Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 9

Date: Sep 18, 2015 09:00 AM

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Abstract:

Gravitational waves

$R = 2R \Rightarrow R = 0$

$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$

Vacuum Einstein equations (ignore Λ) : $G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

to linear order in $h_{\mu\nu}$, $R_{\mu\nu} = \frac{1}{2}\left(h^{\sigma}_{\nu,\rho\sigma} - h_{\mu\nu,}{}^{\sigma}{}_{\sigma} - h^{\sigma}{}_{\sigma,\rho\nu} - h_{\rho\sigma,}{}^{\sigma}{}_{\nu} \right)$

$\eta^{\sigma\alpha}\partial_\sigma\partial_\alpha = \partial^\sigma\partial_\sigma = \Box$ (Minkowski)

$= \frac{1}{2}\left(-\Box h_{\rho\nu} + h^{\sigma}_{\nu,\rho\sigma} + h^{\sigma}_{\rho,\nu\sigma} - h^{\sigma}{}_{\sigma,\rho\nu} \right)$

$\left(h^{\sigma}_{\nu,\sigma} - \frac{1}{2} h^{\sigma}{}_{\sigma,\nu} \right)_{,\rho} + \left(h^{\sigma}_{\rho,\sigma} - \frac{1}{2} h^{\sigma}{}_{\sigma,\rho} \right)_{,\nu}$

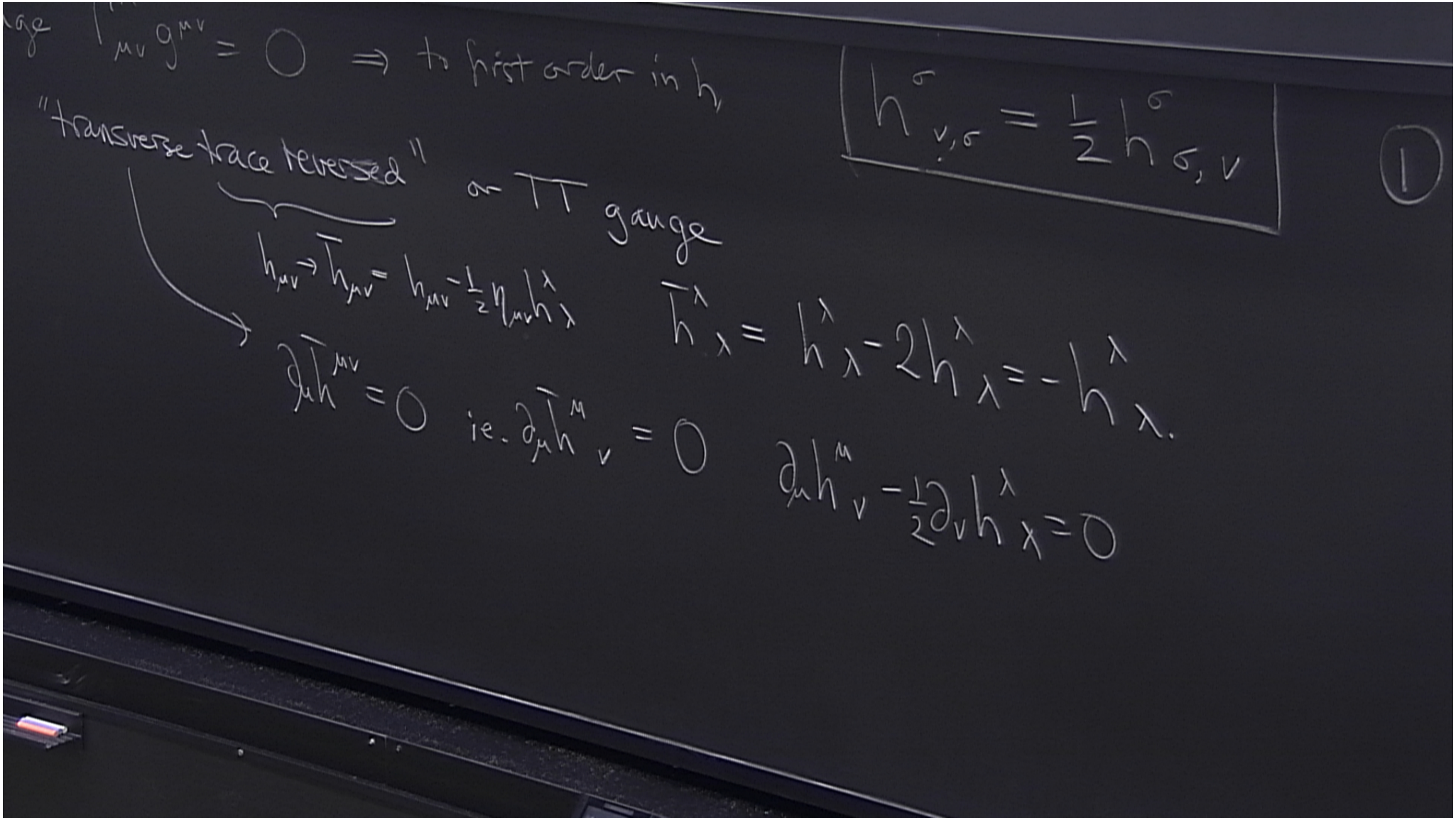
(Minkowski)

$$\left(h^{\sigma}_{\nu\sigma} - \tfrac{1}{2} h^{\sigma}_{\sigma,\nu}\right)_{,\rho} + \left(h^{\sigma}_{\rho\sigma} - \tfrac{1}{2} h^{\sigma}_{\sigma,\rho}\right)_{,\nu}$$

harmonic gauge $\Gamma^{\lambda}_{\mu\nu} g^{\mu\nu} = 0 \Rightarrow$ to first order in h $\quad \boxed{h^{\sigma}_{\nu,\sigma} = \tfrac{1}{2} h^{\sigma}_{\sigma,\nu}}$ (1)

"transverse trace reversed" or TT gauge

$h_{\mu\nu} \to \bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h^{\lambda}_{\lambda}$ $\quad \bar{h}^{\lambda}_{\lambda} = h^{\lambda}_{\lambda} - 2h^{\lambda}_{\lambda} = -h^{\lambda}_{\lambda}$.


$\partial_\mu g^{\mu\nu} = 0 \Rightarrow$ to first order in h
$h^\sigma_{\nu,\sigma} = \frac{1}{2} h^\sigma_{\sigma,\nu}$ (1)
"transverse trace reversed" or TT gauge
$h_{\mu\nu} \to \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda_\lambda$
$\bar{h}^\lambda_\lambda = h^\lambda_\lambda - 2h^\lambda_\lambda = -h^\lambda_\lambda$.
$\partial_\mu \bar{h}^{\mu\nu} = 0$ ie. $\partial_\mu \bar{h}^\mu_\nu = 0$
$\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda = 0$

$\eta^{\sigma\alpha}\partial_\alpha\partial_\sigma = \partial^\sigma\partial_\sigma = \Box$ (Minkowski)

$= \frac{1}{2}\left(-\Box h_{\rho\nu} + h^{\sigma}_{\nu,\rho\sigma} + h^{\sigma}_{\rho,\nu\sigma} - h^{\sigma}_{\sigma,\rho\nu}\right)$

$\left(h^{\sigma}_{\nu,\sigma} - \frac{1}{2}h^{\sigma}_{\sigma,\nu}\right)_{,\rho} + \left(h^{\sigma}_{\rho,\sigma} - \frac{1}{2}h^{\sigma}_{\sigma,\rho}\right)_{,\nu}$

Harmonic gauge $\Gamma^{\lambda}_{\mu\nu}g^{\mu\nu} = 0 \Rightarrow$ to first order in h, $\boxed{h^{\sigma}_{\nu,\sigma} = \frac{1}{2}h^{\sigma}_{\sigma,\nu}}$ (1)

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$\partial_\mu\bar{h}^{\mu\nu} = 0$ ie. $\partial_\mu\bar{h}^{\mu}_{\nu} = 0$ $\quad \partial_\mu h^{\mu}_{\nu} - \frac{1}{2}\partial_\nu h^{\lambda}_{\lambda} = 0$

In TT gauge, $\boxed{R_{\mu\nu} = -\frac{1}{2}\Box h^{TT}_{\mu\nu}}$

Harmonic gauge $\Gamma^{\lambda}_{\mu\nu} g^{\mu\nu} = 0 \Rightarrow$ to first or

"transverse trace reversed" or TT

$h_{\mu\nu} \to \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\lambda}{}_{\lambda}$

$\partial_\mu \bar{h}^{\mu\nu} = 0$ ie. $\partial_\mu \bar{h}^{\mu}$

$\Box = -\partial_0^2 - \underline{k}^2$

In TT gauge, $\boxed{R_{\mu\nu} = -\frac{1}{2}\Box h^{TT}_{\mu\nu}}$

$\epsilon_{\mu\nu}(\underline{k})\, e^{ik\cdot x} + \epsilon^{*}_{\mu\nu}(\underline{k})\, e^{-ik\cdot x}\Big)$

$k\cdot x = k_\mu x^\mu = k_0 x^0 + \underline{k}\cdot\underline{x}$

$= -k^0 x^0 + \underline{k}\cdot\underline{x}$

$\partial_\mu \to ik_\mu$ or $-ik_\mu$

$\Box = 0 \Rightarrow k^2 = \underline{k}^2 \Rightarrow k^0 = |\underline{k}|$

$\Big) \Rightarrow k_\mu \epsilon^\mu{}_\nu = \frac{1}{2} k_\nu \epsilon^\mu{}_\mu$

$\epsilon_{\mu\nu}$ $10 - 4 = 6$ free parameters?

$\Longrightarrow$ waves travelling at c

general solution $h_{\mu\nu}(x^\alpha) = \sum_{\underline{k}} \left(\epsilon_{\mu\nu}(\underline{k})\, e^{ik\cdot x} + \epsilon^*_{\mu\nu}(\underline{k}) \right)$

$\Box x^\mu = 0$

gauge conditions : $\partial_\mu \to ik_\mu$ or $-ik_\mu$

$① \Rightarrow \quad k_\mu \epsilon^\mu{}_\nu = \frac{1}{2} k_\nu \epsilon^\mu{}_\mu$

remaining gauge freedom $x'^\mu = x^\mu + \epsilon^\mu(x)$, with $\Box \epsilon^\mu = 0$.

