Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 12

Date: Sep 23, 2015 09:00 AM

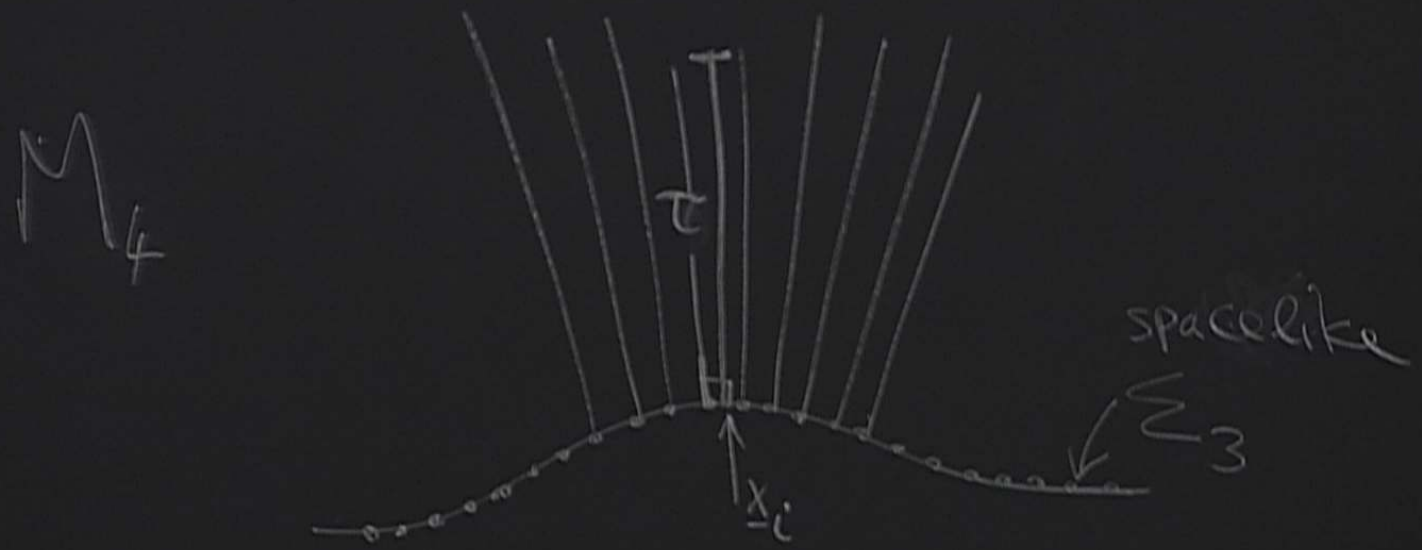
URL: http://pirsa.org/15090039

Abstract:

Synchronous coordinates

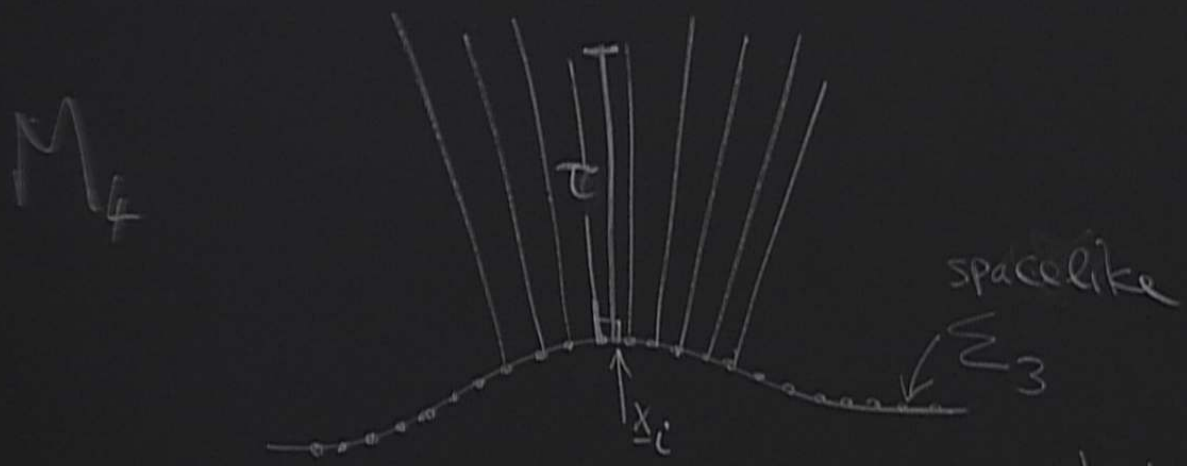
M_4

Synchronous coordinates

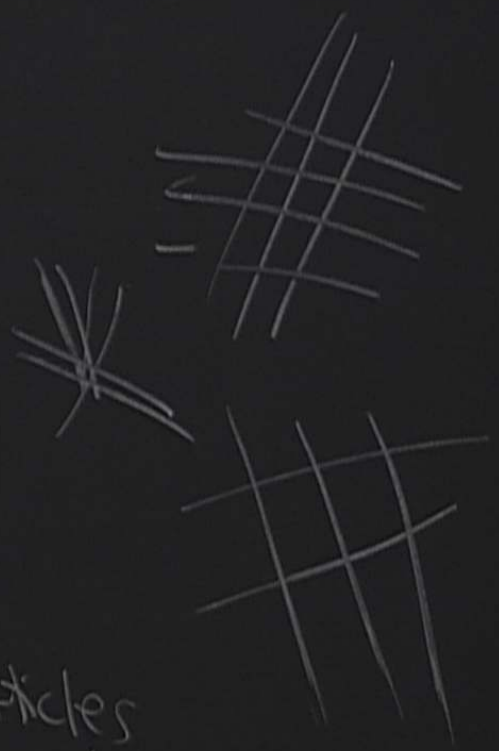


$$\Rightarrow x^\mu = (\tau, x_i)$$

Synchronous coordinates



$\Rightarrow x^\mu = (\tau, \underline{x}_i)$

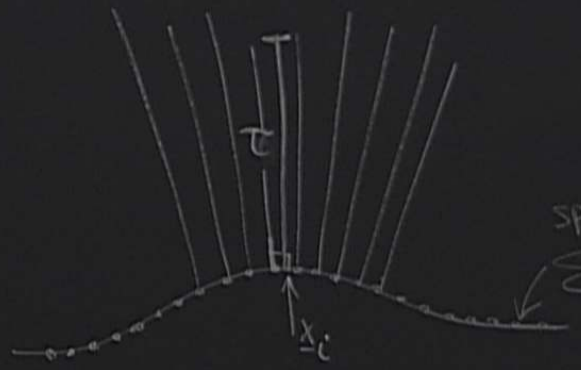


- bad if particles intersect!

