

Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 13

Date: Sep 24, 2015 09:00 AM

URL: <http://pirsa.org/15090040>

Abstract:

Kruskal extension of the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} + r^2 d\Omega^2 \quad ; \quad r_s = \frac{2GM}{c^2}$$

- most general spherically symmetric solution of the vacuum Einstein equations $R_{\mu\nu} = 0$.

What happens at r_s ? For $r < r_s$, "r" is a timelike coordinate, "t" is spacelike.

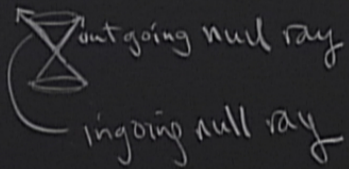
metric is singular
in those coordinates

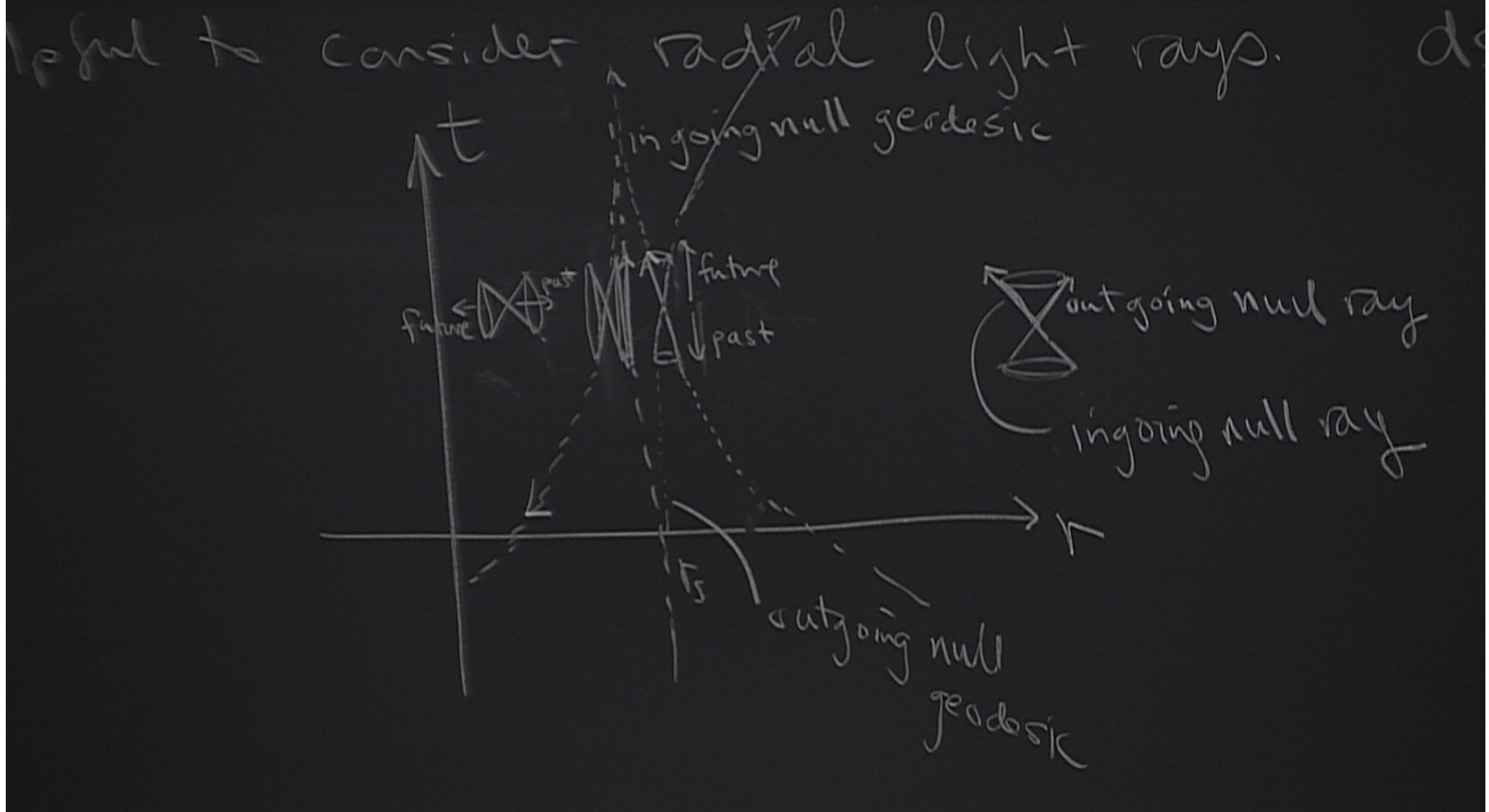
Outside b.hole metric is indept of t $\rightarrow$ static.

Inside b.hole, metric depends on r, $\Rightarrow$ not static.

