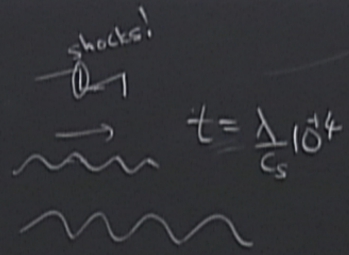
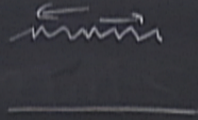


Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 11

Date: Sep 22, 2015 09:00 AM

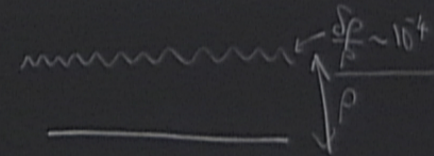
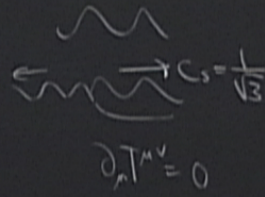
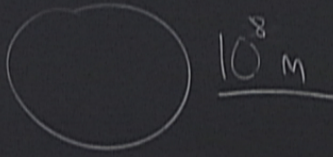
URL: <http://pirsa.org/15090038>

Abstract:



$$3 \times 10^2 \quad 10^{-10} \text{ s} \\ \rightarrow \underline{3 \times 10^2 \text{ m}}$$

$$10^{14}$$



Simplest solution of vacuum Einstein equations $R_{\mu\nu} = 0$

- analog of Coulomb field in electrostatics

Assume spherical symmetry.

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- analog of Coulomb field in electrostatics

Assume spherical symmetry.

Line element depends only on rotational invariants.
 t, dt

$$\underline{x}' = \underline{O} \underline{x} \quad \begin{matrix} \swarrow \underline{O}^T \underline{O} = 1 \\ \text{rotation} \\ \text{matrix} \end{matrix}$$

Simplest solution of vacuum Einstein equations $R_{\mu\nu} = 0$

- analog of Coulomb field in electrostatics

Assume spherical symmetry.

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$$t, dt, r^2 = \underline{x} \cdot \underline{x}, \quad r dr = \underline{x} \cdot d\underline{x}, \quad d\underline{x}^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$\underline{x}' = O \underline{x}$ $O^T O = 1$
rotation matrix
spherical polars

$$\Rightarrow d\ell^2 = -C dt^2 - 2D dr dt + E dr^2 + F r^2 d\Omega_2^2, \text{ where } C, D, E \text{ and } F \text{ are functions of } t \text{ and } r.$$

$d\Omega_2^2$ line element on S^2

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1) Simplify by redefining $r' = \sqrt{F} r$, then delete prime

$$\Rightarrow d\ell^2 = -C dt^2 - 2D dt dr + E dr^2 + r^2 d\Omega_2^2 \text{ with same new } C, D, E$$

2)

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 rotation matrix
 spherical polars

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$$\Rightarrow d\ell^2 = -C dt^2 - 2D dt dr + E dr^2 + r^2 d\Omega_2^2 \text{ with same new } C, D, E$$

2) remove D by setting

$$dt' = \underset{\text{arbitrary}}{f(t, r)} (C dt + D dr)$$

$$dt'^2 = f^2 (C^2 dt^2 + 2CD dr dt + D^2 dr^2)$$

only obstacle is $= \partial_t t'(t, r) dt + \partial_r t'(t, r) dr$

$$d\ell^2 = - \frac{dt'^2}{f^2} + (D^2 + E) dr^2 + r^2 d\Omega_2^2$$

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$$\underline{x}' = \underline{O} \underline{x}$$

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1) Simplify by redefining $r' = \sqrt{F} r$, then delete prime

$$\Rightarrow d\ell^2 = -C dt^2 - 2D dt dr + E dr^2 + r^2 d\Omega_2^2 \text{ with some new } C, D, E$$

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$$\Rightarrow \text{can I solve } \partial_t t' = fC, \partial_r t' = fD \text{ for some } f.$$