Synchronous coordinates

M_4

$\Rightarrow g_{0\mu} = -\delta_{0\mu}$



$\Rightarrow x^\mu = (\tau, \underset{\sim}{x}_i)$

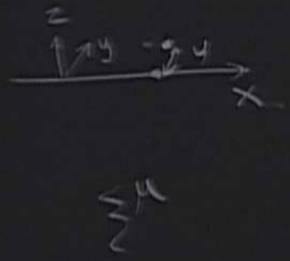
- bad if particles intersect!

g-wave in locally inertial coordinates

$$-dt^2 + dx^2 + dy^2(1 + a(x-t)) + dz^2(1 - a(x-t))$$

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu} \implies h'_{\mu\nu} = h_{\mu\nu} - \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}$$

g-wave in locally inertial coordinates

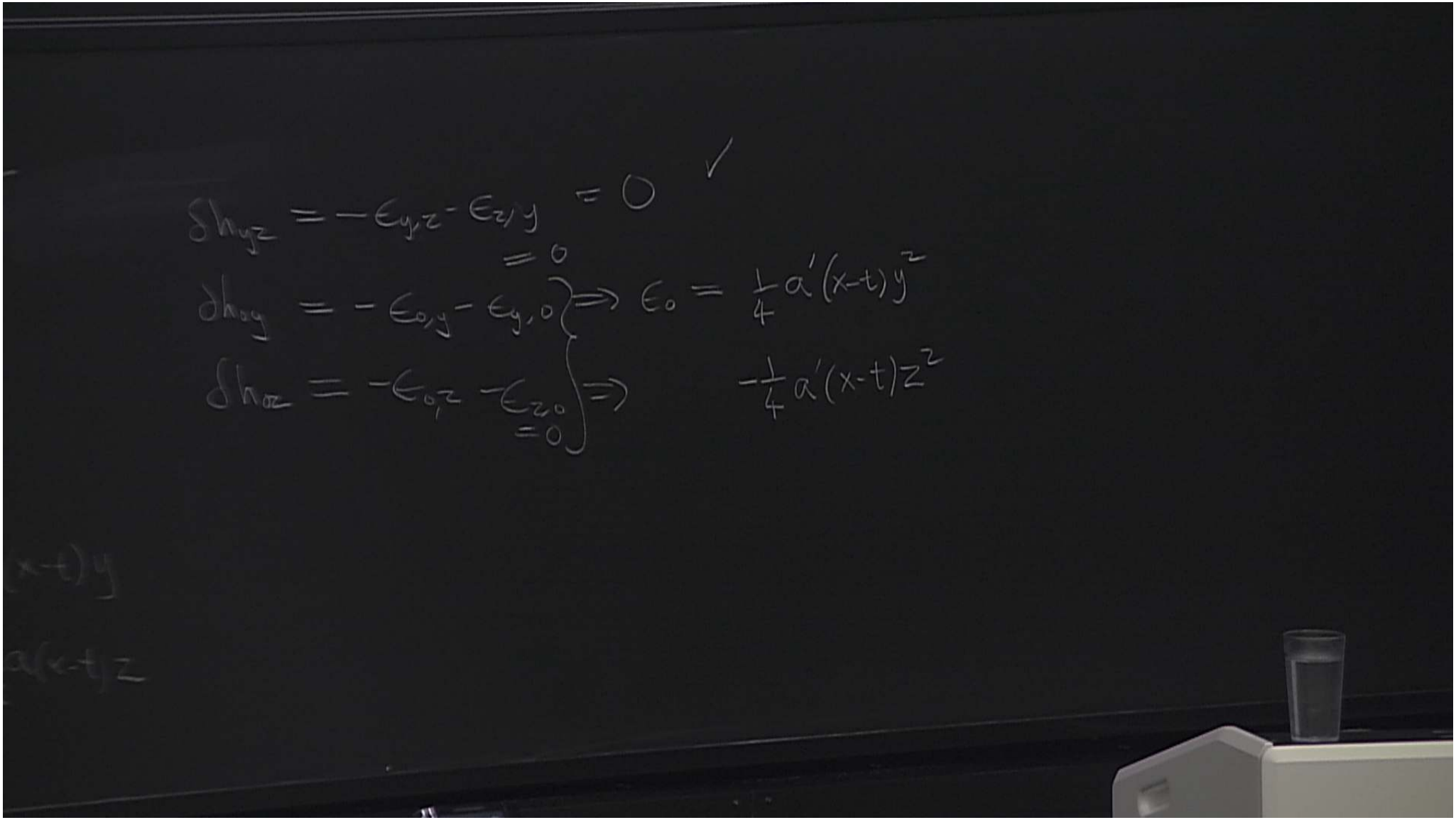


$$-dt^2 + dx^2 + dy^2(1 + a(x-t)) + dz^2(1 - a(x-t))$$

$$x'^\mu = x^\mu + \epsilon^\mu(x) \Rightarrow h'_{\mu\nu} = h_{\mu\nu} - \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}$$

$\Rightarrow$ flat space + small corrections.

$$\delta h_{yy} = -2\epsilon_{y,y} = -a(x-t) \Rightarrow \epsilon_y = \frac{1}{2}a(x-t)\,y$$


$\delta h_{yz} = -\epsilon_{y,z} - \epsilon_{z,y} = 0$ ✓
$= 0$
$\delta h_{xy} = -\epsilon_{0,y} - \epsilon_{y,0}$ $\Rightarrow \epsilon_0 = \frac{1}{4} a'(x-t) y^2$
$\delta h_{xz} = -\epsilon_{0,z} - \epsilon_{z,0}$ $\Rightarrow$ $-\frac{1}{4} a'(x-t) z^2$
$= 0$

$$\delta h_{yz} = -\epsilon_{y,z} - \epsilon_{z,y} = 0 \quad \checkmark$$
$$= 0$$

$$\delta h_{0y} = -\epsilon_{0,y} - \epsilon_{y,0} \Rightarrow \epsilon_0 = \frac{1}{4} a'(x-t) y^2$$

$$\delta h_{0z} = -\epsilon_{0,z} - \underbrace{\epsilon_{z,0}}_{=0} \Rightarrow \quad -\frac{1}{4} a'(x-t) z^2$$

Similarly $\epsilon_x = -\frac{1}{4} a'(x-t) y^2 + \frac{1}{4} a'(x-t) z^2$

$\Rightarrow$ all components of $h_{\mu\nu}$ are $o(y^2)$ or $o(z^2)$ or zero.

$$-dt^2 + dx^2 + dy^2\,(1 + a(x-t)) + dz^2\,(1 - a(x-t))$$

$$x'^\mu = x^\mu + \epsilon^\mu(x) \Rightarrow \quad h'_{\mu\nu} = h_{\mu\nu} - \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}$$

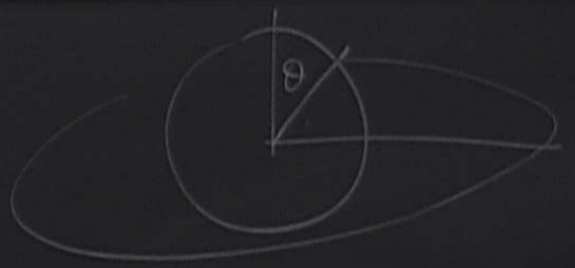
$\Rightarrow$ flat space + small corrections

$$\delta h_{yy} = -2\epsilon_{y,y} = -a(x-t) \Rightarrow \epsilon_y = \tfrac{1}{2}a(x-t)\,y$$

$$\delta h_{zz} = -2\epsilon_{z,z} = +a(x-t) \Rightarrow \epsilon_z = -\tfrac{1}{2}a(x-t)\,z$$

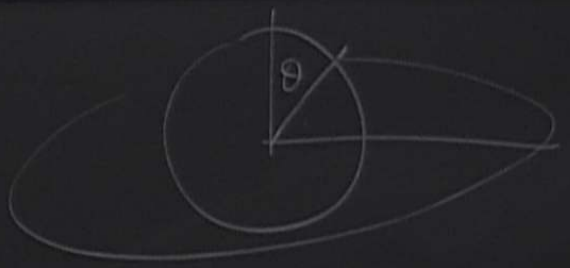
$$y' = \left(1 + \tfrac{1}{2}a(x-t)\right) y$$