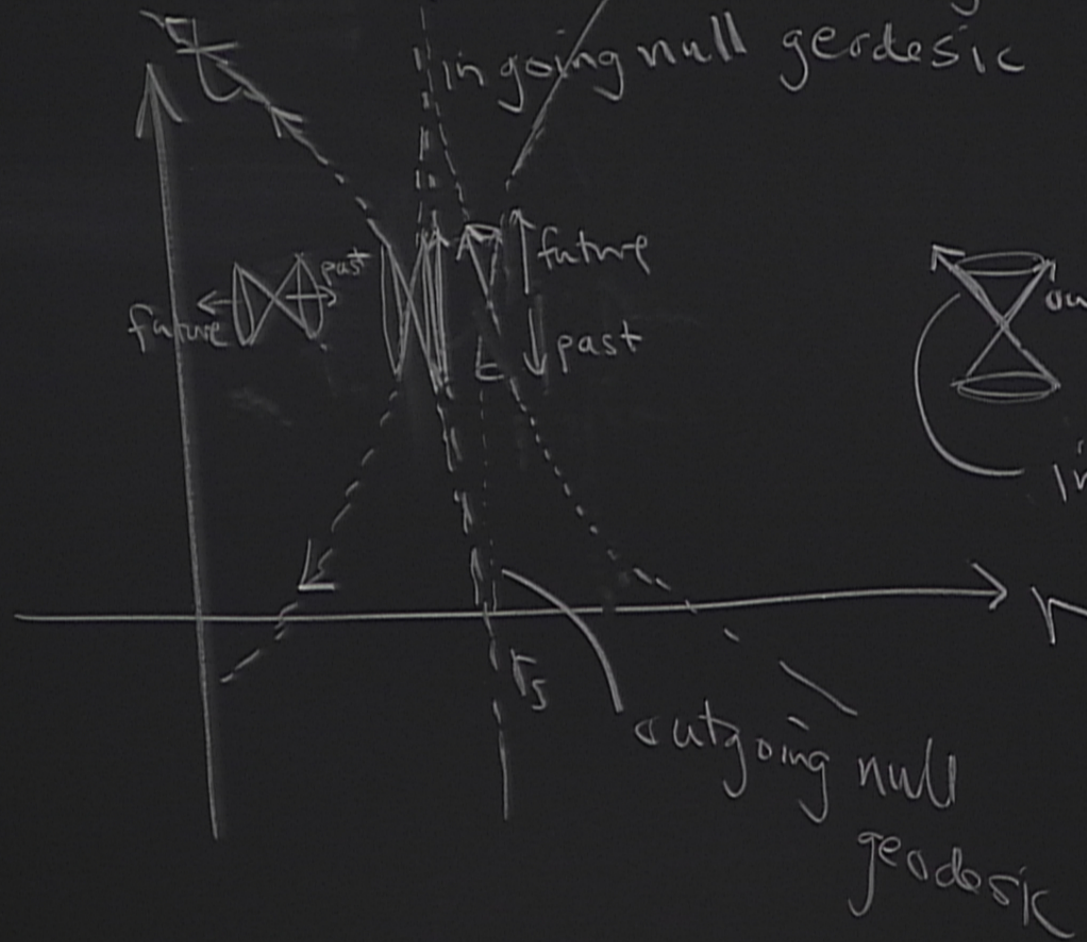
nc depends on r , $\Rightarrow$ not static

for radial light rays. $ds^2 = 0 \Rightarrow dt = \pm \frac{dr}{(1 - \frac{r_s}{r})} = \pm dr \left(1 + \frac{r_s}{r - r_s} \right)$





helpful to consider radial light rays.



$$= 0 \Rightarrow dt = \pm \frac{dr}{(1 - \frac{r_s}{r})} = \pm dr \left(1 + \frac{r_s}{r - r_s} \right)$$

integrate

$$t = \pm \left(r + r_s \ln \left(\frac{r}{r_s} - 1 \right) \right) + \text{const} \quad r > r_s$$

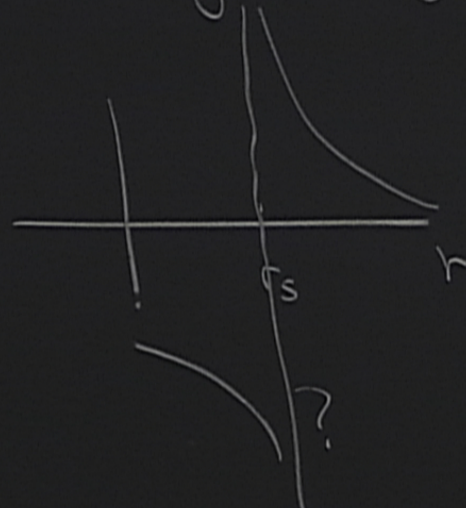
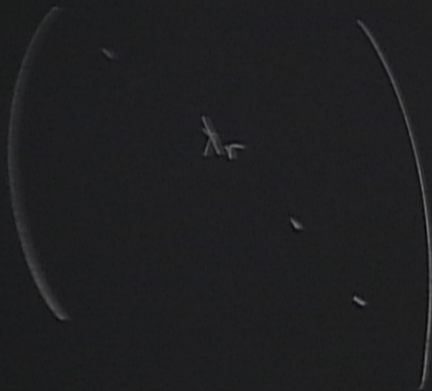
$$= \pm \left(r + r_s \ln \left(1 - \frac{r}{r_s} \right) \right) + \text{const} \quad r < r_s$$

Ingoing null ray ① $t = -r - r_s \ln \left(\frac{r}{r_s} - 1 \right) + \text{const} \rightarrow \infty$ as $r \rightarrow r_s$ from above

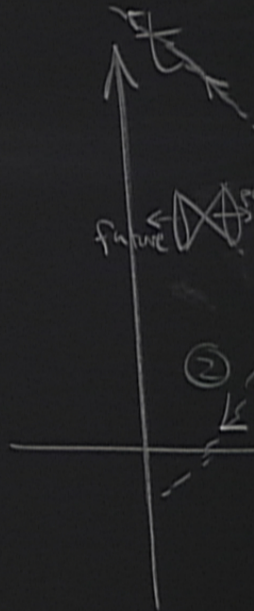
② $= -r - r_s \ln \left(1 - \frac{r}{r_s} \right) + \text{const} \rightarrow \infty$ as $r \rightarrow r_s$ from below

to study the geometry, helpful to consider

$$ds^2 = 0$$



? $g = \infty$
 $\Rightarrow$ cannot say
 whether
 $R_{\mu\nu} = 0$ here



Let us define $v = t + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$ $\leftarrow$ to ensure v is real
a new coordinate