$\mathcal{L}_\epsilon$: $h'_{\mu\nu} = h_{\mu\nu} - \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}$

$$\sum_{\underline{k}} \left(\epsilon_{\mu\nu}(\underline{k})\, e^{ik.x} + \epsilon^{*}_{\mu\nu}(\underline{k})\, e^{-ik.x} \right)$$

$$k.x = k_\mu x^\mu = k_0 x^0 + \underline{k}.\underline{x}$$
$$= -k^0 x^0 + \underline{k}.\underline{x}$$

$\partial_\mu \to ik_\mu$ or $-ik_\mu$

$\Box = 0 \Rightarrow k^0{}^2 = \underline{k}^2 \Rightarrow k^0 = |\underline{k}|$

① $\Rightarrow$ $k_\mu \epsilon^\mu{}_\nu = \frac{1}{2} k_\nu \epsilon^\mu{}_\mu$

$\epsilon_{\mu\nu}$ 10 − 4 = 6 free parameters?

, with $\Box \epsilon^\mu = 0$.

$-\epsilon_{\nu,\mu}$: ①' $- \epsilon^\sigma{}_{\nu,\sigma} - \epsilon_{\nu,\sigma}{}^\sigma = - \epsilon^\sigma{}_{;\sigma\nu}$ ⇒ ① still valid if $\epsilon_{\nu,\sigma}{}^\sigma = 0$ ie. $\Box \epsilon_r = 0$,

order in h and ϵ)

$\Rightarrow$ waves travelling at c

general solution $h_{\mu\nu}(x^\alpha) = \sum_{\underline{k}} \left(\epsilon_{\mu\nu}(\underline{k}) e^{ik\cdot x} + \epsilon^*_{\mu\nu}(\underline{k}) \right.$

$\Box x^\mu = 0$ gauge conditions: $\partial_\mu \to ik_\mu$ or $-ik_\mu$

$① \Rightarrow k_\mu \epsilon^\mu{}_\nu = \frac{1}{2} k_\nu \epsilon^\mu{}_\mu$

remaining gauge freedom $x'^\mu = x^\mu + \epsilon^\mu(x)$, with $\Box \epsilon^\mu = 0$.

$\mathcal{L}_\epsilon$: $h'_{\mu\nu} = h_{\mu\nu} - \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}$: $①' - \epsilon^\sigma{}_{,\nu\sigma} - \epsilon_{\nu,\sigma}{}^\sigma$

(to linear order in h and ϵ)

takes $\epsilon_{\mu\nu}(k)$ to

$C_\nu(k) + k_\nu C_\mu(k)$

so we really have

$10 - 4 - 4 = 2$ remaining gauge invariant massless waves

(to linear order in h and ϵ)

Choose $\epsilon_\mu = \sum_{\underline{k}} \left(i C^\mu(\underline{k}) e^{ik.x} - i C^{*\mu}(\underline{k}) e^{-ik.x} \right)$

then, a harmonic coord transformation $x'^\mu \to x^\mu + \epsilon^\mu$, takes $\epsilon_{\mu\nu}(\underline{k})$

$$\epsilon'_{\mu\nu}(\underline{k}) = \epsilon_{\mu\nu}(\underline{k}) + k_\mu C_\nu(\underline{k}) + k_\nu C_\mu(\underline{k})$$

Explicitly, let us choose $\underline{k}$ to be in the z direction : $k^\mu = (k, 0, 0, k)$

(every other wave related by a rotation)

(to linear order in h and ϵ)

$$C^{\mu}(k)\, e^{ik\cdot x} - i\, C^{*\mu}(k)\, e^{-ik\cdot x}\Big)$$

coord transformation $x'^{\mu} \to x^{\mu} + \epsilon^{\mu}$, takes $\epsilon_{\mu\nu}(k)$ to

$$\epsilon'_{\mu\nu}(k) = \epsilon_{\mu\nu}(k) + k_{\mu} C_{\nu}(k) + k_{\nu} C_{\mu}(k)$$

so we really have

$10 - 4 - 4 = 2$ rema

invariant massless wa

be in the z direction : $k^{\mu} = (k, 0, 0, k)$

ated by a rotation) $k_{\mu} = (-k, 0, 0, k)$

$$k^\mu \epsilon_{\mu\nu} = \frac{1}{2} k_\nu \epsilon^\mu{}_\mu$$

$\nu = 0$ $\quad k(\epsilon_{00} + \epsilon_{30}) = -\frac{1}{2} k(-\epsilon_{00} + \epsilon_{11} + \epsilon_{22} + \epsilon_{33})$

3 $\quad k(\epsilon_{03} + \epsilon_{33}) = +\frac{1}{2} k($ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $)$

1 $\quad k(\epsilon_{01} + \epsilon_{31}) = 0$

2 $\quad k(\epsilon_{02} + \epsilon_{32}) = 0$

$\epsilon_{00} + \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$) A

$=$) B

$\epsilon_{01} = -\epsilon_{31}$

$\epsilon_{02} = -\epsilon_{32}$

A+B $\epsilon_{03} = -\frac{1}{2}(\epsilon_{00} + \epsilon_{33})$

A-B $\epsilon_{22} = -\epsilon_{11}$

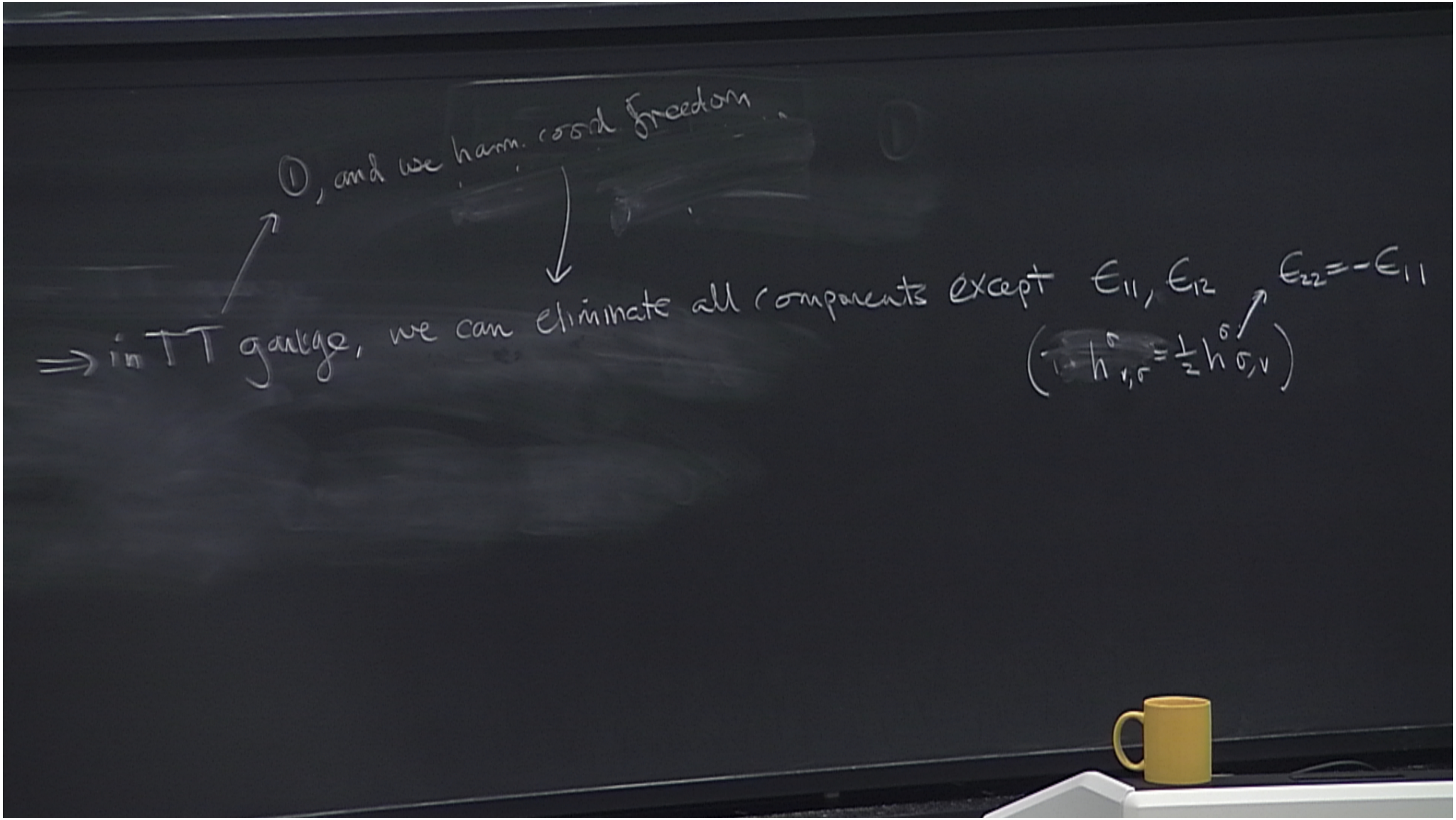
} eliminates $\epsilon_{01}, \epsilon_{02}, \epsilon_{03}, \epsilon_{22}$

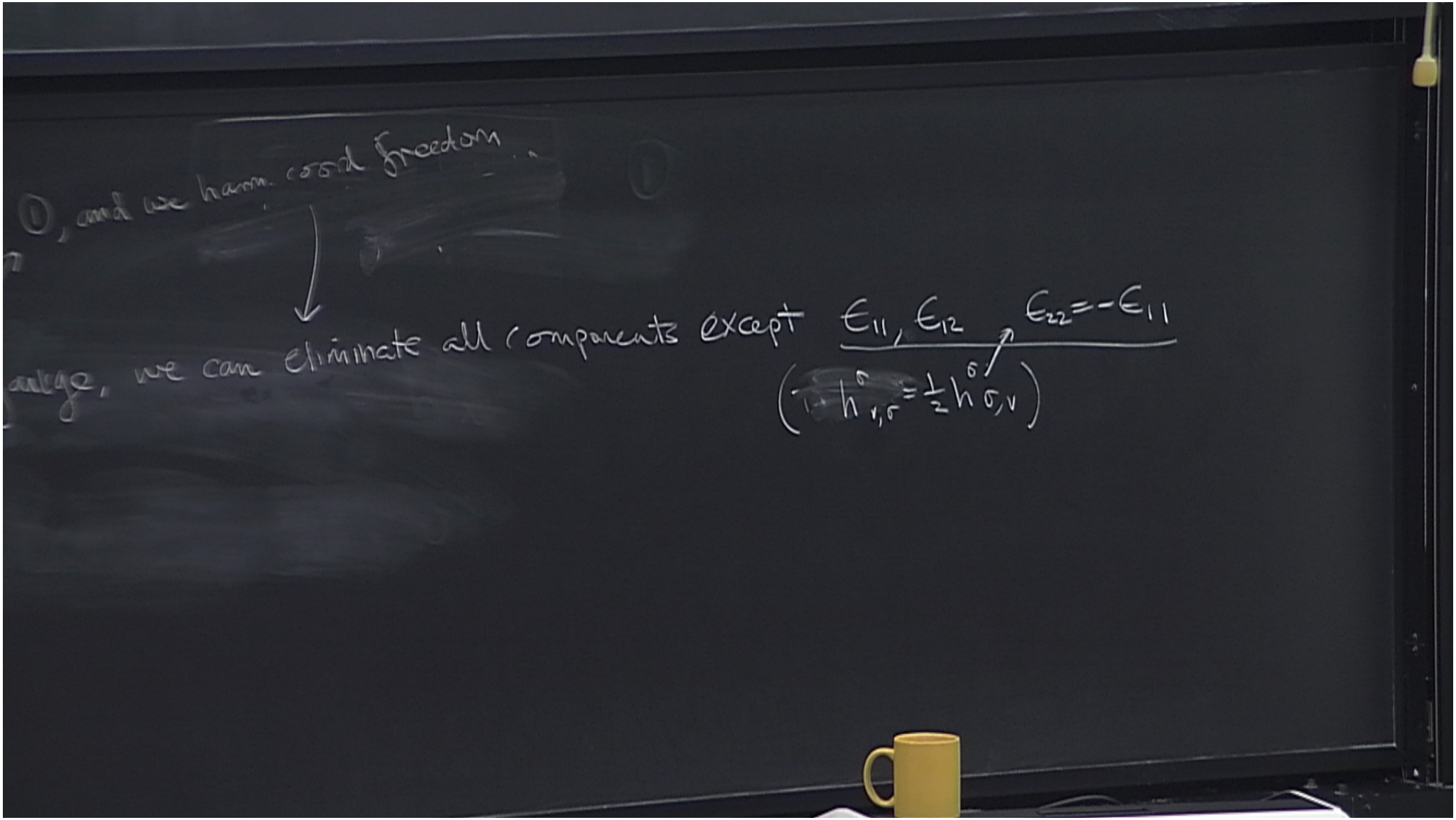
harmonic coordtrans.