Particle in Schwarzschild



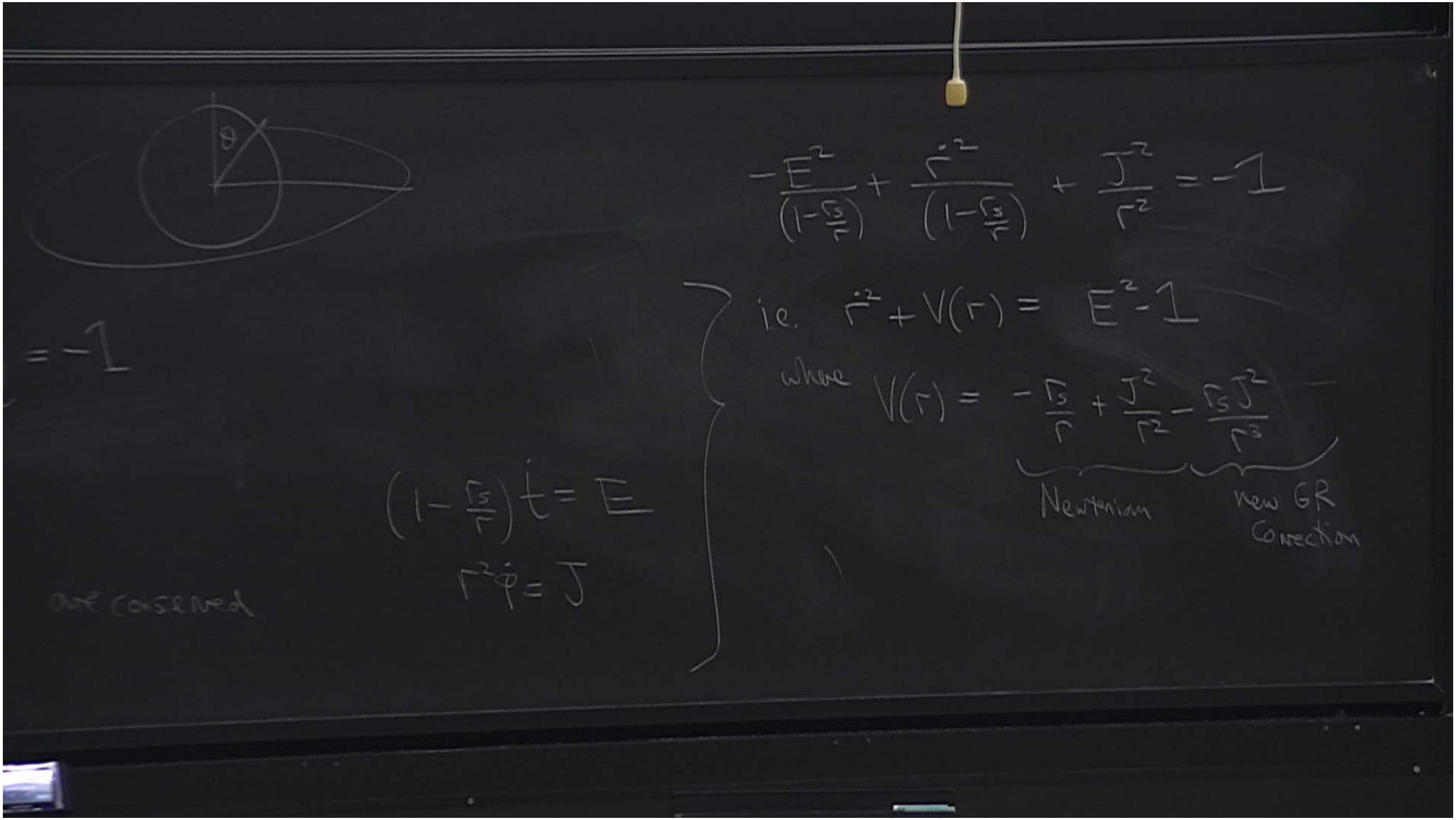
$$-\left(1-\frac{r_s}{r}\right)\dot{t}^2+\left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2+r^2\underbrace{\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right)}_{\theta=\pi/2 \quad r^2\dot{\varphi}^2}=-1$$

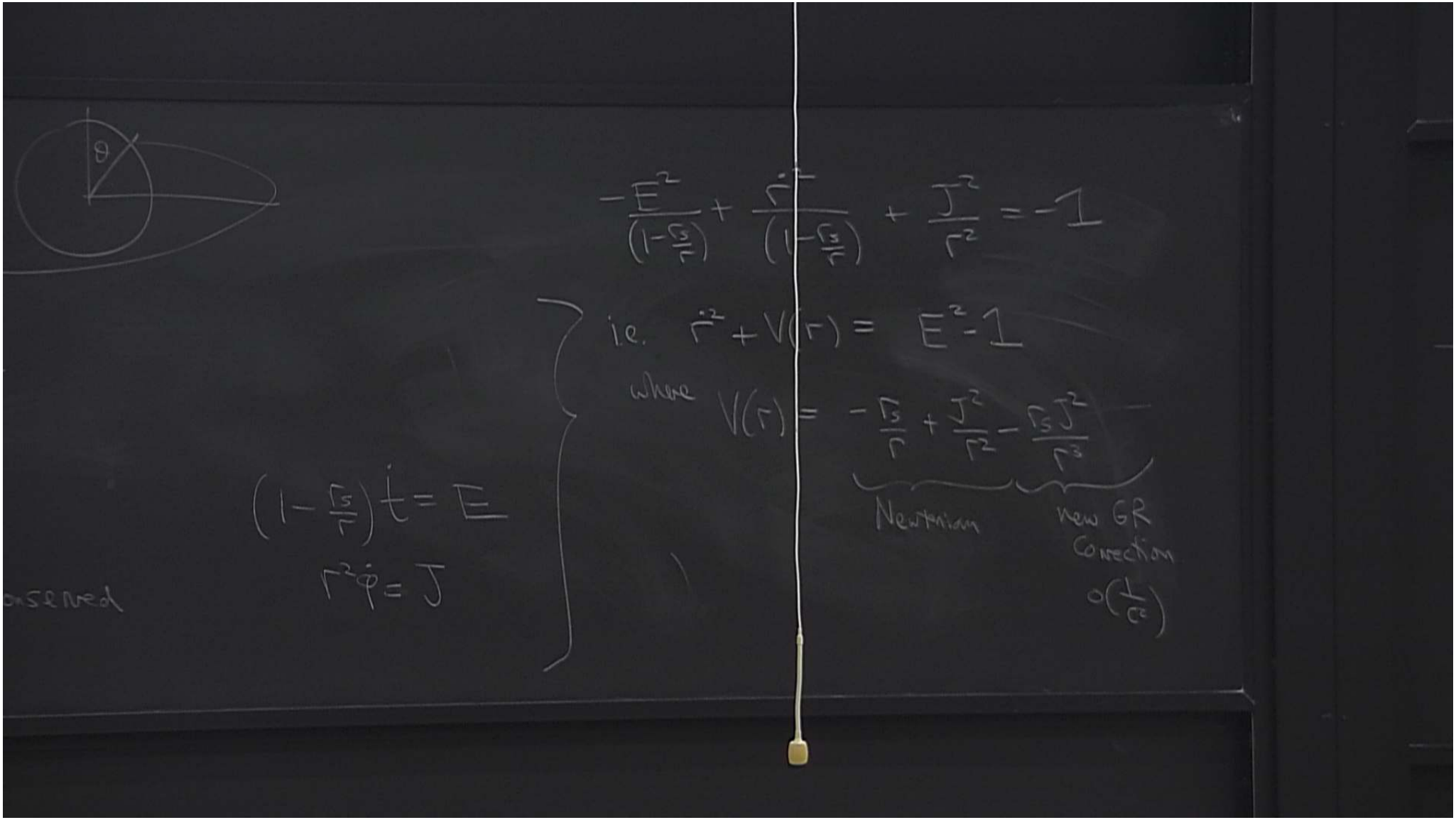
Particle in Schwarzschild



$$-\left(1-\frac{r_s}{r}\right)\dot{t}^2+\left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2+r^2\underbrace{\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right)}=-1$$