$$dr = dt + \frac{dr}{1 - r_s/r} = 0 \text{ on ingoing rays.}$$

$$dt = dv - \frac{dr}{1 - r_s/r}$$

Let us define $v = t + r + r_s \ln |$
 a new coordinate

$$dv = dt + \frac{dr}{1 - r_s/r} = 0$$

$$\begin{aligned} & -\alpha dt^2 + \frac{dr^2}{\alpha} \\ & \parallel \\ & (dv - \alpha' dr) \\ & -\alpha dv^2 + 2dvdr - \frac{1}{\alpha} dr^2 + \frac{dr^2}{\alpha} \end{aligned}$$

$$dt = dv - \frac{dr}{1 - r_s/r}$$

$$\Rightarrow ds^2 = -dv^2 \left(1 - \frac{r_s}{r}\right) + 2dvdr + r^2 d\Omega^2$$

$$v = t + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$$

↖ to ensure v is real

$$dv = dt + \frac{dr}{1 - r_s/r} = 0 \text{ on ingoing rays.}$$

$$\frac{dr}{1 - r_s/r}$$

$$-dv^2 \left(1 - \frac{r_s}{r}\right) + 2dvdr + r^2 d\Omega_2^2$$

nonsingular at $r = r_s$!

$$\frac{dr}{1 - \frac{r_s}{r}}$$

$$-dr^2 \left(1 - \frac{r_s}{r}\right) + 2drdr + r^2 d\Omega^2$$

no problem!

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r^2 \\ r_s \sin^2 \theta \end{pmatrix}$$

nonsingular at $r = r_s$!

to ensure v is finite

$$dv = dt + \frac{dr}{1 - \frac{r_s}{r}} = 0 \text{ on ingoing rays.}$$

$$- \frac{dr}{1 - \frac{r_s}{r}}$$

$$= -dr^2 \left(1 - \frac{r_s}{r}\right) + 2drdr + r^2 d\Omega^2$$

No problem!

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_s^2 \\ r_s^2 \sin^2 \theta \end{pmatrix}$$

Eddington-Finkelstein
nonsingular at $r = r_s$!

— still singular at $r = 0$

$$\alpha = 1 - \frac{r_s}{r}$$

$$\det = 1$$

$$\begin{pmatrix} \frac{r_s}{r} - 1 & 1 \\ 1 & 0 \end{pmatrix}$$

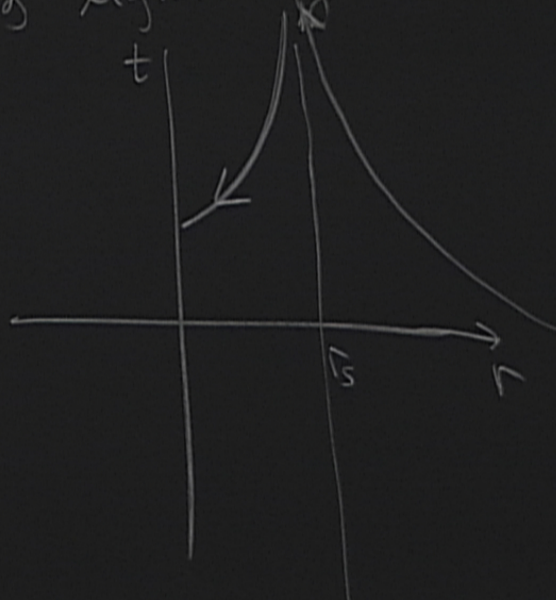
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

no problem!

- still singular

Ingoing light ray in E-F coords : $v = \text{const} \rightarrow dv = 0$

- determines the arrow of time



$\det = 1$

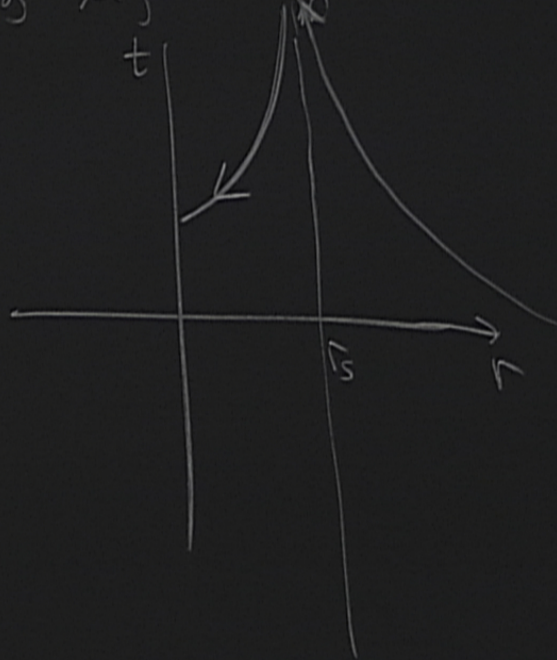
Ingoing light ray in E-F coords : $u = \text{const}$

- determines the arrow of

$u = \text{const}$

$$= t + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$$

$$t(r) = u - r - r_s \ln \left| \frac{r}{r_s} - 1 \right|$$



null rays,

define $u = t - \left(r + r_s \ln \left| \frac{r}{r_s} - 1 \right| \right)$: $du = 0$ on outgoing

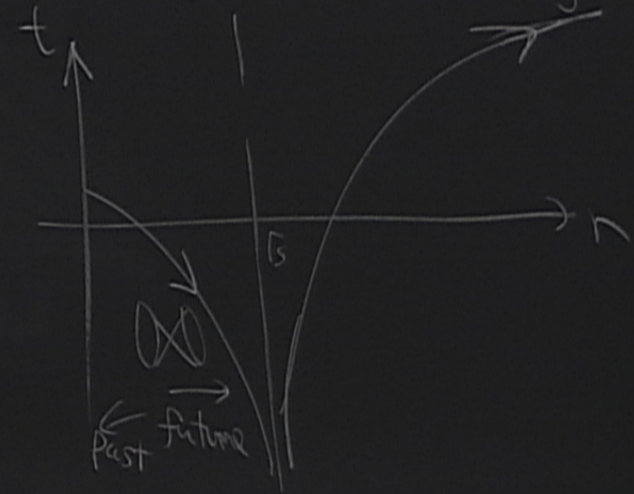
$$ds^2 = - \left(1 - \frac{r_s}{r} \right) du^2 - 2 du dr + r^2 d\Omega^2 \quad \text{also nonsingular}$$

ing null rays.