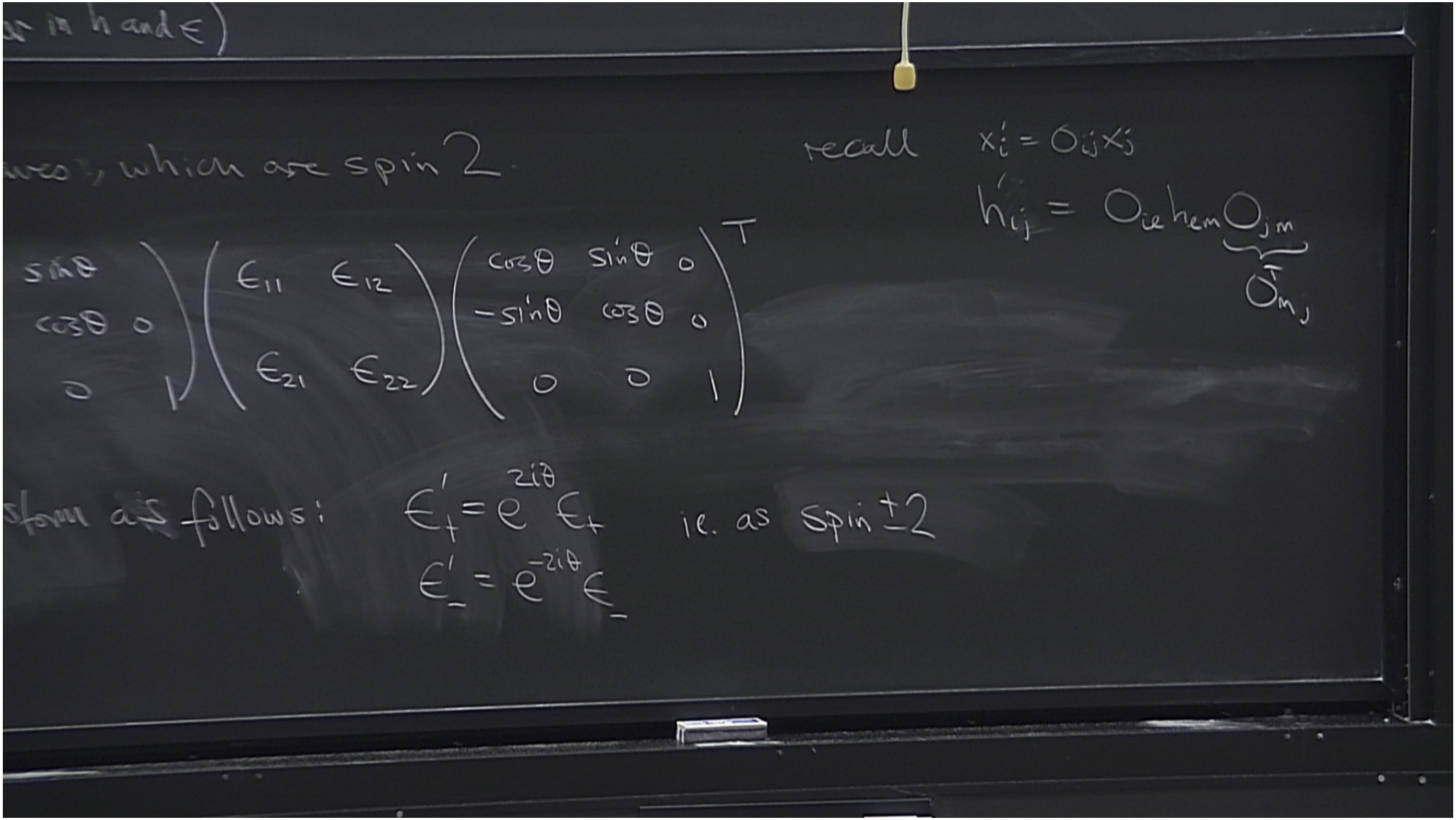
$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu$$

$$\epsilon'_{00} = \epsilon_{00} - 2kC_0$$
$$\epsilon'_{13} = \epsilon_{13} + kC_1$$
$$\epsilon'_{23} = \epsilon_{23} + kC_2$$
$$\epsilon'_{33} = \epsilon_{33} + 2kC_3$$
$$\epsilon'_{11} = \epsilon_{11}$$
$$\epsilon'_{12} = \epsilon_{12}$$

} set these zero.


① and we harm. coord. freedom
⇒ in TT gauge, we can eliminate all components except $\epsilon_{11}, \epsilon_{12}$ $\epsilon_{22} = -\epsilon_{11}$
$h^0_{i,0} = \frac{1}{2} h^0_{0,i}$


①, and we have coord freedom
antage, we can eliminate all components except $\epsilon_{11}, \epsilon_{12}$, $\epsilon_{22} = -\epsilon_{11}$
$h^{\sigma}_{\mu\sigma} = \frac{1}{2} h^{\sigma}_{\sigma\nu}$


in h and ϵ)
which are spin 2
recall $x_i' = O_{ij} x_j$
$h'_{ij} = O_{ie} h_{em} \underbrace{O_{jm}}_{O^T_{mj}}$
$\begin{pmatrix} \sin\theta & & \\ \cos\theta & 0 & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^T$
form as follows: $\epsilon_+' = e^{2i\theta} \epsilon_+$ i.e. as spin $+2$
$\epsilon_-' = e^{-2i\theta} \epsilon_-$

spin 2.

recall $x_i' = O_{ij} x_j$

$h_{ij}' = O_{ie} h_{em} \underbrace{O_{jm}}_{O^T_{mj}}$

$$\begin{pmatrix} \epsilon_{12} \\ \epsilon_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^T$$

$\epsilon_+' = e^{2i\theta} \epsilon_+$

$\epsilon_-' = e^{-2i\theta} \epsilon_-$

i.e. as spin ± 2 components of a massless d.o.f. with helicity 2.

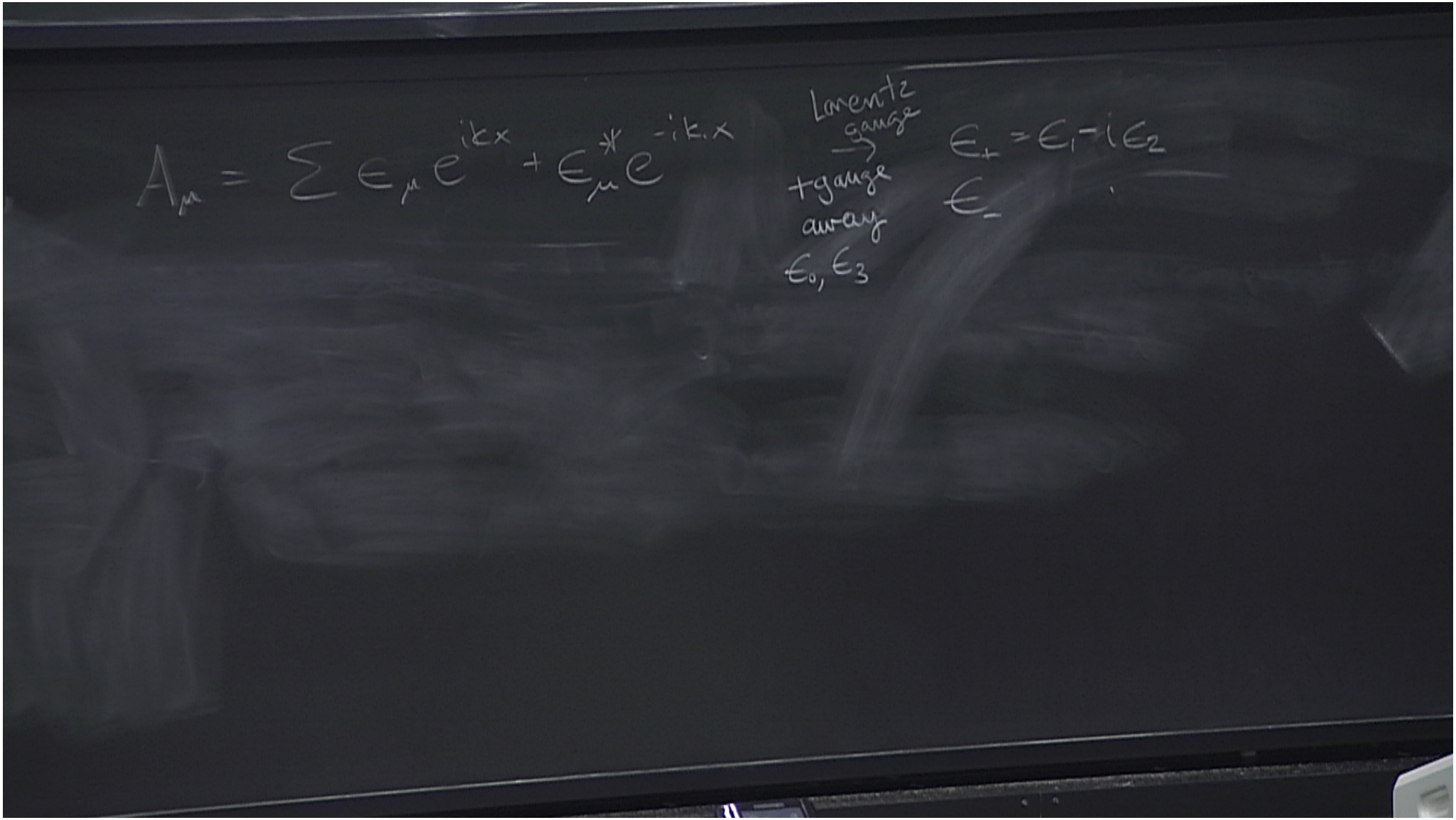
(to linear order in h and ϵ)