Convenient to choose $\theta=\pi/2$ $\quad r^2\dot{\varphi}^2$


θ
$-\frac{E^2}{(1-\frac{r_s}{r})} + \frac{\dot{r}^2}{(1-\frac{r_s}{r})} + \frac{J^2}{r^2} = -1$
$= -1$
i.e. $\dot{r}^2 + V(r) = E^2 - 1$
where $V(r) = -\frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3}$
Newtonian
new GR correction
$(1-\frac{r_s}{r})\dot{t} = E$
$r^2\dot{\varphi} = J$
are conserved


θ
$-\frac{E^2}{(1-\frac{r_s}{r})} + \frac{\dot{r}^2}{(1-\frac{r_s}{r})} + \frac{J^2}{r^2} = -1$
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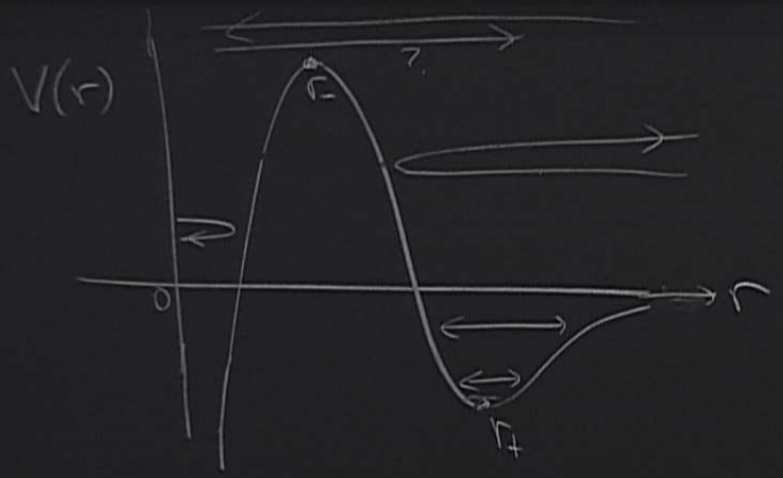
Newtonian
new GR correction
$O(\frac{1}{c^2})$
$(1-\frac{r_s}{r})\dot{t} = E$
$r^2\dot{\varphi} = J$
onserved

$= E$ $= L$



$$\frac{\partial V}{\partial r} = \frac{r_s}{r^2} - 2\frac{J^2}{r^3} + 3r_s\frac{J^2}{r^4} = 0$$

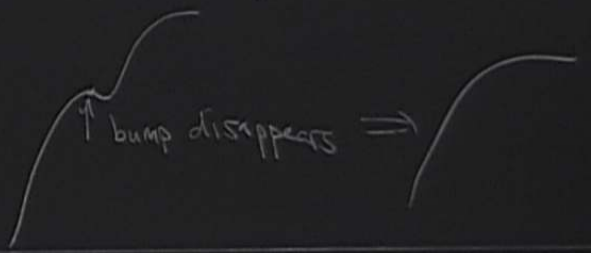
$$\Rightarrow r_\pm = \frac{J^2}{r_s}\left(1 \pm \sqrt{1 - 3\frac{r_s^2}{J^2}}\right)$$

as you decrease J, potential changes shape

$J = \sqrt{3}r_s$

$r_+ = 3r_s$

Innermost stable circular orbit.



$\delta h_{yy} = -2\epsilon_{y,y} = -a(x-t) \Rightarrow \epsilon_y = \frac{1}{2}a(x-t)y$

$\delta h_{zz} = -2\epsilon_{z,z} = +a(x-t) \Rightarrow \epsilon_z = -\frac{1}{2}a(x-t)z \Rightarrow$ all components of $h_{\mu\nu}$ are $o(y^2)$, or $o(z$

$y' = \left(1 + \frac{1}{2}a(x-t)\right)y$

Einstein: what is the effect of the GR correction * on planetary orbits

(* down by $\frac{1}{c^2}$)

$$\dot{r}^2 - \frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3} = \text{const}$$

Solve for $r(\varphi)$: $' = \frac{d}{d\varphi}$

$$r'^2 = \frac{\dot{r}^2}{\dot{\varphi}^2} = \frac{\dot{r}^2 r^4}{J^2} \rightarrow \frac{r'^2}{r^4} - \frac{r_s}{J^2 r} + \frac{1}{r^2} - \frac{r_s}{r^3} = \text{const}$$

Set $u = \frac{1}{r}$ $\quad u'^2 - \frac{r_s}{J^2}u + u^2 - r_s u^3 = \text{const}$

$\delta h_{yy} = -2\epsilon_{y,y} = -a(x-t) \Rightarrow \epsilon_y = \frac{1}{2}a(x-t)y$ Sim'y $\epsilon_x = -\frac{1}{4}a'(x-t)y^2 + \frac{1}{4}a'(x-t)z^2$

$\delta h_{zz} = -2\epsilon_{z,z} = +a(x-t) \Rightarrow \epsilon_z = -\frac{1}{2}a(x-t)z$ $\Rightarrow$ all components of $h_{\mu\nu}$ are $o(y^2)$ or $o(z^2)$

$y' = \left(1 + \frac{1}{2}a(x-t)\right)y$

Einstein: what is the effect of the GR correction * on planetary orbits

$\dot{r}^2 - \frac{r_s}{r} + \frac{J^2}{r^2} - \frac{r_s J^2}{r^3} = \text{const}$ ← down by $\frac{1}{c^2}$