define $u = t - \left(r + r_s \ln \left| \frac{r}{r_s} - 1 \right| \right)$: $du = 0$ on out

$ds^2 = - \left(1 - \frac{r_s}{r} \right) du^2 - 2 du dr + r^2 d\Omega_2^2$ also nonsingular

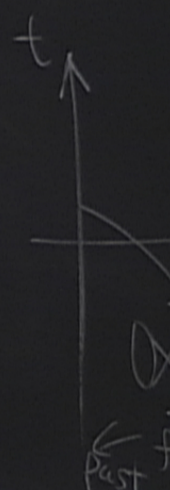
$u = \text{const} \Rightarrow t = u + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$



$$ds^2 = -\left(1 - \frac{r_s}{r}\right) du^2 - 2 du dr + r^2 d\Omega^2$$

$$u = \text{const} \Rightarrow t = u + r + r_s \ln \left| \frac{r}{r_s} - 1 \right|$$

$\Rightarrow$ Outgoing and ingoing photons disagree on the arrow of time?

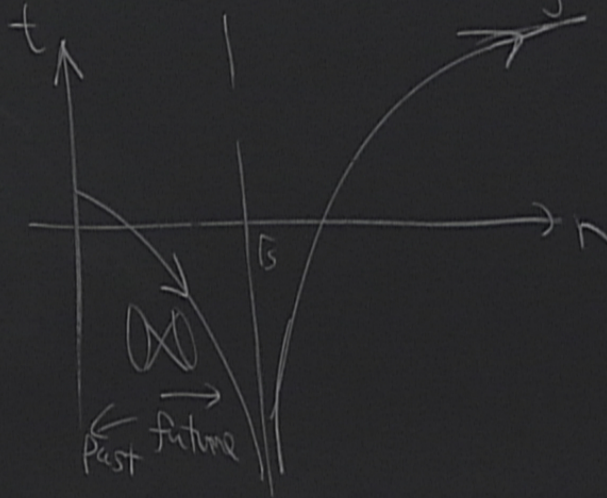


$$u = t - \left(r + r_s \ln \left| \frac{r}{r_s} - 1 \right| \right) \quad ; du = 0 \text{ on outgoing null rays}$$

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) du^2 - 2 du dr + r^2 d\Omega_2^2 \quad \text{also nonsingular}$$

$$u = t - \left(r + r_s \ln \left| \frac{r}{r_s} - 1 \right| \right)$$

as disagree on
?



sions of region $r < r_s$, each with its own arrow of time.

$$\left. \begin{aligned} &= t + r + r_s \ln\left(\frac{r}{r_s} - 1\right) \\ &= t - r - r_s \ln\left(\frac{r}{r_s} - 1\right) \end{aligned} \right\} * \quad \frac{dr}{du} = \begin{cases} dt + \frac{dr}{1 - r_s/r} \end{cases}$$

$$s^2 = -\left(1 - \frac{r_s}{r}\right) du^2 + r^2 d\Omega_2^2$$

$$\left(\frac{r}{r_s} - 1\right) = \frac{r - u}{2r_s} \Rightarrow \frac{r}{r_s} - 1 = e^{-r/r_s} e^{\frac{r-u}{2r_s}} \Rightarrow 1 - \frac{r}{r_s} = \frac{r}{r_s} e^{-r/r_s} e^{\frac{r-u}{2r_s}}$$

→ clue that there are two different versions of region $r > r_s$

Kruskal
PR 119, 1743 (1960)

For $r > r_s$ define

$$v = t + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$$

$$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$$

$$\Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) du dv + r^2 d\Omega^2$$

$$\text{from } *, \quad r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \frac{v - u}{2} \Rightarrow \frac{r}{r_s} - 1 = e^{\frac{v - u}{2r_s}}$$

versions of region $r < r_s$, each with its own arrow of time.

$$\begin{aligned} v &= t + r + r_s \ln\left(\frac{r}{r_s} - 1\right) \\ u &= t - r - r_s \ln\left(\frac{r}{r_s} - 1\right) \end{aligned} \quad * \quad \left. \vphantom{\begin{aligned} v &= t + r + r_s \ln\left(\frac{r}{r_s} - 1\right) \\ u &= t - r - r_s \ln\left(\frac{r}{r_s} - 1\right) \end{aligned}} \right\} \quad \frac{dr}{du} = \begin{cases} dt \pm \frac{dr}{1 - \frac{r_s}{r}} \end{cases}$$

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) du dv + r^2 d\Omega_2^2$$

$$s \ln\left(\frac{r}{r_s} - 1\right) = \frac{v - u}{2} \Rightarrow \frac{r}{r_s} - 1 = e^{-\frac{r}{r_s}} e^{\frac{v - u}{2r_s}} \Rightarrow 1 - \frac{r_s}{r} = \frac{r}{r_s} e^{-\frac{r}{r_s}} e^{\frac{v - u}{2r_s}}$$

$$\frac{-u}{2r} du dr + r^2 d\Omega^2$$

$$= r e^{-\frac{u}{2r}}, V = r e^{+\frac{v}{2r}} \quad) \quad du dv = \left(-\frac{1}{2} e^{-\frac{u}{2r}} du \right) \left(\frac{1}{2} e^{+\frac{v}{2r}} dv \right)$$

$$\Rightarrow ds^2 = - \frac{r_s}{r} e^{-\frac{r}{r_s}} \underbrace{e^{\frac{v-u}{2r_s}} du dv}_{\substack{u = r_s e^{-\frac{u}{2r_s}}, \\ v = r_s e^{+\frac{v}{2r_s}}}} + r^2 d\Omega_2^2$$

$$ds^2 = + 4 \frac{r_s}{r} e^{-\frac{r}{r_s}} du dv + r^2 d\Omega^2$$

$$\frac{u}{r_s} du dr + r^2 d\Omega^2$$

$$r_s e^{-\frac{u}{2r_s}}, V = r_s e^{+\frac{v}{2r_s}} \quad) \quad du dv = \left(-\frac{1}{2} e^{-\frac{u}{2r_s}} du \right) \left(\frac{1}{2} e^{+\frac{v}{2r_s}} dv \right)$$