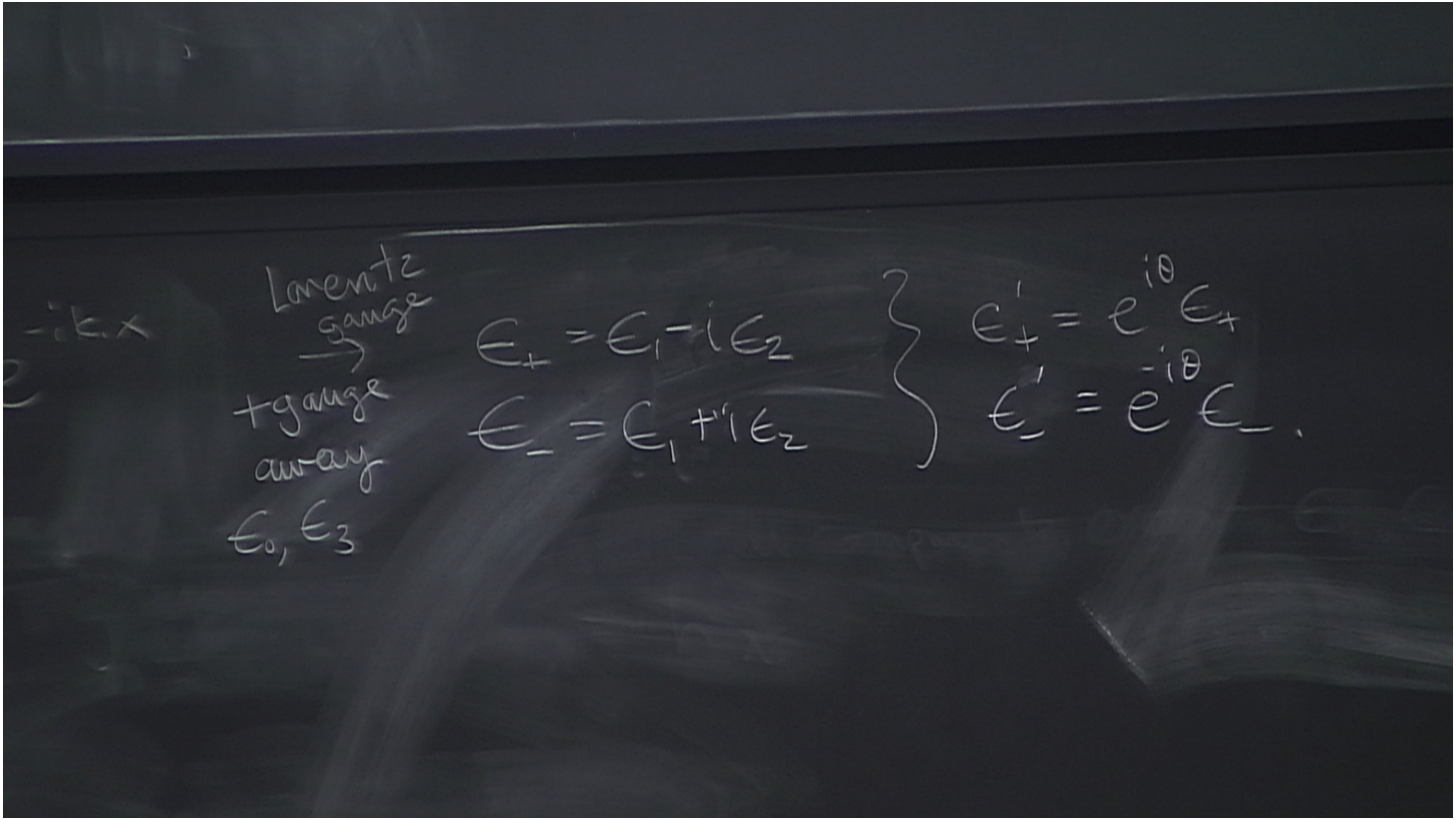
reedom describe transverse tensor waves, which are spin 2

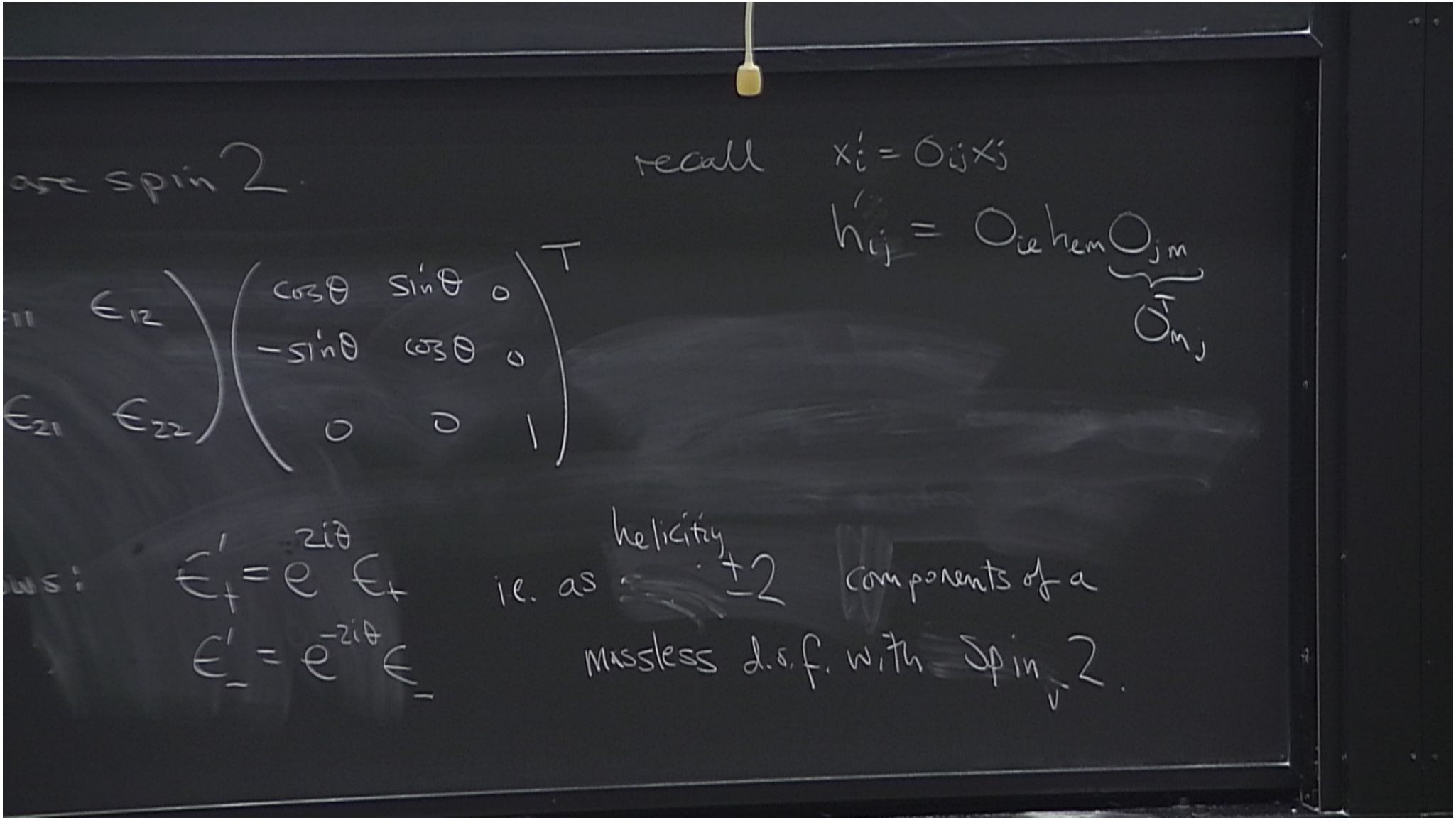
about the z axis $\quad \epsilon'_{ij} = \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^T$

$\Rightarrow$ Show that $\epsilon_+ = \epsilon_{11} - i\epsilon_{12}$ transform as follows: $\epsilon'_+ = e^{2i\theta}\epsilon_+$ i.e. a

$\epsilon_- = \epsilon_{11} + i\epsilon_{12}$ $\qquad \epsilon'_- = e^{-2i\theta}\epsilon_-$


$A_\mu = \sum \epsilon_\mu e^{ik\cdot x} + \epsilon_\mu^* e^{-ik\cdot x}$
Lorentz gauge
+gauge away
ϵ_0, ϵ_3
$\epsilon_+ = \epsilon_1 - i\epsilon_2$
ϵ_-


$e^{-ik\cdot x}$
Lorentz gauge
→
+gauge away
ϵ_0, ϵ_3
$\epsilon_+ = \epsilon_1 - i\epsilon_2$
$\epsilon_- = \epsilon_1 + i\epsilon_2$
$\epsilon_+' = e^{i\theta}\epsilon_+$
$\epsilon_-' = e^{-i\theta}\epsilon_-$.


are spin 2.
recall $x_i' = O_{ij} x_j$
$h'_{ij} = O_{ie} h_{em} O_{jm}$
O^T_{mj}
$\begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^T$
ows: $\epsilon_+' = e^{2i\theta} \epsilon_+$
$\epsilon_-' = e^{-2i\theta} \epsilon_-$
i.e. as helicity ± 2 components of a
massless d.o.f. with spin 2.

do same thing for E&M, $A_\mu = \sum \epsilon_\mu$

photon has spin 1 → helicity ±1.

graviton " " 2 → " ±2

$$A_\mu = \sum \epsilon_\mu e^{ik\cdot x} + \epsilon_\mu^* e^{-ik\cdot x}$$

Lorentz gauge → + gauge away ϵ_0, ϵ_3

$$\epsilon_+ = \epsilon_1 - i\epsilon_2$$
$$\epsilon_- = \epsilon_1 + i\epsilon_2$$