Solve for $r(\varphi)$: $' = \frac{d}{d\varphi}$

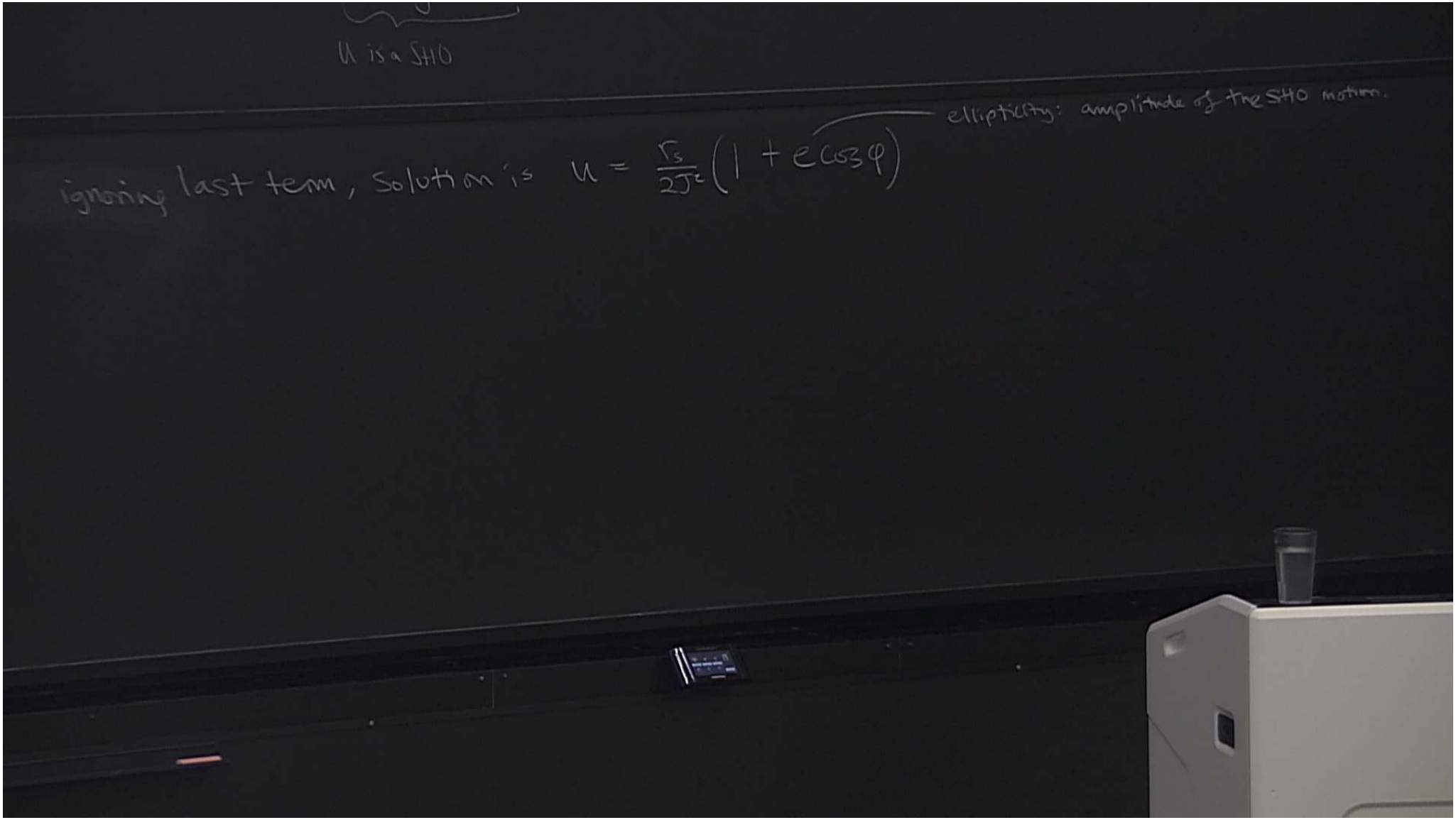
$r'^2 = \frac{\dot{r}^2}{\dot{\varphi}^2} = \frac{\dot{r}^2 r^4}{J^2}$ →

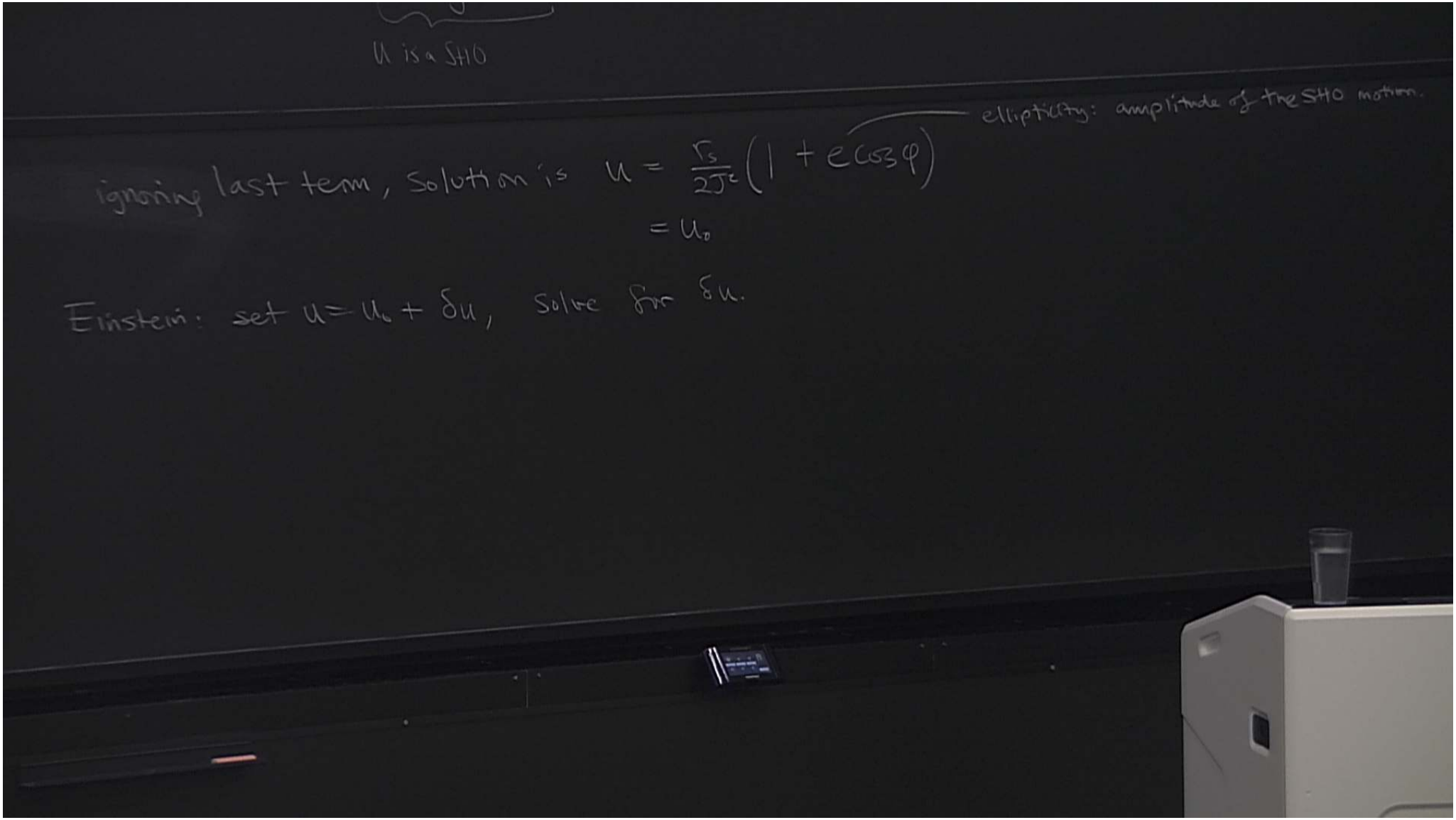
$\frac{r'^2}{r^4} - \frac{r_s}{J^2 r} + \frac{1}{r^2} - \frac{r_s}{r^3} = \text{const}$

Set $u = \frac{1}{r}$

$u'^2 - \frac{r_s}{J^2}u + u^2 - r_s u^3 = \text{const}$

u is a SHO


u is a SHO
ignoring last term, Solution is $u = \frac{r_s}{2J^2}(1 + e\cos\varphi)$
ellipticity: amplitude of the SHO motion.


u is a SHO
ignoring last term, solution is $u = \frac{r_s}{2J^2}(1 + e\cos\varphi) = u_0$
ellipticity: amplitude of the SHO motion.
Einstein: set $u = u_0 + \delta u$, solve for δu.

