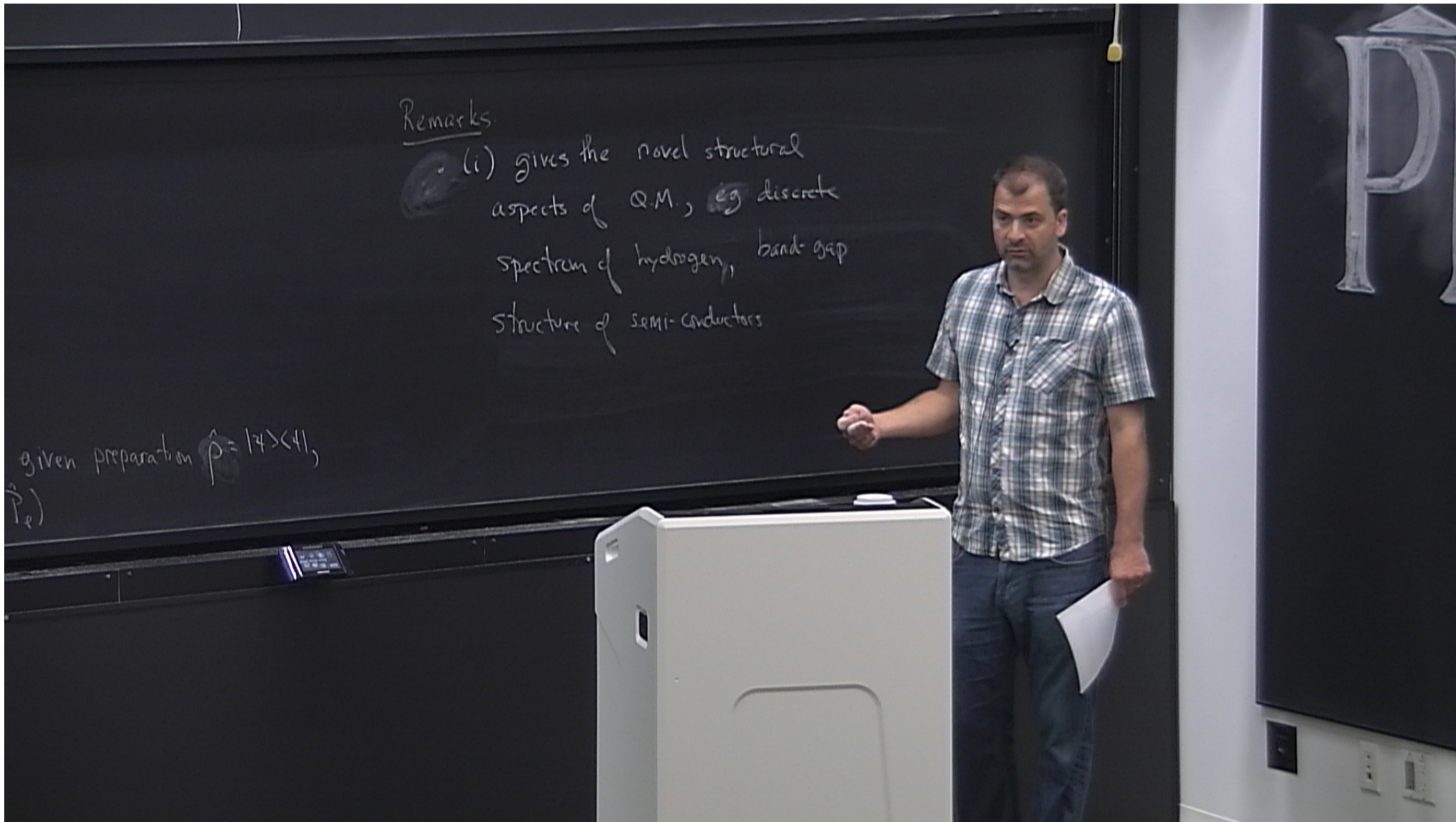
procedure is described by

a self-adjoint operator  $\hat{A}$  s.t.