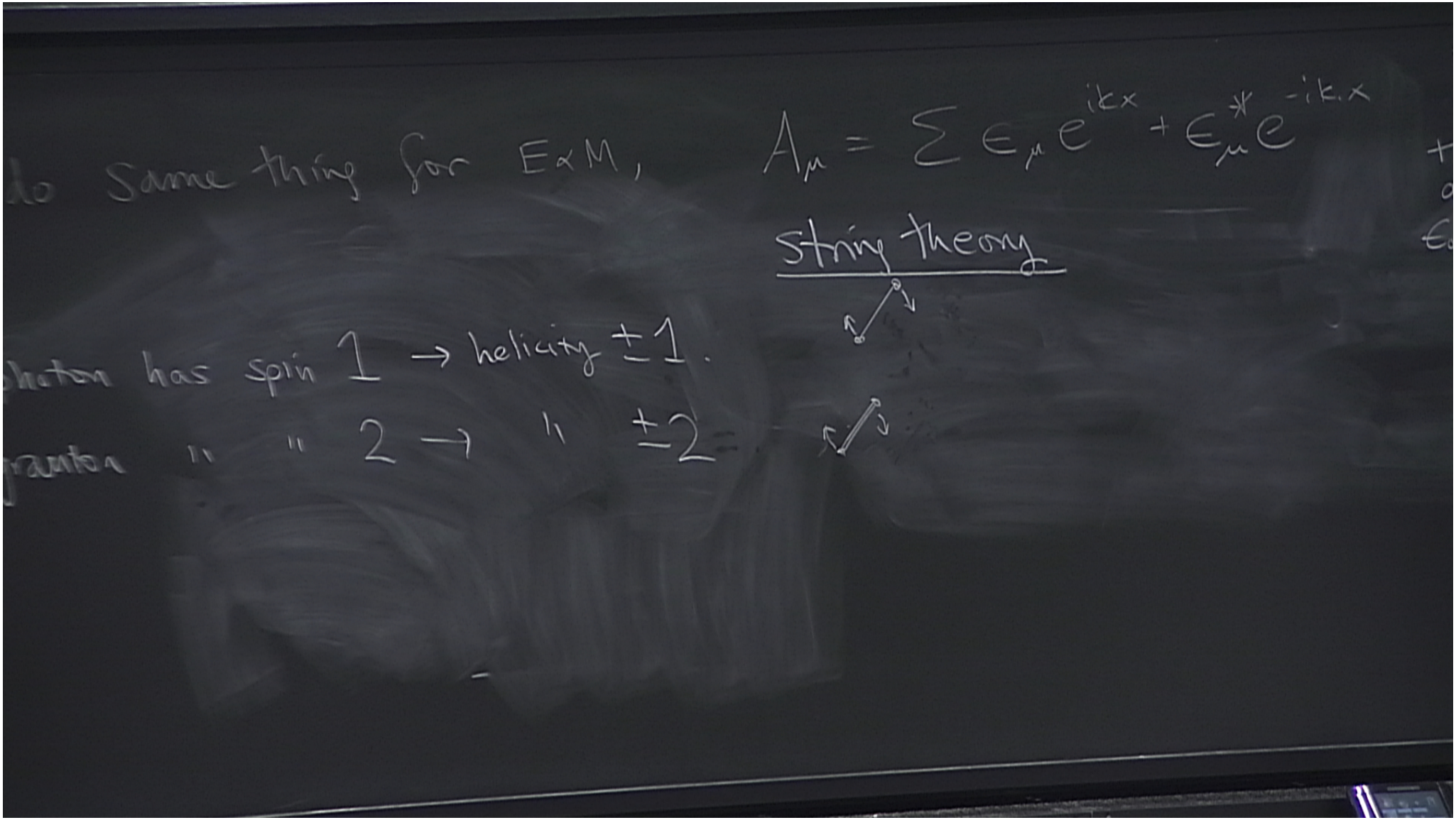
$$\left.\begin{matrix} \epsilon_+' = \\ \epsilon_-' = \end{matrix}\right\}$$

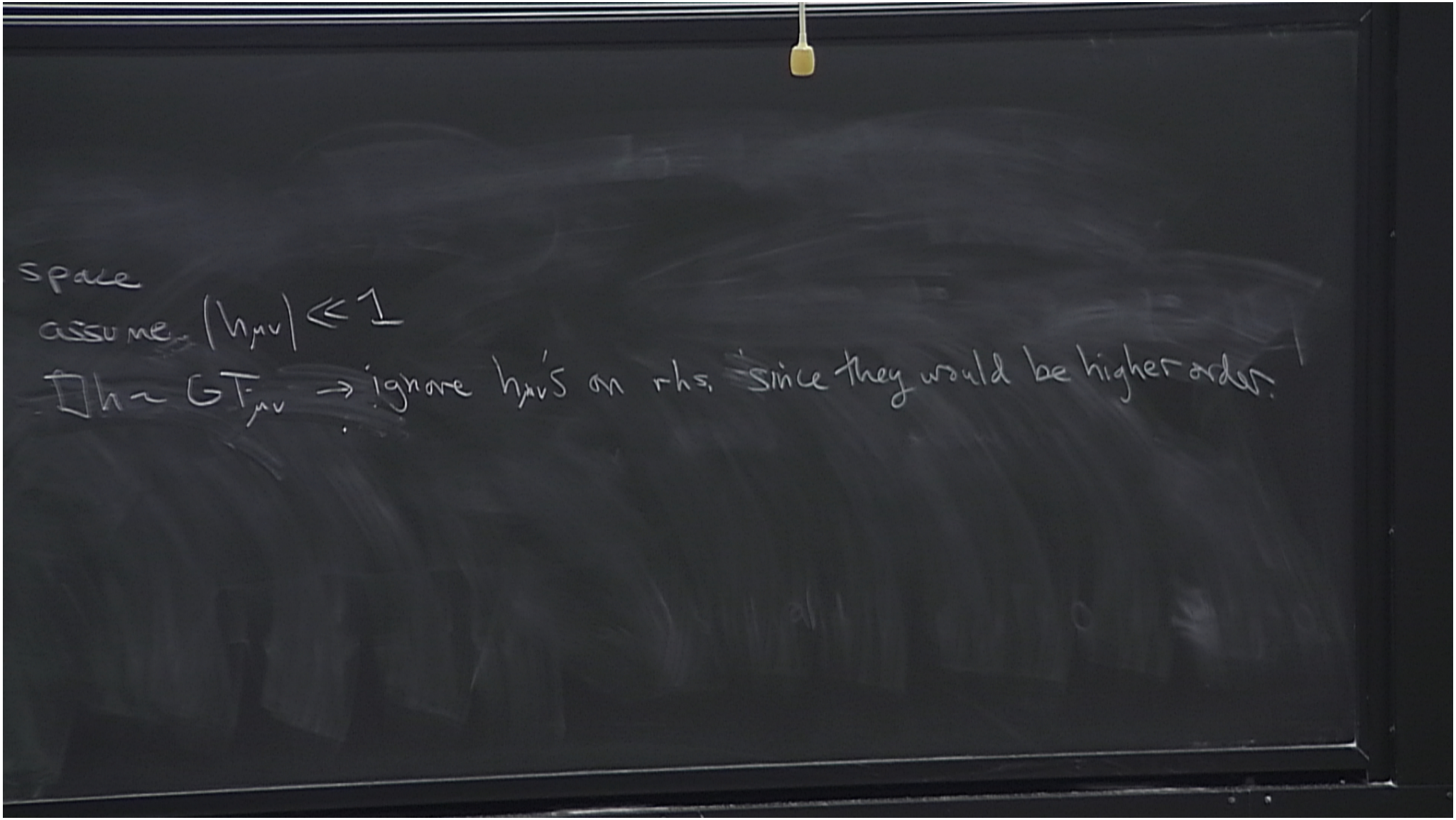
± 1



± 2




do same thing for E&M,
$A_\mu = \sum \epsilon_\mu e^{ik\cdot x} + \epsilon_\mu^* e^{-ik\cdot x}$
String theory
photon has spin 1 → helicity ±1.
graviton " " 2 → " ±2


space
assume $|h_{\mu\nu}| \ll 1$
$\Box h \sim G T_{\mu\nu}$ → ignore $h_{\mu\nu}$'s on rhs. since they would be higher order

Gravitational Radiation

In harmonic gauge $\quad R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\lambda}_{\ \lambda}\right)$

· source: assume localised

space

assume $|h_{\mu\nu}| \ll 1$

$\Box h \sim G T_{\mu\nu}$ → ignore $h_{\mu\nu}$'s on rhs. since they would be higher order.

i.e. replace metric on rhs by $\eta_{\mu\nu}$.

ntational Radiation

· source: assume localised in space

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\lambda}{}_{\lambda}\right) \quad (*)$$

assume

$\Box h \sim$

– "linearised gravity"

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\lambda}{}_{\lambda} \;:\; (*) \Rightarrow \boxed{\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}}$$

$$\left(G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} = 8\pi G T_{\mu\nu}\right)$$

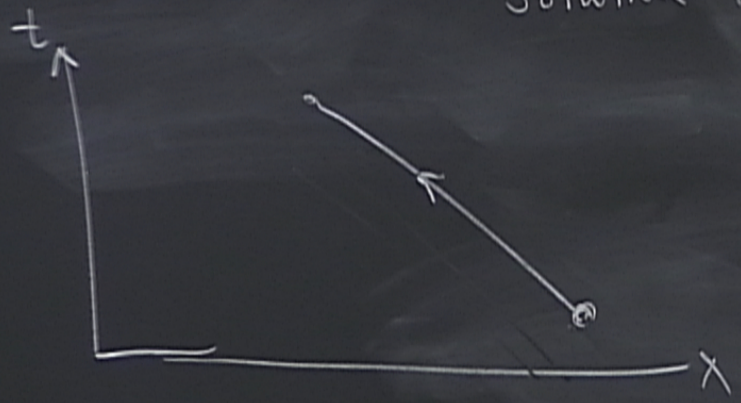
$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\lambda}{}_{\lambda} \quad : (*) \Rightarrow \boxed{\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}}$$

$$\left(G_{\mu\nu} = -\frac{1}{2} D \bar{h}_{\mu\nu} = 8\pi G T_{\mu\nu} \right)$$

Solution is $\bar{h}_{\mu\nu} = -4G \int \frac{d^3 \underline{x}'}{|\underline{x} - \underline{x}'|} T_{\mu\nu}\left(t - |\underline{x} - \underline{x}'|, \underline{x}'\right)$

$$\left(G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} = 8\pi G T_{\mu\nu}\right)$$

Solution is $\bar{h}_{\mu\nu}(t,\underline{x}) = -4G\int \frac{d^3\underline{x}'}{|\underline{x}-\underline{x}'|} T_{\mu\nu}\left(t-|\underline{x}-\underline{x}'|, \underline{x}'\right)$