...ve for $r(\varphi)$:

$' = \frac{d}{d\varphi}$

$$r'^2 = \frac{\dot r^2}{\dot\varphi^2} = \frac{\dot r^2 r^4}{J^2} \rightarrow \underbrace{\frac{r'^2}{r^4} - \frac{r_s}{J^2 r} + \frac{1}{r^2} - \frac{r_s}{r^3}} = \text{const}$$

Set $u = \frac{1}{r}$

$$\neq \underbrace{u'^2 - \frac{r_s}{J^2}u + u^2}_{u \text{ is a SHO}} - r_s u^3 = \text{const}$$

ellipticity: am...

ignoring last term, solution is $u = \frac{r_s}{2J^2}\left(1 + e\cos\varphi\right)$

$= u_0$

$-\frac{x^2}{2}$

$+x=0$

Einstein: set $u = u_0 + \delta u$, solve for δu.

$$\frac{\frac{d}{d\varphi}(\#)}{u'}$$

$$\Longrightarrow \quad u'' = -\left(u - \frac{r_s}{2J^2}\right) + \frac{3r_s}{2}u^2 \quad \downarrow u_0$$

$u = u_0 + \delta u$

$$\delta u'' = -\delta u + \frac{3r_s}{2}\left(\frac{r_s}{2J^2}\right)^2\left(1 + e\cos\varphi\right)^2$$

Set $u = \frac{1}{r}$

$$\neq \underbrace{u'^2 - \frac{r_s}{J^2}u + u^2}_{u \text{ is a SHO}} - r_s u^3 = \text{const}$$

ignoring last term, solution is $u = \frac{r_s}{2J^2}(1 + e\cos\varphi)$ — ellipticity: amplitude of the SHO

$= u_0$

Einstein: set $u = u_0 + \delta u$, solve for δu.

$$\Rightarrow u'' = -\left(u - \frac{r_s}{2J^2}\right) + \frac{3r_s}{2}\underbrace{u^2}_{\downarrow u_0}$$

$u = u_0 + \delta u$

$$\delta u'' = -\delta u + \frac{3r_s}{2}\left(\frac{r_s}{2J^2}\right)^2(1 + e\cos\varphi)^2$$

Noether)

$P_t = g_{tt}\dot{t}$, $P_\varphi = g_{\varphi\varphi}\dot{\varphi}$ are conserved

$= E$ $\quad = J$

$r^2\dot{\varphi} =$ $\qquad r^2\dot{\varphi} = J$

Correction $o\left(\frac{1}{c^2}\right)$ *

Important term is the resonant one $\propto \cos\varphi$, this causes secular growth of δu.

$$\delta u'' = -\delta u + \frac{3}{4}\frac{r_s^3}{J^4} e\cos\varphi$$

$$\Rightarrow \delta u \sim \frac{3 r_s^3 e}{8J^4}\,\varphi \sin\varphi$$

secular growth

c.f. $\frac{r_s}{2J^2}\left(1 + e\cos(\varphi(1-\epsilon))\right)$

$= u_0 + \frac{r_s}{2J^2} e\epsilon\varphi\sin\varphi$

with $\epsilon = \frac{3r_s^2}{4J^2}$

rder

secular growth

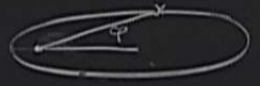
$= U_0 + \frac{\Gamma_3}{2J^2} e\epsilon\varphi \sin\varphi$

with ϵ

With ϵ correction, maxima/minima of u are separated not by $\Delta\varphi = 2\pi$, but by $\Delta\varphi$

Instead of ellipse

Newton



Einstein

⇒ advance of the perihelion $\Delta\varphi = 2$

$=$

for Mercury

$=$

known from astronomy to $\pm 1''$.

minima of u are separated not by $\Delta\varphi = 2\pi$, but by $\Delta\varphi = 2\pi(1+\epsilon)$

Einstein

$\Rightarrow$ advance of the perihelion $\Delta\varphi = 2\pi\epsilon$

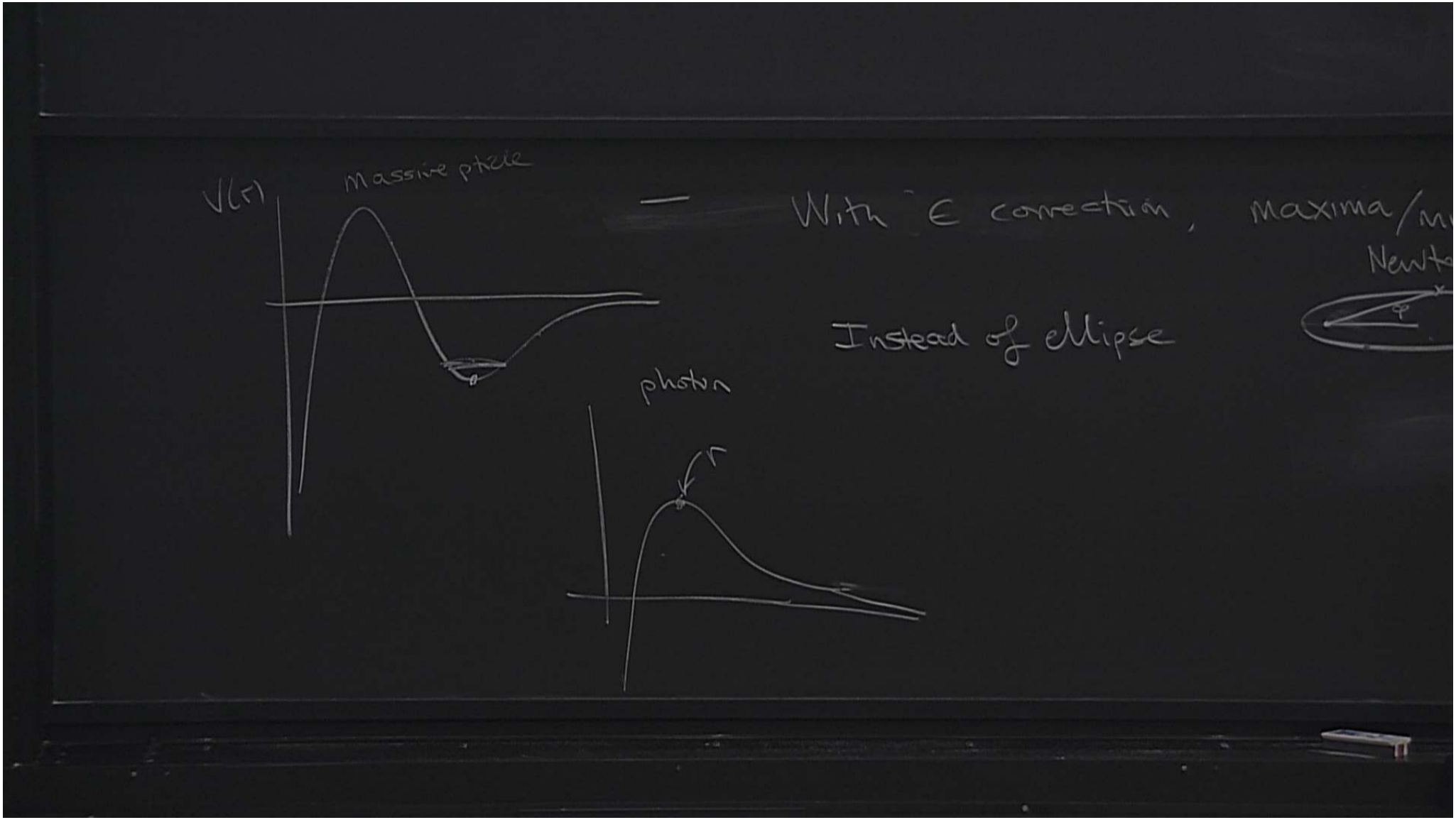
$= 5\times10^{-7}$ radians per orbit

for Mercury

$= 43''$ per century!

known from GR astronomy to $\pm 1''$.