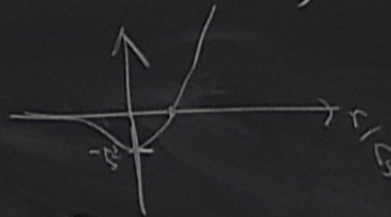
$$dV + r^2 d\Omega^2$$

fishy! (but had to be!)

Past lecture !!!

$$ds^2 = (-dT^2 + dX^2) \frac{4r_s}{r} e^{-r/r_s} + r^2 d\Omega_z^2$$

$$U.V = X^2 - T^2 = r_s^2 e^{\frac{V-U}{2r_s}} = r_s^2 e^{r/r_s} \left(\frac{r}{r_s} - 1 \right) \Rightarrow \text{defines } r \text{ in terms of } X \text{ and } T.$$



finally, set

$$U = X - T$$

$$V = X + T$$

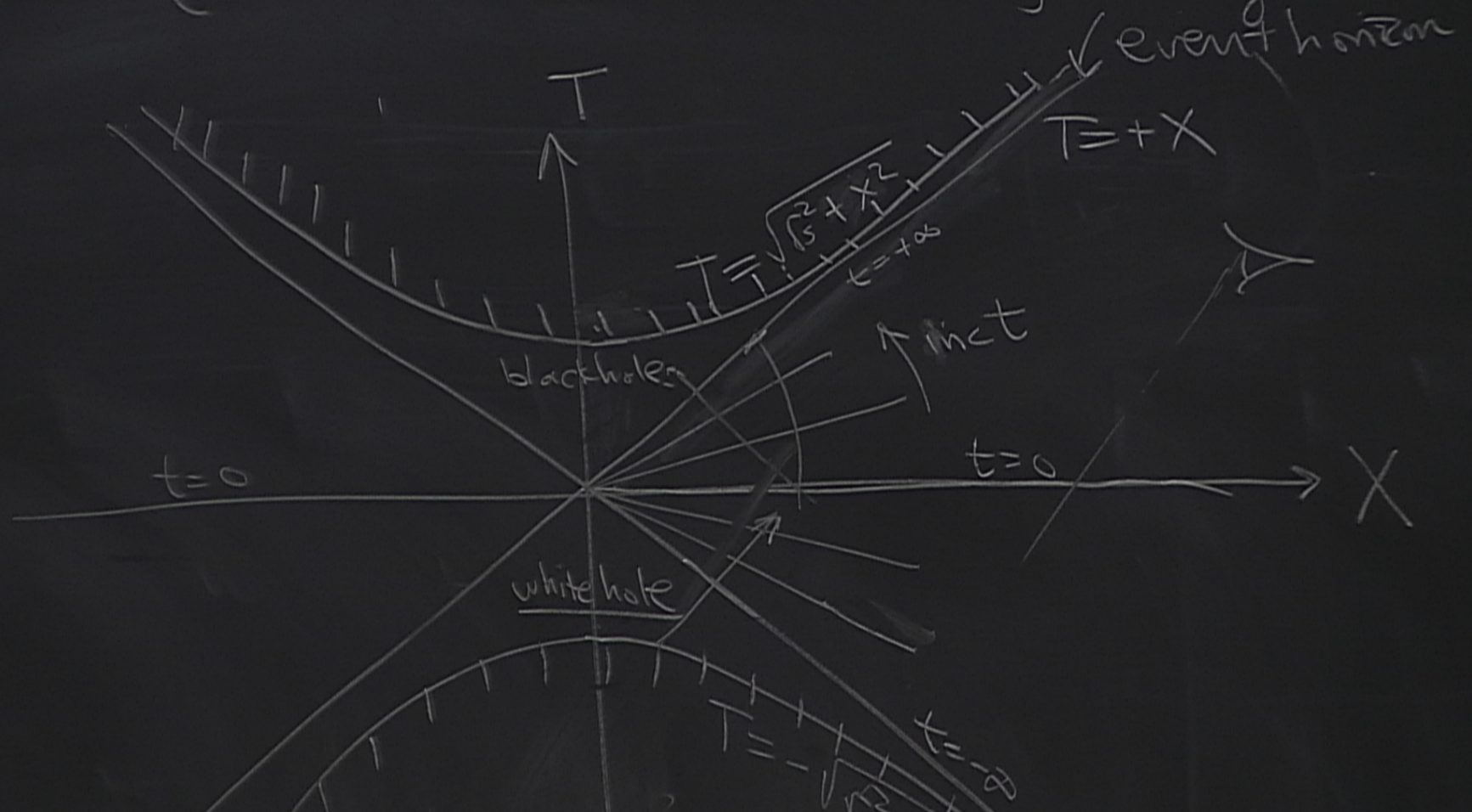
$$ds^2 = (-dT^2 + dX^2) \frac{4r_s}{r} e^{-\frac{r}{r_s}}$$