$h_{\mu\nu} = 8\pi G T_{\mu\nu}$)

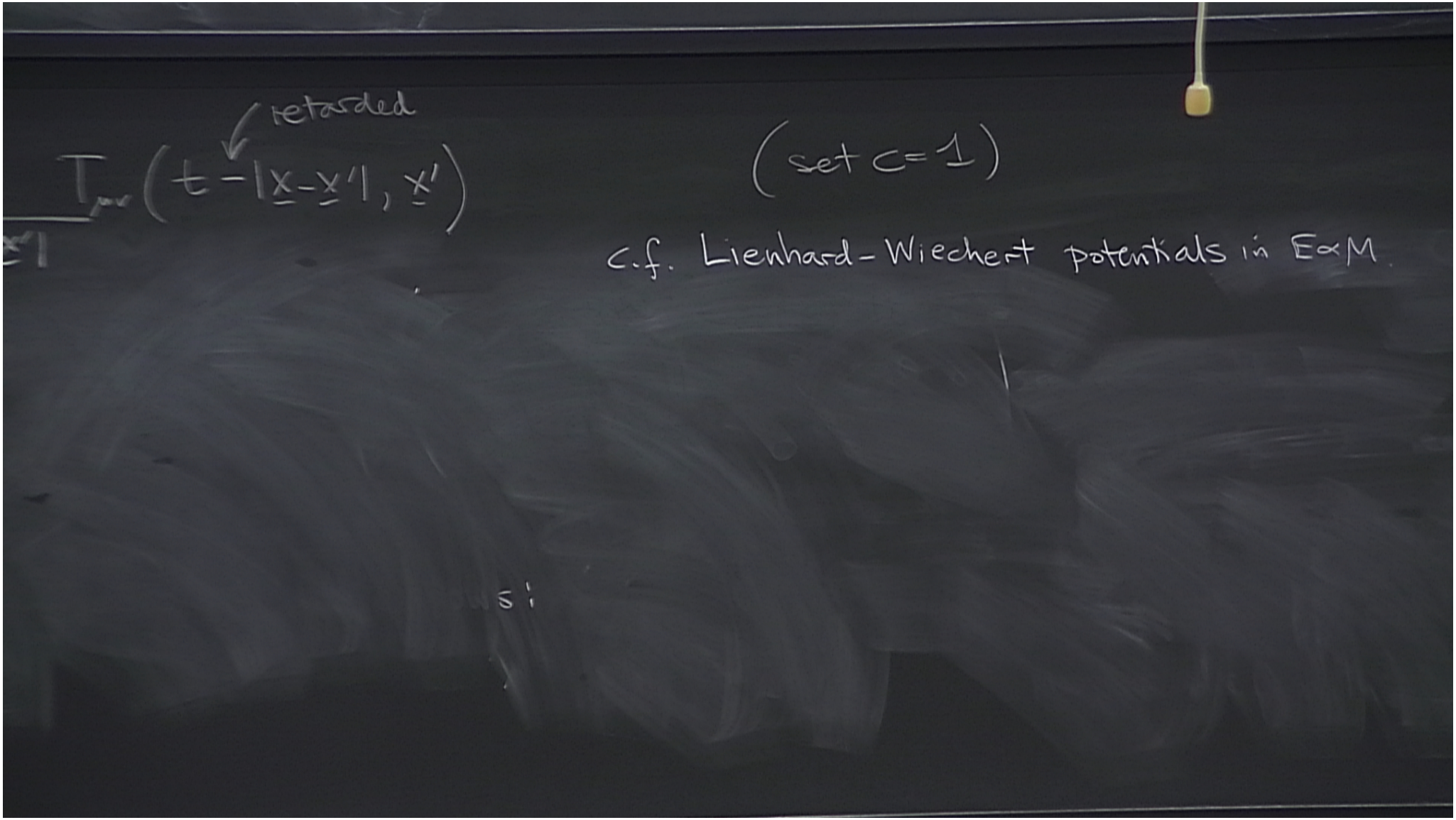
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$$t,\underline{x}) = -4G \int \frac{d^3\underline{x}'}{|\underline{x}-\underline{x}'|} T_{\mu\nu}\left(t - |\underline{x}-\underline{x}'|, \underline{x}'\right)$$

(set c=

c.f. Lienhard-Wi

s:


retarded
$T_{\mu\nu}\left(t - |\underline{x} - \underline{x}'|, \underline{x}'\right)$
(set $c = 1$)
c.f. Lienhard-Wiechert potentials in E&M

(set $c=1$)

c.f. Lienhard-Wiechert potentials in E&M.

$$r = |\underline{x}|$$

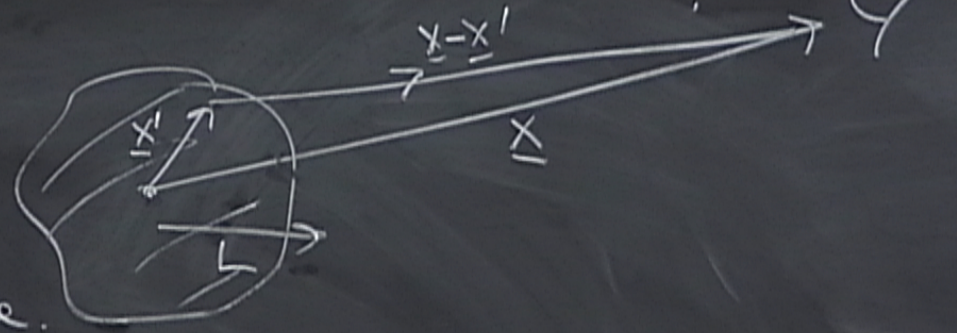
$$\sqrt{(\underline{x}-\underline{x}')^2} = r\sqrt{1 - 2\frac{\underline{x}\cdot\underline{x}'}{r^2} + \frac{\underline{x}'^2}{r^2}} \approx r\left(1 - \frac{\underline{x}\cdot\underline{x}'}{r^2} + \frac{1}{2}\frac{\underline{x}'^2}{r^2}\right) \text{etc.}$$

$$\bar{h}_{\mu\nu}(t,\underline{x}) = -4G\int \frac{d^3\underline{x}'\; T_{\mu\nu}\left(t - |\underline{x}-\underline{x}'|, \underline{x}'\right)}{|\underline{x}-\underline{x}'|}$$

retarded

c.f. Lienh

expand in $\frac{|\underline{x}'|}{|\underline{x}|}$



$\sqrt{(\underline{x}-\underline{x}')^2}$

i.e. we are far from source.

$$\Rightarrow \bar{h}_{\mu\nu} \simeq \frac{4G}{r} = \int d^3\underline{x}'\, T_{\mu\nu}(t-r, \underline{x}')$$

(expansion in $1/r$)

(set $c=1$)

c.f. Lienhard-Wiechert potentials in E&M.

$r = |\underline{x}|$

$$\sqrt{(\underline{x}-\underline{x}')^2} = r\sqrt{1 - 2\frac{\underline{x}\cdot\underline{x}'}{r^2} + \frac{\underline{x}'^2}{r^2}} \approx r\left(1 - \frac{\underline{x}\cdot\underline{x}'}{r^2} + \frac{1}{2}\frac{\underline{x}'^2}{r^2}\right) \text{etc.}$$

$$h_{\mu\nu}(t,\underline{x}) = -4G\int \frac{d^3\underline{x}'\; T_{\mu\nu}\left(t-|\underline{x}-\underline{x}'|,\underline{x}'\right)}{|\underline{x}-\underline{x}'|}$$

retarded

c.f. L

$\sqrt{(\underline{x}-\underline{x}'$

expand in $\frac{|\underline{x}'|}{|\underline{x}|}$



i.e. we are far from source.

$$\Rightarrow h_{\mu\nu} \simeq \frac{4G}{r}\int d^3\underline{x}'\, T_{\mu\nu}\left(t-r,\underline{x}'\right)$$

(expansion in L/r)

OK if source varies only on scale L.

(set $c=1$)

c.f. Lienhard-Wiechert potentials in E&M.

$r=|\underline{x}|$

$$\sqrt{(\underline{x}-\underline{x}')^2} = r\sqrt{1-2\frac{\underline{x}\cdot\underline{x}'}{r^2}+\frac{\underline{x}'^2}{r^2}} \approx r\left(1-\frac{\underline{x}\cdot\underline{x}'}{r^2}+\frac{1}{2}\frac{\underline{x}'^2}{r^2}\right) \text{etc.}$$

$$\Rightarrow \bar{h}_{kl} \sim \frac{G}{r}\int T_{kl}\, d^3\underline{x}$$

$$\frac{1}{2}\frac{d^2}{dt^2}\int d^3\underline{x}\, x^k x^\ell T^{00}(t,\underline{x})$$

$$= \frac{1}{2}\int d^3\underline{x}\, \partial_n \partial_m T^{nm}\, x^k x^\ell$$

for any $T^{\mu\nu}$ obeying $\partial_\mu T^{\mu\nu} = 0$

$$\partial_0 T^{00} = -\partial_i T^{i0}$$

$$\partial_0 T^{i0} = -\partial_j T^{ij}$$

$$= \frac{1}{2}\frac{d^2}{dt^2}\int d^3\underline{x}\, x^k x^\ell T^{00}(t,\underline{x})$$

$$= \frac{1}{2}\int d^3\underline{x}\, \partial_n\partial_m T^{nm}\, x^k x^\ell$$

$$= -\frac{1}{2}\int d^3\underline{x}\, \partial_m T^{nm}\left(\delta^{kn}x^\ell + \delta^{\ell n}x^k\right)$$