(Mercury has several orbits/yr)


V(r)
massive ptcle
With ϵ correction, maxima/m
Newto
Instead of ellipse
photon
r

$u = \frac{r_s}{2J^2}\left(1 + e\cos\varphi\right)$ ellipticity: amplitude of the SHO motion.

$= u_0$

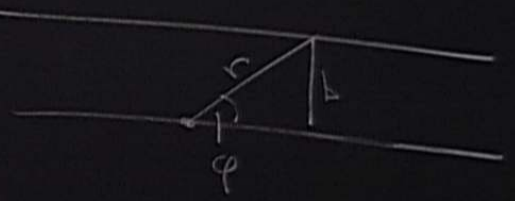
$\frac{b}{r} = \sin\varphi$

ive for δu.

photon $u'^2 + u^2 - r_s u^3 = \text{const}$

$+ 3\frac{r_s}{2}u^2$

$u = b\sin\varphi = \frac{1}{r}$



$\frac{3r_s}{2}\left(\frac{r_s}{2J^2}\right)^2\left(1 + e\cos\varphi\right)^2$

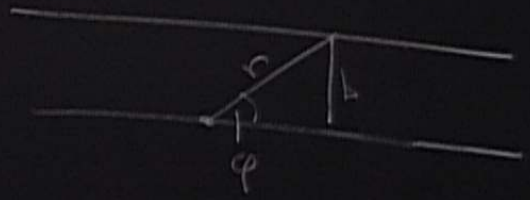
$u = \frac{r_s}{2J^2}(1 + e\cos\varphi)$ ellipticity: amplitude of the SHO motion.

$= u_0$

$\frac{b}{r} = \sin\varphi$

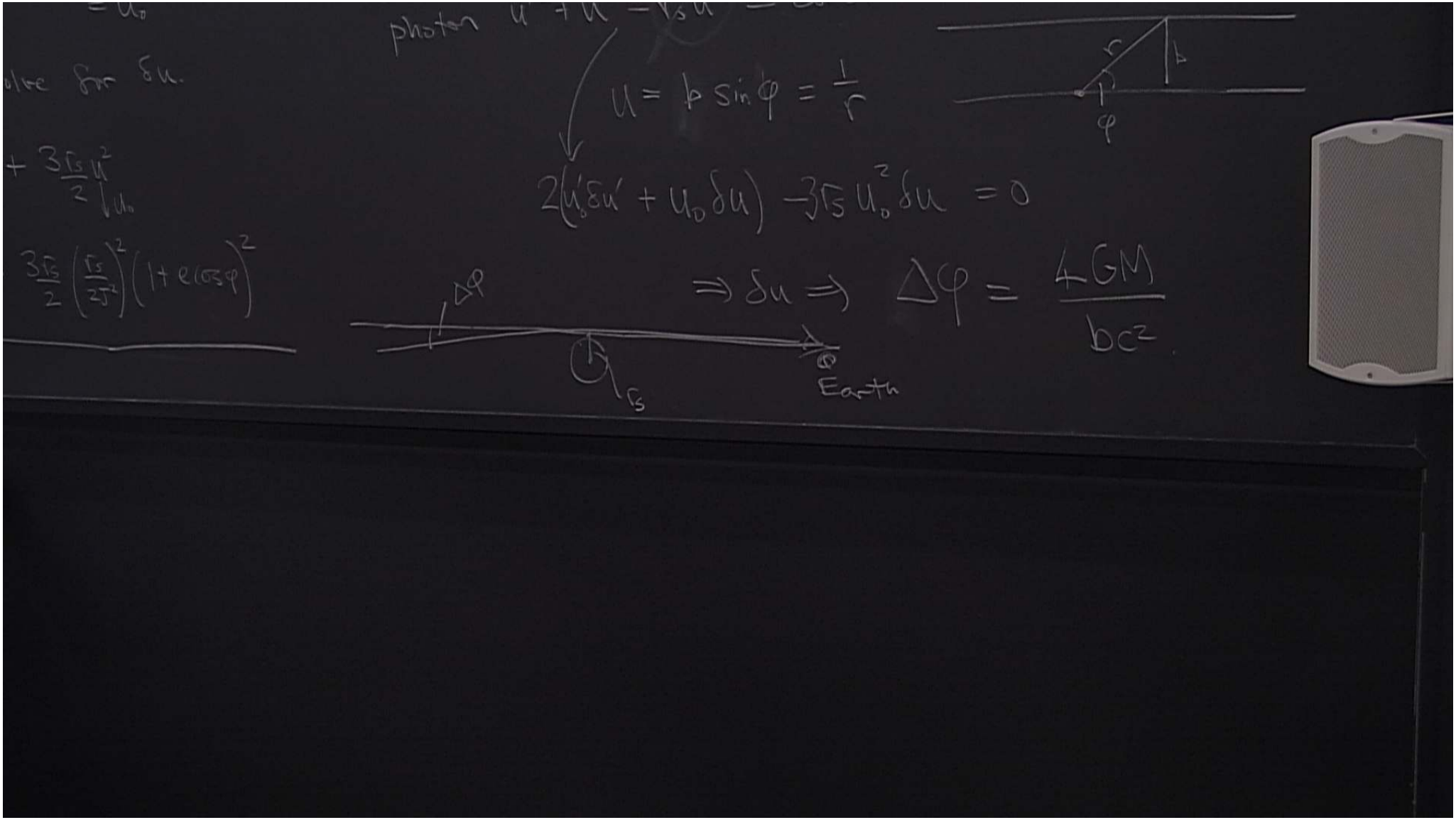
photon $u'^2 + u^2 - r_s u^3 = \text{const}$