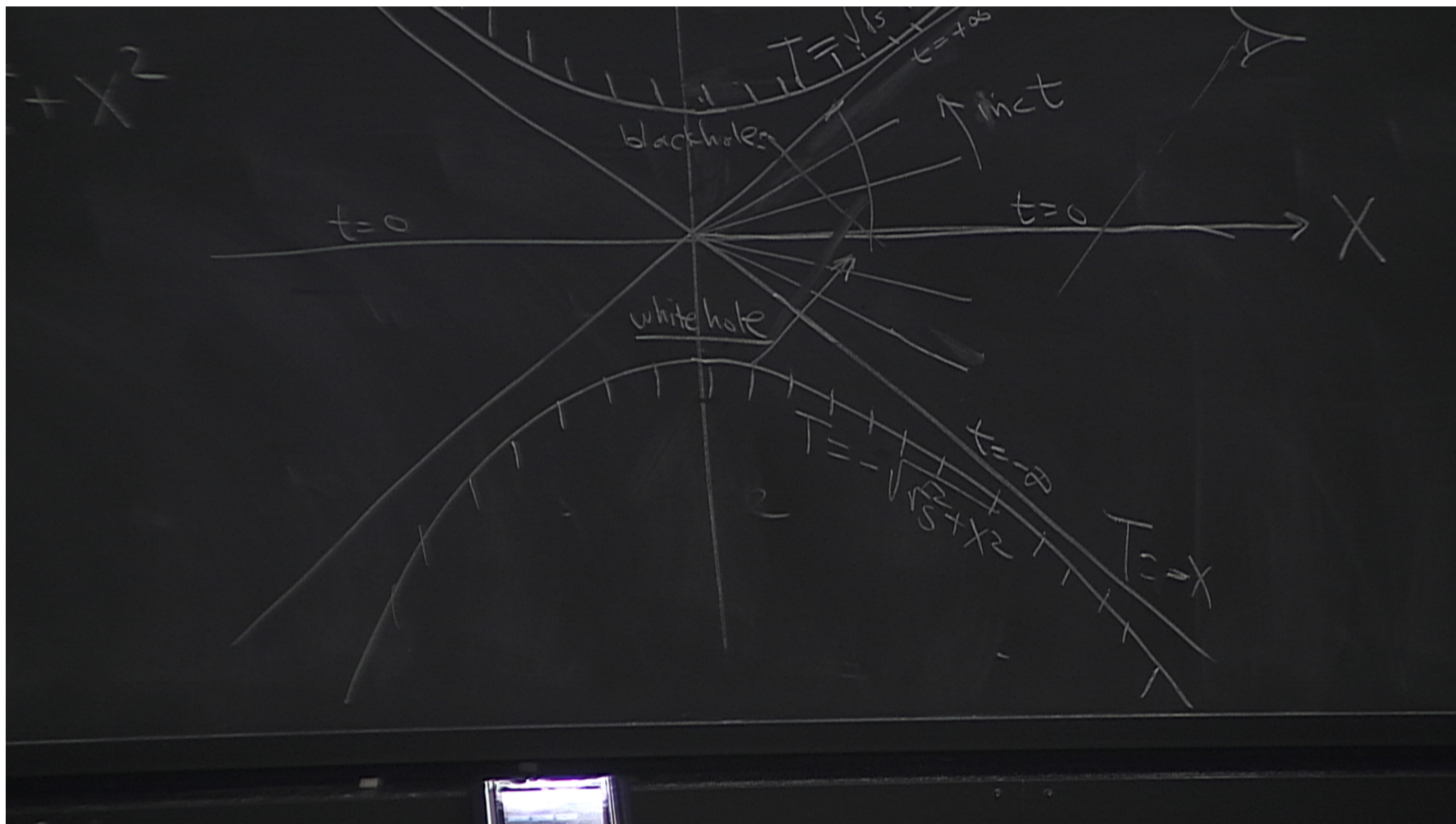
$$U \cdot V = X^2 - T^2 = r_s^2 e^{\frac{V-U}{2r_s}} = r$$

$$\left[\begin{aligned} t = \frac{u+v}{2} &= r_s \ln\left(\frac{V}{U}\right) = r_s \ln\left(\frac{X+T}{X-T}\right) \Rightarrow \text{defines } t(X, T) \\ &\Rightarrow T = X \tanh\left(\frac{t}{2r_s}\right) \end{aligned} \right.$$

($r=0$ is a real singularity: $R_{\mu\nu\rho\sigma} \sim \frac{1}{r^3}$)

($r=0$ is a real singularity:





Collapse of a star

Eddington-Finkelstein

recall $ds^2 = -\left(1 - \frac{r_s}{r}\right) dv^2 + 2dvdr + r^2 d\Omega_2^2$

define $t' = v - r$ as a new "time" coordinate

$$dv = dt' + dr$$

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt'^2 + \left(1 + \frac{r_s}{r}\right) dr^2 + \frac{2r_s}{r} dr dt' + r^2 d\Omega_2^2$$

$$= dt' + dr$$

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt'^2 + \left(1 + \frac{r_s}{r}\right) dr^2 + \frac{2r_s}{r} dr dt'$$