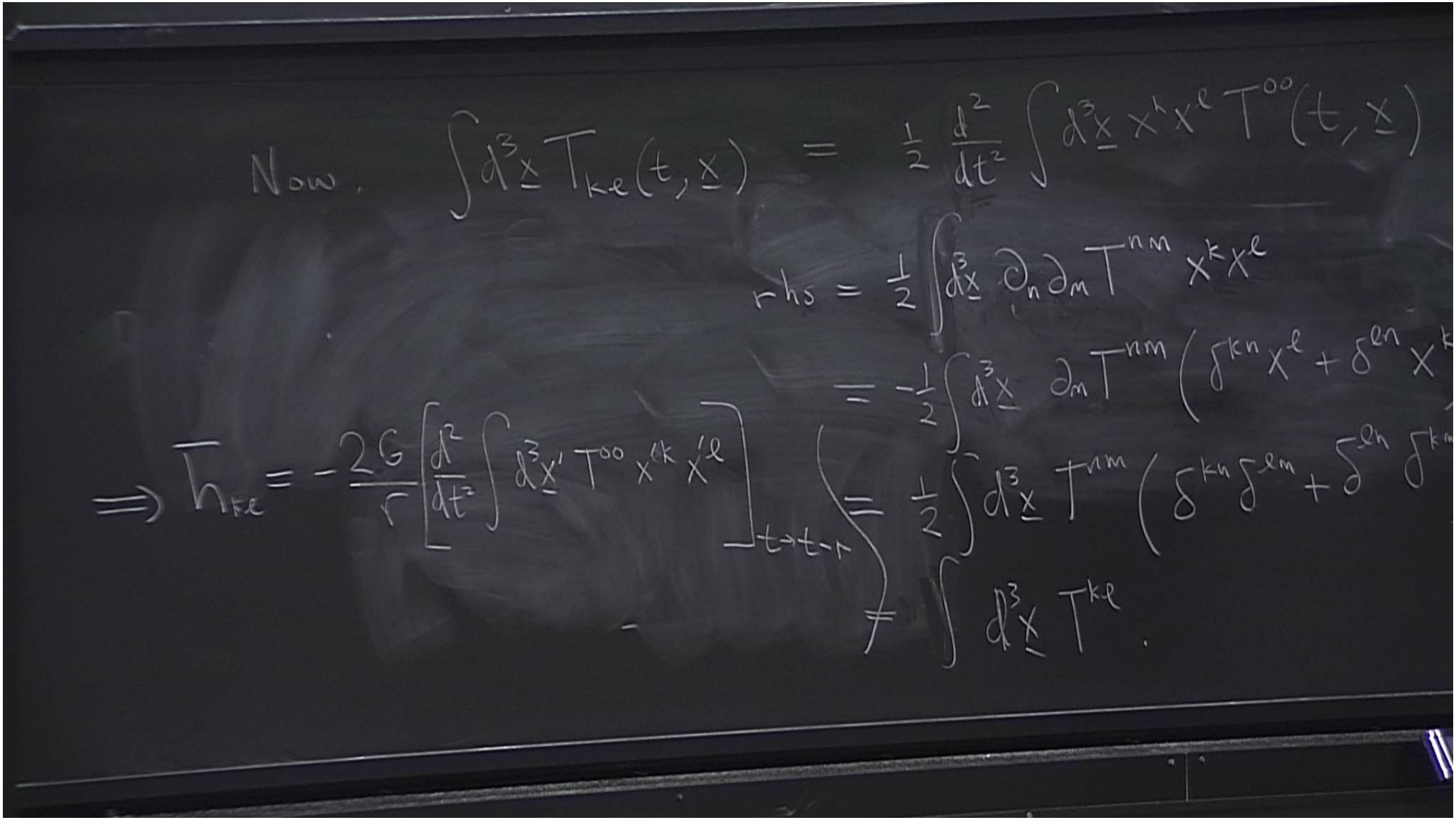
$$= \frac{1}{2}\int d^3\underline{x}\, T^{nm}\left(\delta^{kn}\delta^{\ell m} + \delta^{\ell n}\delta^{km}\right)$$

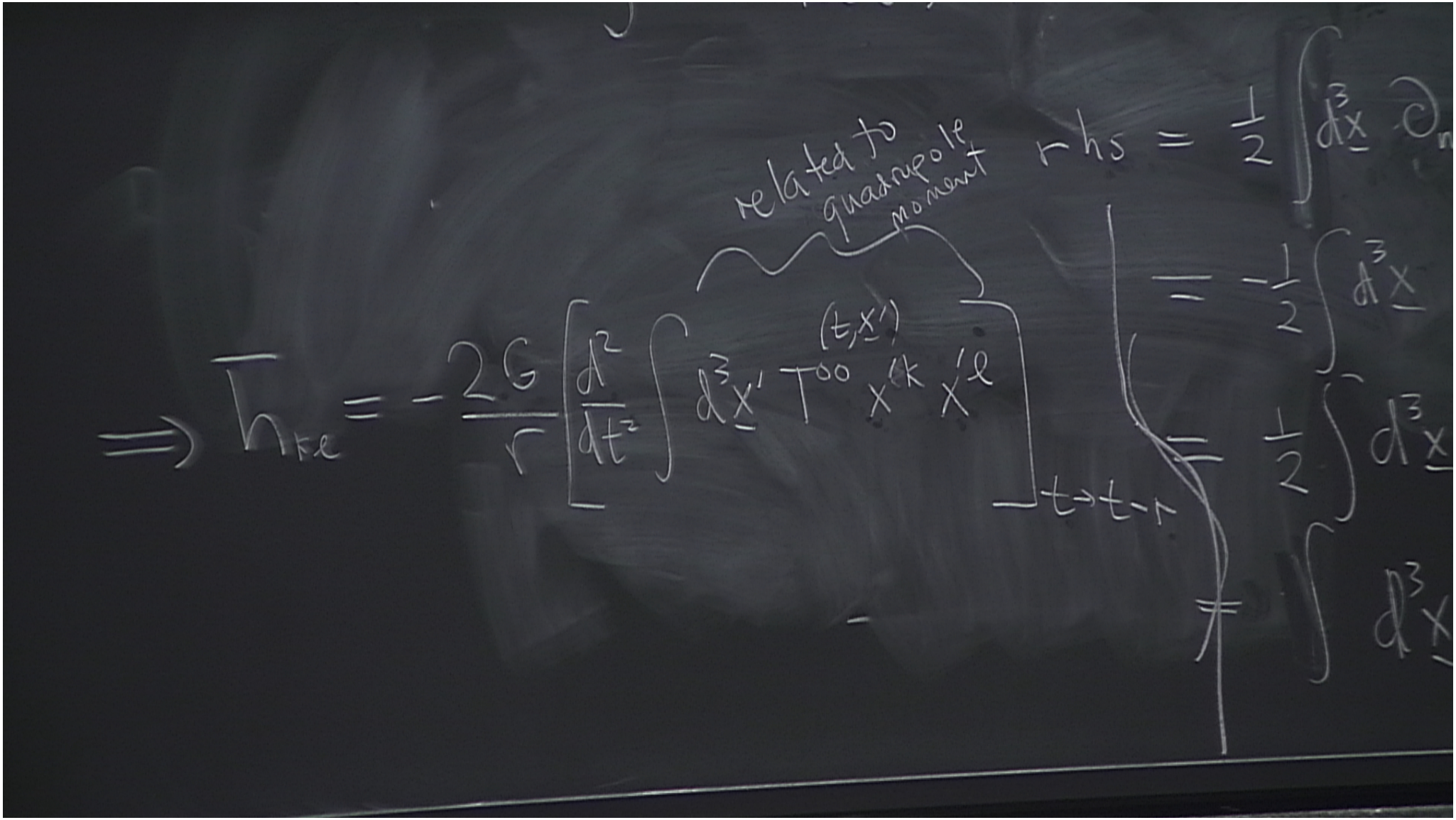
$$= \int d^3\underline{x}\, T^{k\ell}.$$

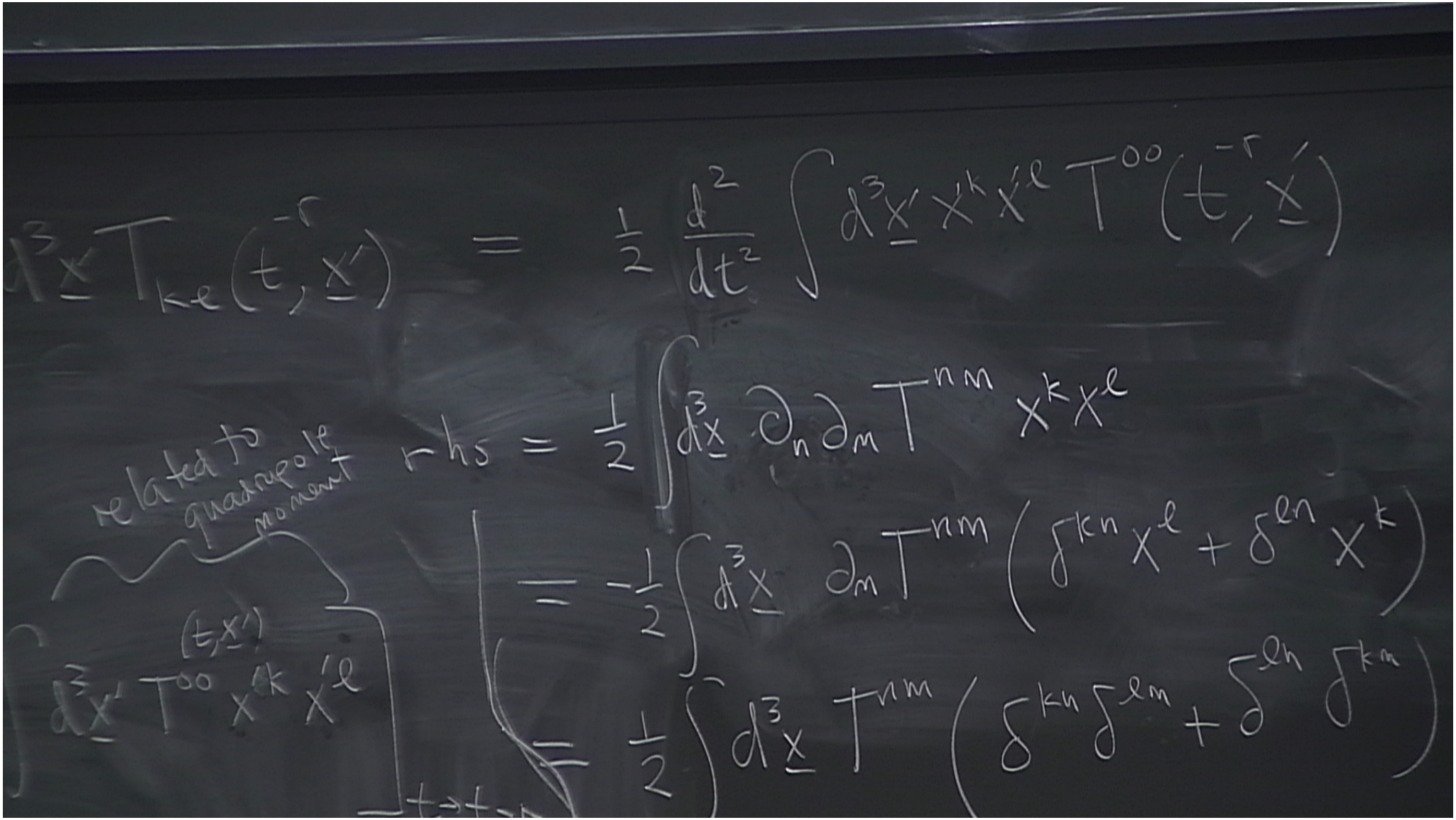
for any $T^{\mu\nu}$ obeying $\partial_\mu T^{\mu\nu} = 0$