(i) the set of physically observable outcomes are prescribed by  $\{a_e\}$

(ii) the probability of any outcome  $a_e$ , given preparation  $\hat{\rho} = |\psi\rangle\langle\psi|$ , is prescribed by  $P(a_e) = \text{Tr}(\hat{\rho} \hat{P}_e)$







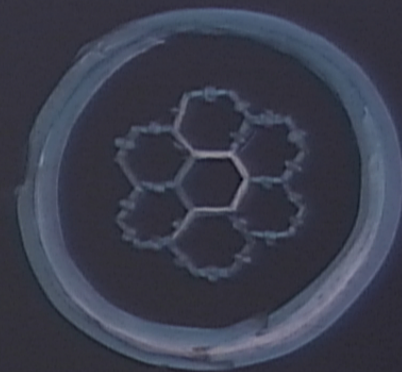
• How do expectation values  
fit into this picture?

$$\langle \hat{A} \rangle \equiv \langle \psi | \hat{A} | \psi \rangle$$

$$= \langle \psi | \sum_e a_e \hat{P}_e | \psi \rangle$$

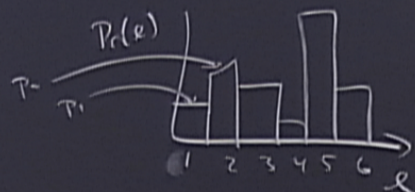
$$= \sum_e a_e \langle \psi | \hat{P}_e | \psi \rangle = \sum_e a_e \text{Tr}(\hat{P}_e |\psi\rangle\langle\psi|)$$

Note.  $\text{Tr}(\hat{\rho} \hat{P}_e) = \text{Tr}(|\psi\rangle\langle\psi| \hat{P}_e)$   
 $= \langle \psi | \hat{P}_e | \psi \rangle$





## Measurement



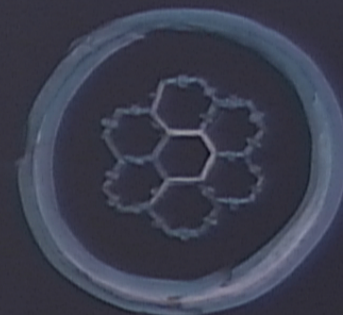
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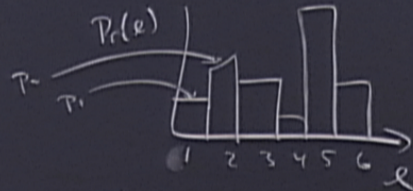
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## Measurement



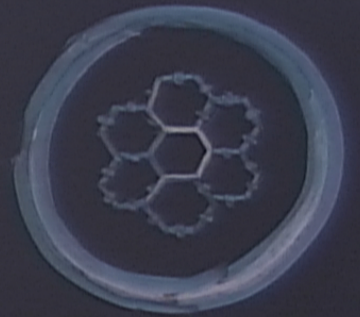
Histogram for an  
ensemble of repetitions  
of the operational procedure  
in Fig 1.

How do expectation values  
fit into this picture?

$$\langle \hat{A} \rangle \equiv \langle 4 | \hat{A} | 4 \rangle$$

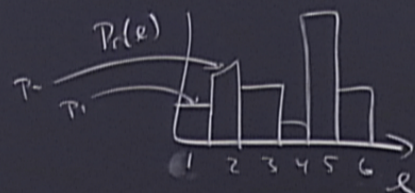
$$= \sum_e a_e \text{Tr}(\hat{\rho}_e |4\rangle\langle 4|)$$

Note:  $\text{Tr}(\hat{\rho} \hat{\rho}_e)$





## Measurement



Histogram for an  
ensemble of repetitions  
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in Fig 1.

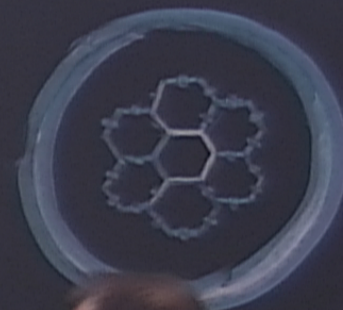
$$\langle A \rangle = \sum_e P_e a_e = 62^m. \text{ How do expectation values}$$

fit into this picture?

$$\begin{aligned} \langle \hat{A} \rangle &\equiv \langle \psi | \hat{A} | \psi \rangle \\ &= \langle \psi | \sum_e a_e \hat{P}_e | \psi \rangle \end{aligned}$$