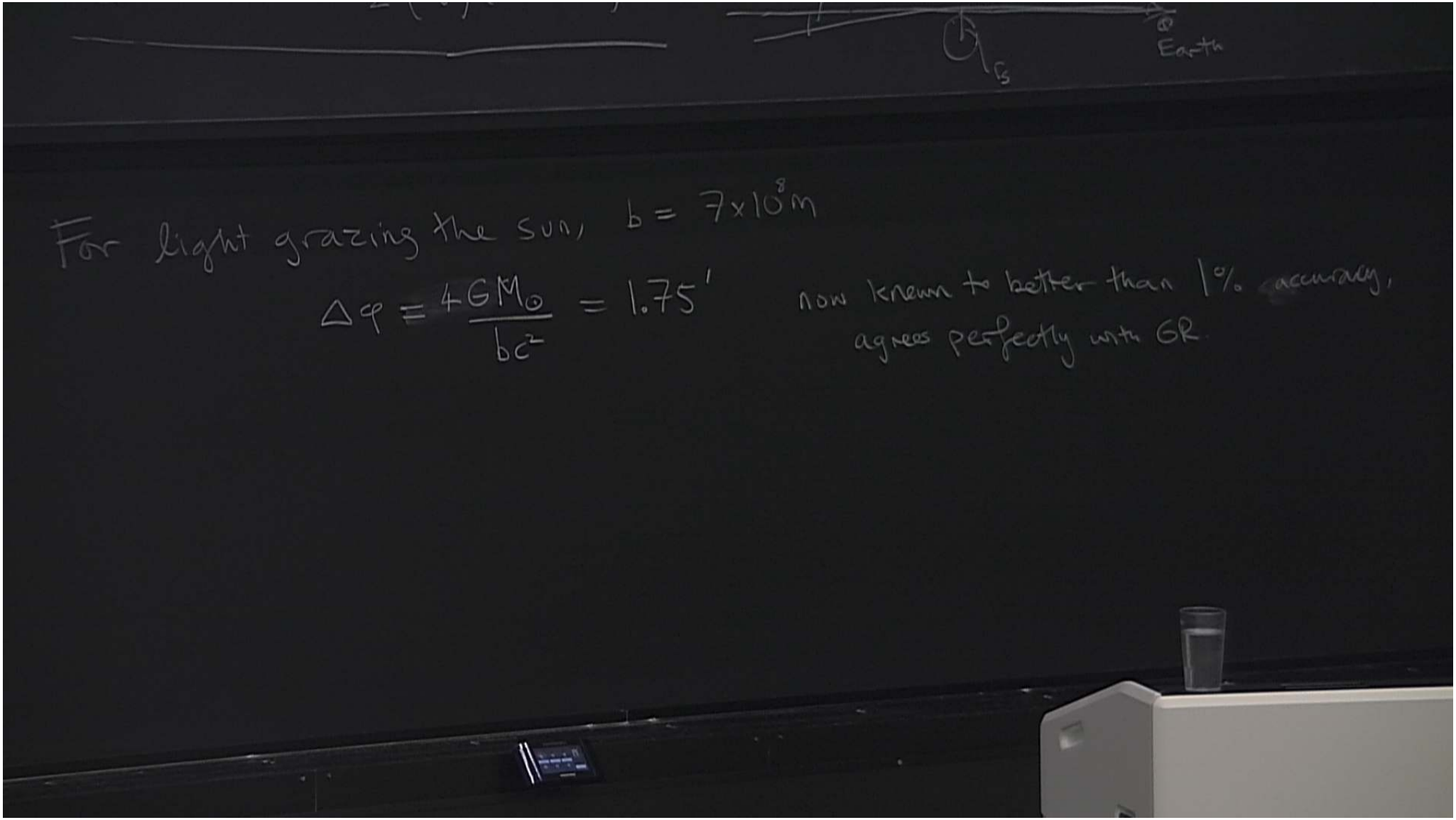
$u = b\sin\varphi = \frac{1}{r}$



$2(u_0'\delta u' + u_0\delta u) - 3r_s u_0^2 \delta u = 0$

$\Rightarrow \delta u \Rightarrow \Delta\varphi = \frac{4GM}{bc^2}$

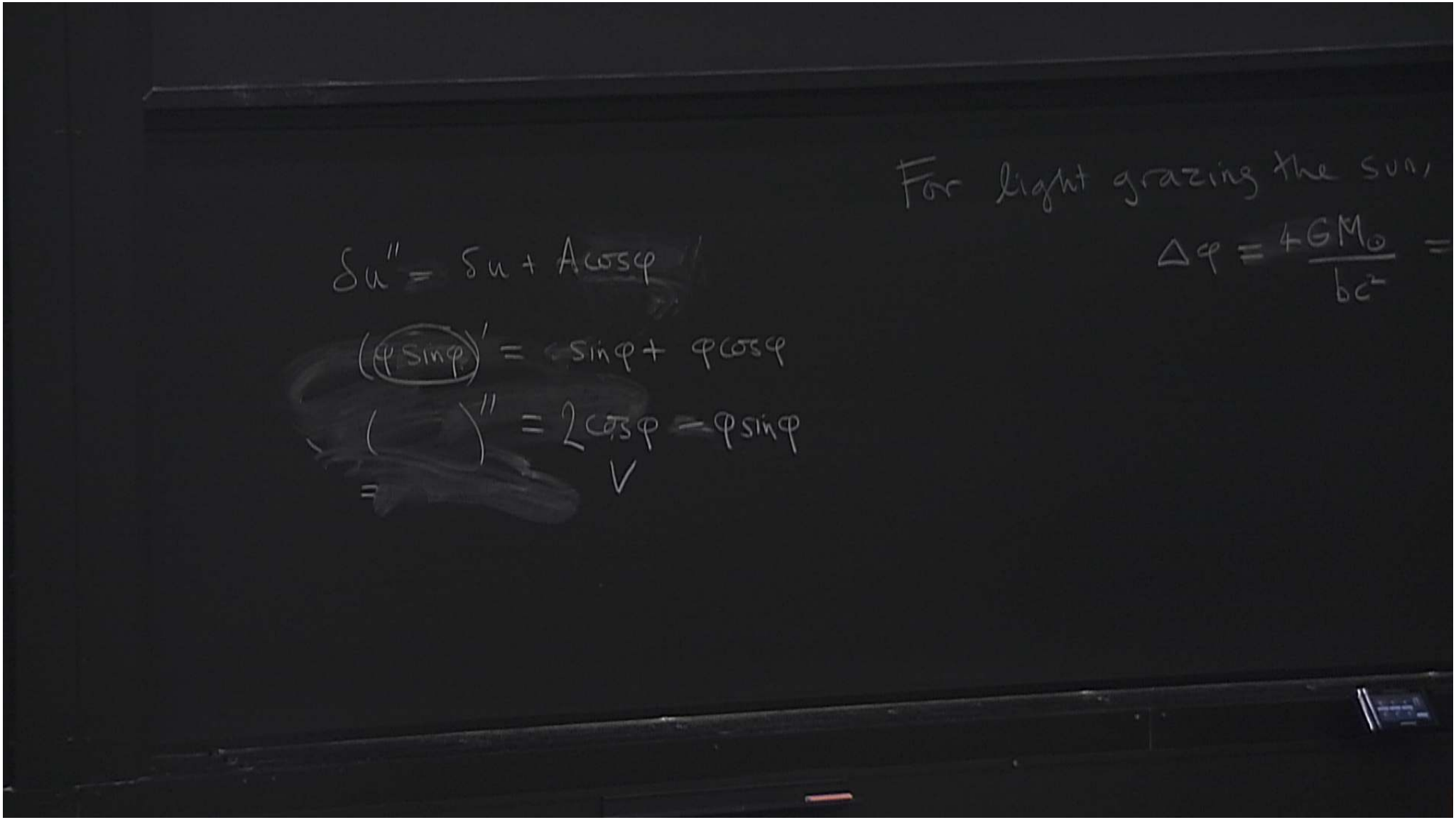

photon
$u = b\sin\varphi = \frac{1}{r}$
$2(u_0\delta u' + u_0\delta u) - 3r_s u_0^2 \delta u = 0$
$\Rightarrow \delta u \Rightarrow \Delta\varphi = \frac{4GM}{bc^2}$
r
b
φ
Δφ
Earth



For light grazing the sun, $b = 7 \times 10^8 m$

$$\Delta\varphi = \frac{4GM_\odot}{bc^2} = 1.75'$$

now known to better than 1% accuracy, agrees perfectly with GR


For light grazing the sun,
$\Delta\varphi = \frac{4GM_\odot}{bc^2} =$
$\delta u'' = \delta u + A\cos\varphi$
$(\varphi \sin\varphi)' = \sin\varphi + \varphi\cos\varphi$
$(\quad)'' = 2\cos\varphi - \varphi\sin\varphi$
$=$
V