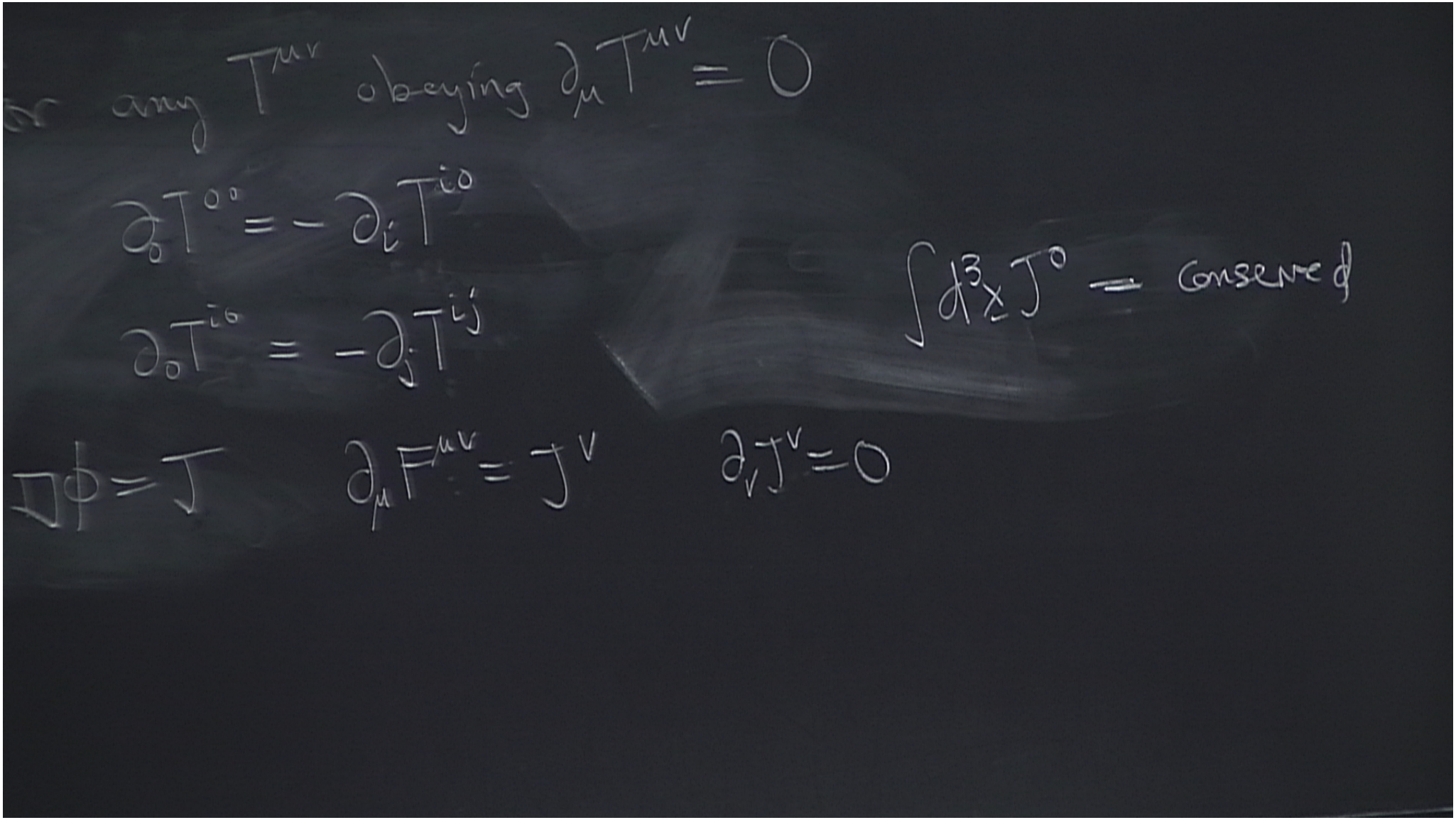
$$\partial_0 T^{00} = -\partial_i T^{i0}$$

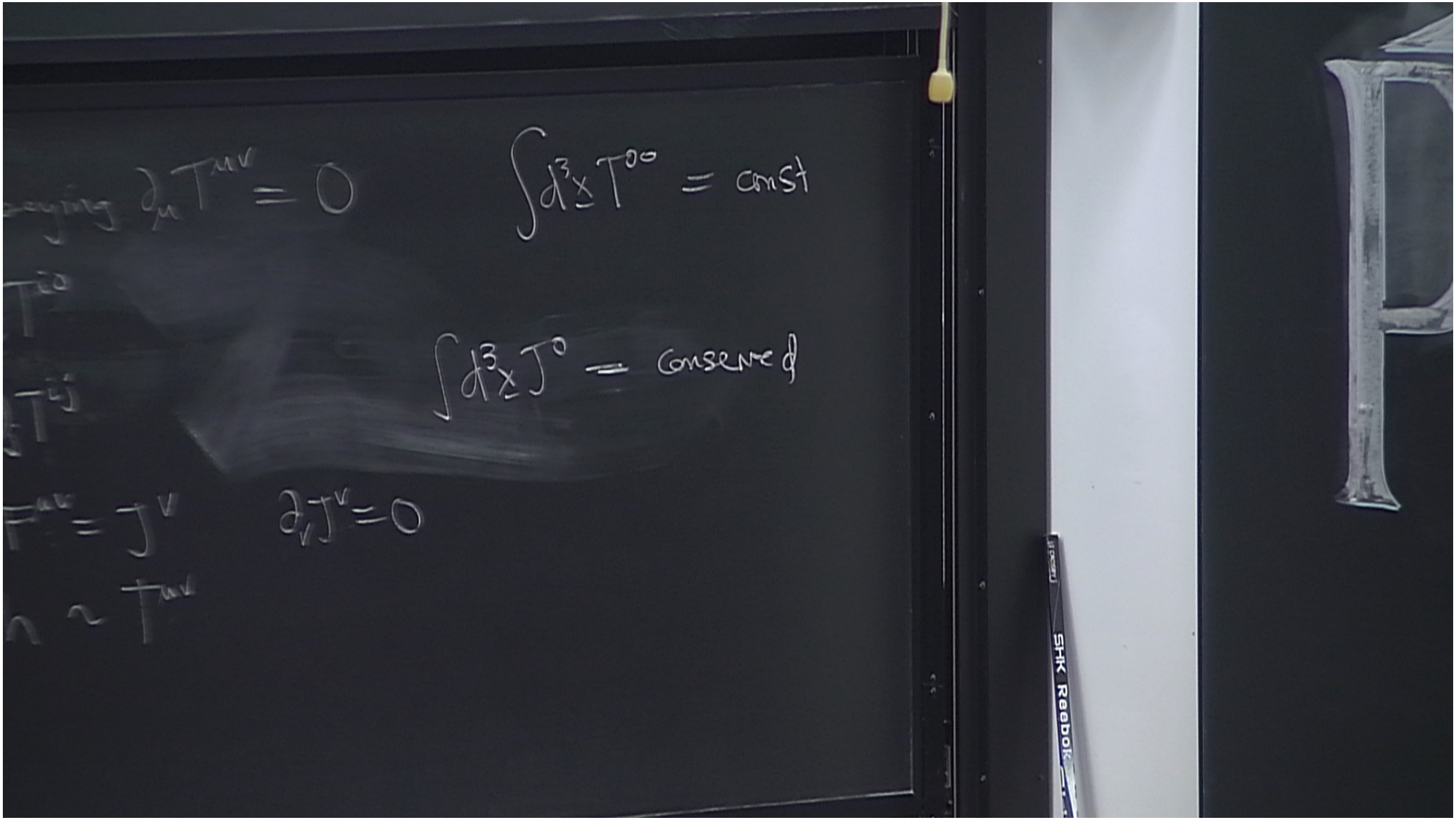
$$\partial_0 T^{i0} = -\partial_j T^{ij}$$


Now, $\int d^3\underline{x}\, T_{ke}(t,\underline{x}) = \frac{1}{2}\frac{d^2}{dt^2}\int d^3\underline{x}\, x^k x^e T^{00}(t,\underline{x})$
$rhs = \frac{1}{2}\int d^3\underline{x}\, \partial_n\partial_m T^{nm}\, x^k x^e$
$= -\frac{1}{2}\int d^3\underline{x}\, \partial_m T^{nm}\left(\delta^{kn} x^e + \delta^{en} x^k\right.$
$= \frac{1}{2}\int d^3\underline{x}\, T^{nm}\left(\delta^{kn}\delta^{em} + \delta^{en}\delta^{km}\right.$
$= \int d^3\underline{x}\, T^{ke}.$
$\Rightarrow h_{ke} = -\frac{2G}{r}\left[\frac{d^2}{dt^2}\int d^3\underline{x}'\, T^{00}\, x'^k x'^e\right]_{t\to t-r}$


related to quadrupole moment
$\Rightarrow h_{TT} = -\frac{2G}{r}\left[\frac{d^2}{dt^2}\int d^3x' T^{00}(t,x') x'^k x'^\ell\right]_{t\to t-r}$
$rhs = \frac{1}{2}\int d^3x \, \partial_n$
$= -\frac{1}{2}\int d^3x$
$= \frac{1}{2}\int d^3x$
$= \int d^3x$


$\int d^3\underline{x}\, T_{ke}(t-r, \underline{x}) = \frac{1}{2}\frac{d^2}{dt^2}\int d^3\underline{x}\, x^k x^e T^{00}(t-r, \underline{x})$
related to quadrupole moment
$rhs = \frac{1}{2}\int d^3\underline{x}\, \partial_n \partial_m T^{nm}\, x^k x^e$
$= -\frac{1}{2}\int d^3\underline{x}\, \partial_m T^{nm}\left(\delta^{kn} x^e + \delta^{en} x^k\right)$
$= \frac{1}{2}\int d^3\underline{x}\, T^{nm}\left(\delta^{kn}\delta^{em} + \delta^{en}\delta^{km}\right)$
$\int d^3\underline{x}\, T^{00}\, x^k x^e$
$(t, \underline{x})$
$t \to t - r$


any $T^{\mu\nu}$ obeying $\partial_\mu T^{\mu\nu} = 0$
$\partial_0 T^{00} = -\partial_i T^{i0}$
$\partial_0 T^{i0} = -\partial_j T^{ij}$
$\int d^3x J^0 =$ Conserved
$\Box\phi = J$
$\partial_\mu F^{\mu\nu} = J^\nu$
$\partial_\nu J^\nu = 0$


$\partial_\mu T^{\mu\nu} = 0$
$\int d^3x\, T^{00} = \text{const}$
$\int d^3x\, J^0 = \text{Conserved}$
$\partial_\nu J^\nu = 0$