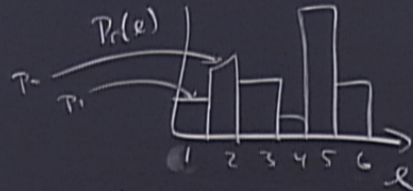
$$= \sum_e a_e \langle \psi | \hat{P}_e | \psi \rangle = \sum_e a_e \langle \psi | \hat{P}_e | \psi \rangle$$

Note.  $\text{Tr}(\hat{\rho} \hat{P}_e) = \text{Tr}(|\psi\rangle\langle\psi| \hat{P}_e)$   
 $= \langle \psi | \hat{P}_e | \psi \rangle$





## Measurement



Histogram for an  
ensemble of repetitions  
of the operational procedure  
in Fig 1.

$$\langle A \rangle = \sum_e p_e a_e = 6.2^m. \text{ How do expectation values}$$

$$\text{chose } p_e = \underline{\underline{Pr(e)}}$$

fit into this picture?

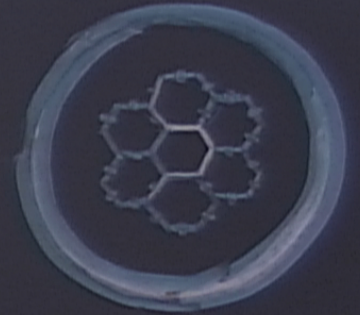
$$\begin{aligned} \langle \hat{A} \rangle &\equiv \langle \psi | \hat{A} | \psi \rangle \\ &= \langle \psi | \sum_e a_e \hat{P}_e | \psi \rangle \end{aligned}$$

$$= \sum_e a_e \langle \psi | \hat{P}_e | \psi \rangle = \sum_e a_e \text{Tr}(\hat{P}_e |\psi\rangle\langle\psi|)$$

$$\begin{aligned} \text{Note: } \text{Tr}(\hat{P}_e \hat{P}_e) &= \text{Tr}(|\psi\rangle\langle\psi| \hat{P}_e) = \sum_m \langle \phi_m | \psi \rangle \langle \psi | \hat{P}_e | \phi_m \rangle \\ &= \langle \psi | \hat{P}_e | \psi \rangle \end{aligned}$$

$$\text{Tr}(\cdot) = \sum_m \langle \phi_m | \cdot | \phi_m \rangle$$

$$\begin{aligned} \text{Choose } \langle \phi_1 | &= \langle \psi | \\ \langle \phi_{m \neq 1} | \psi \rangle &= 0 \end{aligned}$$





## Ideal Transformations

Postulate 3. Ideal transformations (in time)

are generated by a linear operator  $U$

$$s.t. \quad |\psi(t_2)\rangle = U(t_2, t_1) |\psi(t_1)\rangle$$

$$\text{where } U \text{ satisfies } i\hbar \frac{\partial U(t_2, t_1)}{\partial t_2} = \hat{H}(t_2) U(t_2, t_1)$$

with  $\hat{H}$  a self-adjoint operator representing energy observable,

with  $t_i \in \mathbb{R}$  representing time,

$$\text{and } U(t_1, t_1) = \hat{1}$$



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$$\text{General sol}^n \quad U(t_2, t_1) = \underset{\substack{\uparrow \\ \text{time-ordering}}}{T} \exp\left(-i \int_{t_1}^{t_2} \frac{\hat{H}(t)}{\hbar} dt\right)$$

If  $H$  is time-independent

$$\text{then } U(t) = \exp\left(-i \frac{H}{\hbar} t\right)$$

• Postulate 3 implies



We can choose  
any basis  $\{|0\rangle, |1\rangle\}$   
 $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$

Example For an electron  
with spin- $\frac{1}{2}$   
Confined to 2D  
 $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$

Postulate 3 implies

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

outcome  $e_i$ , given preparation  $\hat{\rho} = |\psi\rangle\langle\psi|$ ,  
 $P_i(e_i) = \text{Tr}(\hat{\rho} \hat{P}_i)$



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Postulate 3 implies  
 $i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$   
 $\Rightarrow$  Schrödinger's eq<sup>n</sup>.

outcome  $a_i$ , given preparation  $\hat{\rho} = |\psi\rangle\langle\psi|$ ,  
 $\text{Pr}(a_i) = \text{Tr}(\hat{\rho} \hat{P}_i)$