$\Rightarrow$ radial light rays: $\frac{dt'}{dr} = \frac{r+r_s}{r-r_s}$

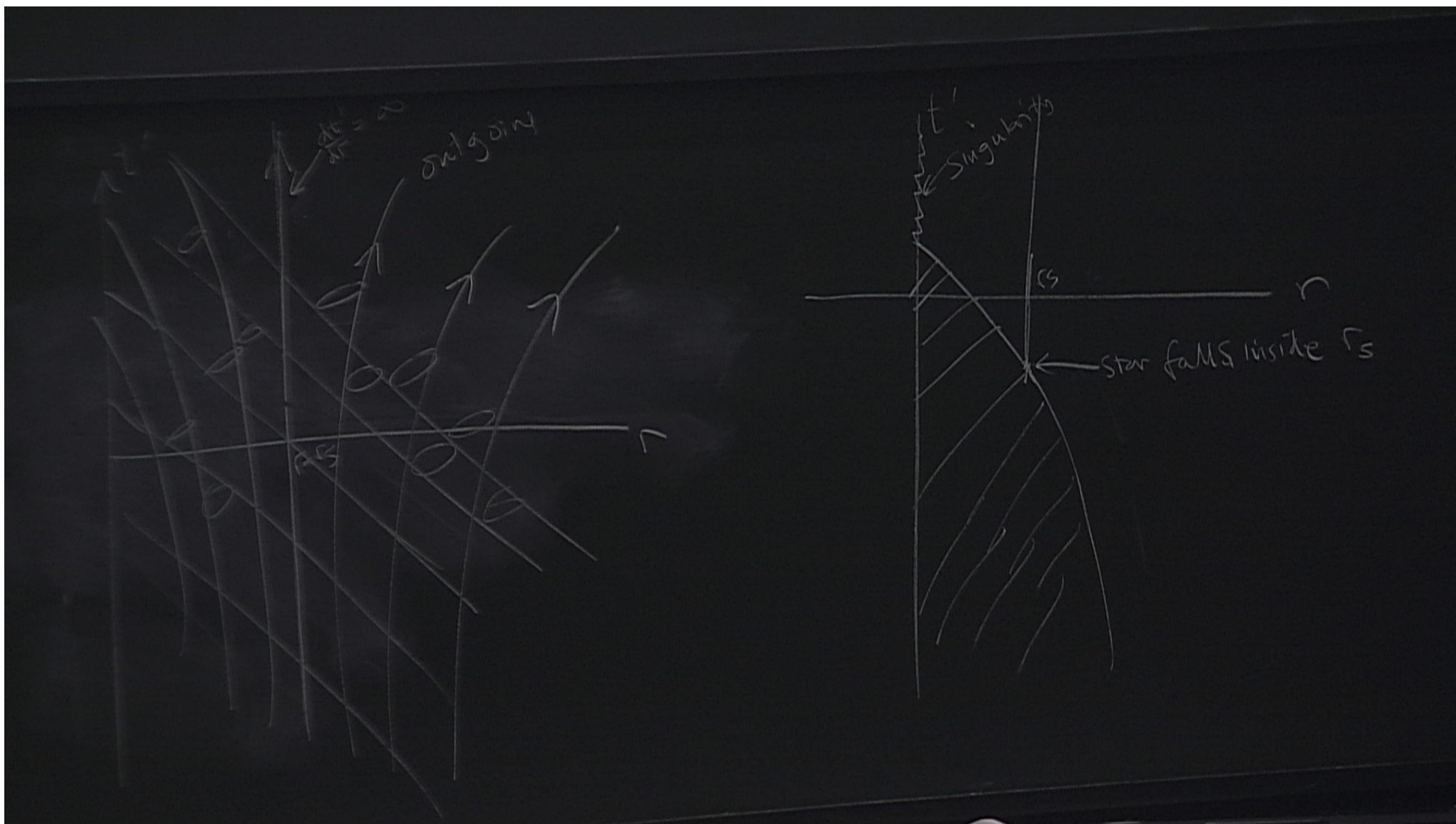
in "time" coordinate

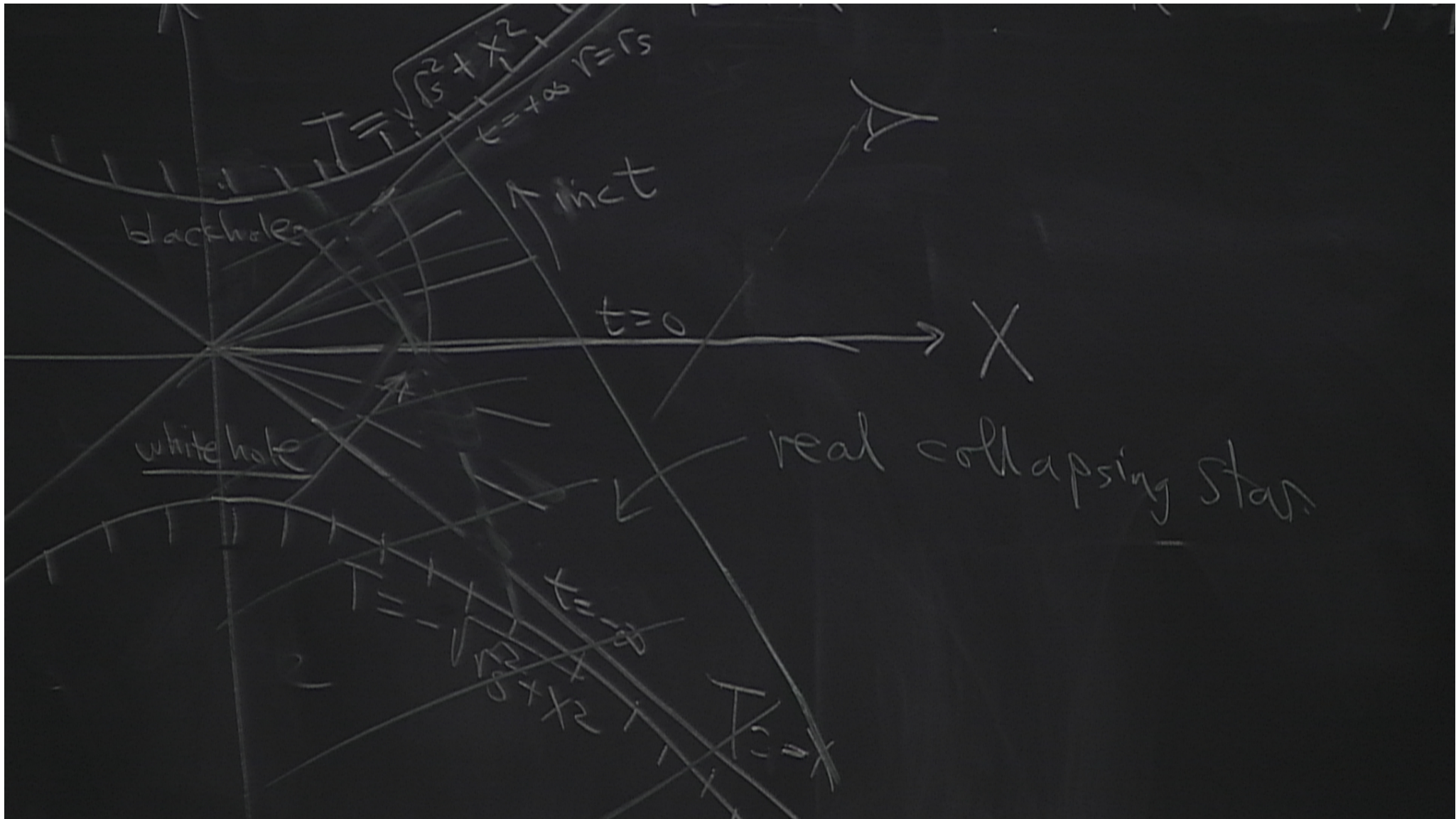
$$-dt'^2 + \left(1 + \frac{r_s}{r}\right) dr^2 + \frac{2r_s}{r} dr dt' + r^2 d\Omega^2$$