Consistency requires

$$\underbrace{\partial_r \partial_t t'} = \partial_t \partial_r t'$$

$\partial_r(fC) = \partial_t(fD)^*$: does this have solutions for f , for arbitrary given C and D ?

yes, locally : assume some $f(0, r)$, then $*$ gives you $\partial_t f = \partial_r(fC) - f \frac{D_r}{D}$

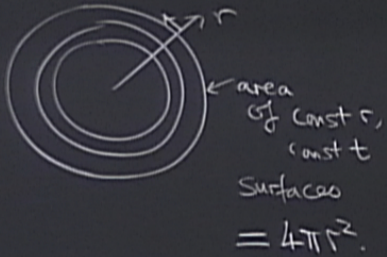
$$\text{eq}^* \Rightarrow \partial_t f(t, r)|_{t=0}$$

$$\partial_t^2 f = \left(f_{tt}, f, C, D, \dots \right)$$

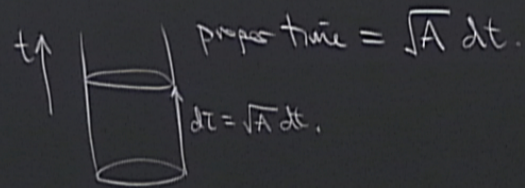
Iterate to get Taylor series for $f(t, r)$.

Iterate to get Taylor series for $f(t, r)$.

$$\Rightarrow ds^2 = -A(t, r)dt^2 + B(t, r)dr^2 + r^2 d\Omega^2$$



; spacetime is foliated by spheres of radius $4\pi r^2$.



Schwarzschild
(1915)

$$\dot{} = \frac{d}{dr}$$

$$\cdot = \frac{d}{dt}$$

assume no torsion

$$\Gamma_{tt}^t = \frac{\dot{A}}{2A}, \quad \Gamma_{rr}^t = \frac{\dot{B}}{2A}, \quad \Gamma_{tr}^t = \frac{A'}{2A}$$

$$\Gamma_{tt}^r = \frac{A'}{2B}, \quad \Gamma_{tr}^r = \frac{\dot{B}}{2B}, \quad \Gamma_{rr}^r = \frac{B'}{2B}, \quad \Gamma_{\theta\theta}^r = -\frac{r}{B}, \quad \Gamma_{\varphi\varphi}^r = -\frac{r \sin^2\theta}{B}$$

$$\Gamma_{r\theta}^\theta = \frac{1}{r}, \quad \Gamma_{\varphi\varphi}^\theta = -\sin\theta \cos\theta, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\varphi\theta}^\varphi = \cot\theta.$$

- with all others either given by symmetry $\Gamma_{\mu\sigma}^\mu = \Gamma_{\sigma\mu}^\mu$ or zero.

$$(1) \quad R_{tt} = \frac{A''}{2B} - \frac{A'B'}{4B^2} + \frac{A'}{Br} - \frac{A'^2}{4AB} - \frac{\ddot{B}}{2B} + \frac{\ddot{B}}{4B^2} + \frac{\dot{A}\dot{B}}{4AB}$$

$$(2) \quad R_{tr} = \frac{\dot{B}}{B} r$$

$$(3) \quad R_{rr} = -\frac{A''}{2A} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} + \frac{B'}{Br} - \frac{\ddot{B}}{2A} + \frac{\dot{A}\dot{B}}{4A^2} + \frac{\ddot{B}}{4AB}$$

$$(4) \quad R_{\theta\theta} = 1 - \frac{1}{B} + r \frac{B'}{2B^2} - \frac{rA'}{2AB}$$

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$$R_{\varphi\varphi} = R_{\theta\theta} \sin^2 \theta, \text{ all other components are zero}$$

yes, locally. assume some $f(t, r)$, then * gives you $\partial_t f$.
 $\partial_t^2 f = \left(f_{tt}, f_{tr}, f_{rr}, \dots \right)$
 $\text{eq}^* \Rightarrow \partial_t f(t, r)|_{t=0}$

Iterate to get Taylor series for $f(t, r)$.