$$dr = -dt' \text{ ingoing}$$

rays

$$dr = dt' \left(\frac{r - r_s}{r + r_s} \right) \text{ outgoing}$$

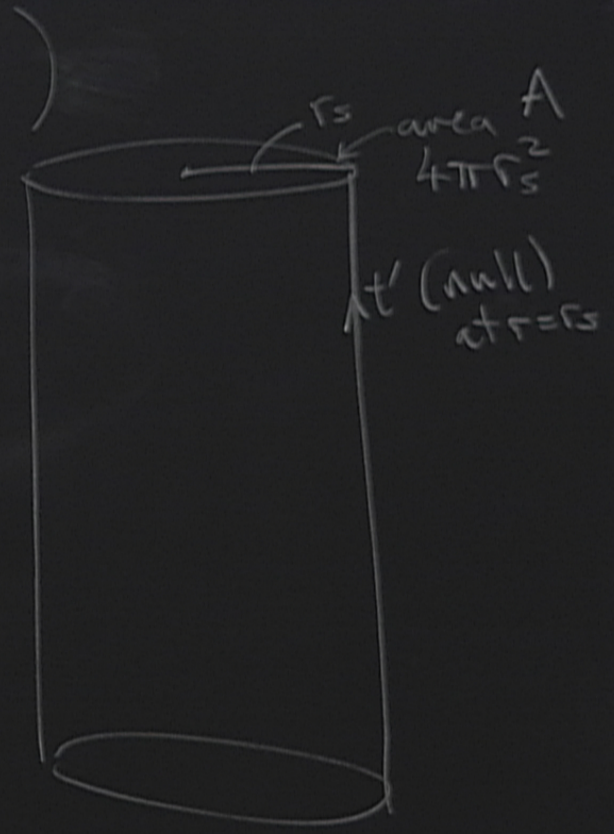




$$R_{\mu\nu\rho\lambda} \sim \frac{1}{r^3} \quad (r \rightarrow \infty)$$

$$R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} = \frac{1}{r^6}$$

X
collapsing star



Area of event horizon $A = 4\pi \left(\frac{2GM}{c^2} \right)^2$

Bekenstein-Hawking: entropy of a black hole $S = \frac{k_B}{4} \frac{A}{l_{Pl}^2} \propto M^2.$

$$l_{Pl} = \sqrt{\frac{G\hbar}{c^3}}$$

Area of event horizon $A = 4\pi (2r_s)^2$

$-dt^2 \left(1 - \frac{r_s}{r}\right) + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} + r^2 d\Omega^2$ Bekenstein-Hawking: entropy of a black hole



Area of event horizon $A = 4\pi ($

$-dt^2\left(1 - \frac{r_s}{r}\right) + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} + r^2 d\Omega^2$ Bekenstein-Hawking: entropy of

observer at $r = \infty$ use t as proper time

observer falling in has proper time τ : $d\tau = dt\sqrt{1 - \frac{r_s}{r}}$
 $\Rightarrow dt = \frac{d\tau}{\sqrt{1 - \frac{r_s}{r}}}$

$$A = 4\pi \left(\frac{2GM}{c^2} \right)^2$$

$$v = t + r + r_s \ln \left(\frac{r}{r_s} - 1 \right)$$

$$t' = t + r_s \ln \left(\frac{r}{r_s} - 1 \right)$$

entropy of a black hole $S = \frac{k_B}{4} \frac{A}{l_{Pl}^2} \propto M^2$

$$l_{Pl} = \sqrt{\frac{G\hbar}{c^3}}$$

$$\frac{r_s}{r}$$

$$dt = \frac{dr}{\sqrt{1 - r_s/r}}$$



$$\tau = \int_0^{r_s} \frac{dr}{\sqrt{1 - \frac{r_s}{r}}}$$

Area of event horizon $A = 4\pi$

$$-dt^2 \left(1 - \frac{r_s}{r}\right) + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} + r^2 d\Omega^2$$

Bekenstein-Hawking: entropy

observer at $r = \infty$ use t as proper time

observer falling in has proper time τ : $d\tau = dt \sqrt{1 - \frac{r_s}{r}}$

$$\Rightarrow dt = \frac{d\tau}{\sqrt{1 - \frac{r_s}{r}}}$$