Solve: (2) $\Rightarrow \tilde{B} = 0 \Rightarrow B = B(r) \Rightarrow$ all time derivatives disappear!

$$\underbrace{\partial_t^2 f(t)}_{B(r)} \quad \text{i.e.} \quad \partial_t^2 = \int dt \sqrt{A}$$

$$R_{tt} + \frac{A}{B} R_{rr} = \frac{A'}{B} + \frac{B'A}{B^2} = 0 \Rightarrow A'B + B'A = (AB)' = 0 \Rightarrow AB = f(t)$$

$$\Rightarrow A = \frac{f(t)}{B} \Rightarrow \frac{A'}{A} = -\frac{B'}{B}$$

$$\Rightarrow R_{\theta\theta} = 1 - \frac{1}{B} + r \frac{B'}{B^2} = 0 \Rightarrow \left(\frac{r}{B} \right)' = \frac{1}{B} - r \frac{B'}{B^2} = 1$$

$$\Rightarrow \frac{r}{B} = r + C, \text{ where, from } (*), \dot{C} = 0.$$

$$\Rightarrow B = \frac{r}{r+C} = \frac{1}{1+C/r}$$

$$A = f(t)/B \xrightarrow{\text{redefine to remove } f} A = 1 + C/r$$

$$(4) \quad R_{\theta\theta} = 1 - \frac{1}{B} + \frac{rB'}{2B^2} - \frac{rA'}{2AB}$$

$$R_{\varphi\varphi} = R_{\theta\theta} \sin^2 \theta, \text{ all other components are zero}$$

→ have proven Birkhoff's theorem, that any spherically symmetric solution of the vacuum Einstein equations must be given, up to a coordinate transformation, by the Schwarzschild metric

$$G \Rightarrow -\frac{2GM}{rc^2}$$

$$-dt^2 \left(1 - \frac{2GM}{rc^2}\right) + \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2 d\Omega^2$$

agrees with Newtonian limit $h_{00} = \frac{2\phi}{c^2} = -\frac{2GM}{rc^2}$

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Numbers: Schwarzschild radius
 $r_s \equiv \frac{2GM}{c^2}$

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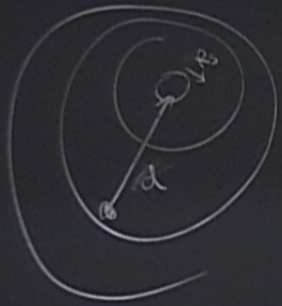
Numbers: Schwarzschild radius

$$r_s \equiv \frac{2GM}{c^2} = 3 \text{ km for } M = M_{\text{Solar}}$$

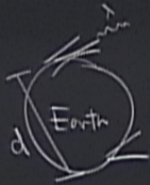
Milky way galaxy has a black hole of $M \approx 4 \times 10^6 M_{\text{Solar}}$ at its centre, $r_s \approx 10^{10} \text{ m}$.

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 its centre, $r_s \approx 10^{10} \text{ m}$.



$$\theta \sim \frac{r_s}{d} \sim \frac{10^{10} \text{ m}}{2 \times 10^{20} \text{ m}} \sim 10^{-10} \text{ radians}$$



Resolution

$$\theta \sim \frac{\lambda}{d} \sim \frac{10^{-3} \text{ m}}{10^7 \text{ m}} \sim 10^{-10}$$

mm wavelength
radio waves

EHT Event Horizon
Telescope

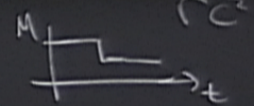
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$$\partial^i f = (f_t, f_r, \dots)$$

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$$\Rightarrow \frac{r}{B} = r + C, \text{ where, from (1), } \dot{C} = 0.$$

$$= \frac{1}{1 + C/r}$$

redefine t to remove f

$$\rightarrow A = 1 + C/r$$