

Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 8

Date: Sep 17, 2015 09:00 AM

URL: <http://pirsa.org/15090035>

Abstract:

$$\underbrace{X_{\mu\nu}(g, \partial g, \partial^2 g)} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$X_{00} \rightarrow -\nabla^2 h_{00}$ in
Newtonian limit

$$X_{\mu\nu} \propto R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \equiv G_{\mu\nu} \quad \text{because this obeys } \nabla_\mu G^{\mu\nu} = 0, \text{ necessary for consistency of } *$$

w/ $\nabla_\mu T^{\mu\nu} = 0.$

$$\underbrace{X_{\mu\nu}(g, \partial g, \partial^2 g)} = \frac{8\pi G}{c^4} T_{\mu\nu} *$$

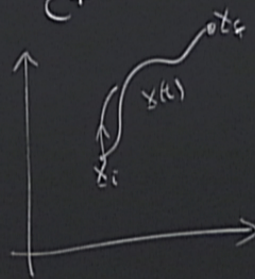
$X_{00} \rightarrow -\nabla^2 h_{00}$ in
Newtonian limit

$$\int d^4x \sqrt{-g} R$$

$$X_{\mu\nu} \propto R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv G_{\mu\nu} \quad \text{becau}$$

$$\underbrace{X_{\mu\nu}(g, \partial g, \partial^2 g)} = \frac{8\pi G}{c^4} T_{\mu\nu} *$$

$X_{00} \rightarrow -\nabla^2 \phi$ in
Newtonian limit



$$X_{\mu\nu} \propto R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv G_{\mu\nu} \quad \text{becau}$$

$$\int d^4x \sqrt{-g} R$$

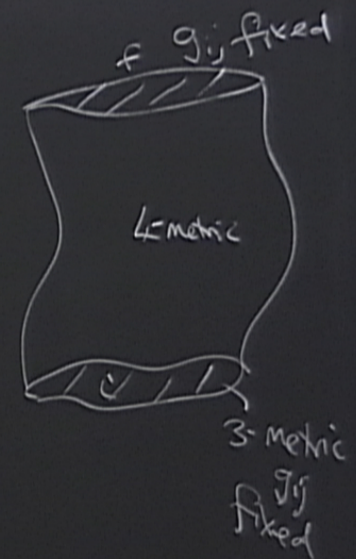
$\Rightarrow \partial_0^2 g$ $\delta x = 0$ at t_i, t_f

$$\delta \left(\int \frac{1}{2} m \dot{x}^2 \right) = 0 \quad \forall \delta x \Rightarrow m \ddot{x} = 0$$

satisfying A.S

$$-\frac{1}{2} \int m \ddot{x} x \, dt \Rightarrow \frac{1}{2} \int m \dot{x}^2 \, dt$$

$$\int m \dot{x} \delta \dot{x} = \left[m \dot{x} \delta x \right]_i^f - \int m \ddot{x} \delta x$$



Question: what is $R_{00} - \frac{1}{2}g_{00}R$

Question: what is $R_{00} - \frac{1}{2}g_{00}R$ in Newtonian limit, assuming the Einstein equation

look at i_j equation $R_{ij} - \frac{1}{2}g_{ij}R \propto T_{ij} \ll T_{00}$ in Newtonian limit
 e.g. $T^{\mu\nu}_{\text{particle}} \propto u^\mu u^\nu$
 $T_{ij} \ll T_{00} \quad \left| \frac{dx^i}{dt} \right| \ll \left| \frac{dt}{dt} \right| c$
 i.e. $\left| \frac{dx^i}{dt} \right| \ll c$

Question: what is $R_{00} - \frac{1}{2}g_{00}R$ in Newtonian limit, assuming the Einstein equation

look at i,j equation $R_{ij} - \frac{1}{2}g_{ij}R \propto T_{ij} \ll T_{00}$ in Newtonian limit

e.g. $T^{\mu\nu}_{\text{particle}} \propto u^\mu u^\nu$

$T_{ij} \ll T_{00}$ $\left|\frac{dx^i}{dt}\right| \ll \left|\frac{dt}{dt}\right|c$
i.e. $\left|\frac{dx^i}{dt}\right| \ll c$

recall $R_{\mu\nu} \sim \partial^2 h \Rightarrow R_{ij} \approx \frac{1}{2}\delta_{ij}R$

$$R_{\mu\nu} = R_{\rho\mu\sigma\nu} g^{\rho\sigma} \Rightarrow R \approx -R_{00} + R_{kk}$$

$R = g^{\mu\nu} R_{\mu\nu}$ at first order in h
can just use η

Question: what is $R_{00} = \frac{1}{2} g_{00} R$ in Newtonian limit, assuming the Einstein equation

look at i,j equation $R_{ij} = \frac{1}{2} g_{ij} R \propto \overset{\text{spatial stress}}{T_{ij}} \ll \overset{\text{energy density}}{T_{00}}$ in Newtonian limit

e.g. $T^{\mu\nu}_{\text{particle}} \propto u^\mu u^\nu$

$T_{ij} \ll T_{00} \quad \left| \frac{dx^i}{dt} \right| \ll \left| \frac{dt}{dt} \right| c$
 i.e. $\left| \frac{dx^i}{dt} \right| \ll c$

recall $R_{\mu\nu} \sim \partial^2 h$

$$R_{\mu\nu} = R_{\rho\mu\sigma\nu} g^{\rho\sigma}$$

$$\Rightarrow R_{ij} \approx \frac{1}{2} \delta_{ij} R$$

$$\Rightarrow R \approx -R_{00} + R_{kk} = -R_{00} + \frac{3}{2} R \Rightarrow \underline{R = 2R_{00}}$$

$$R = g^{\mu\nu} R_{\mu\nu} \text{ at first order in } h$$

can just use η

Question: what is $R_{00} - \frac{1}{2}g_{00}R$ in Newtonian limit, assuming the Einstein equation

look at i, j equation $R_{ij} - \frac{1}{2}g_{ij}R \propto T_{ij} \ll T_{00}$ in Newtonian limit
 spatial stress energy density
 e.g. $T_{\mu\nu}^{\text{particle}} \propto u^\mu u^\nu$
 $T_{ij} \ll T_{00}$

recall $R_{\mu\nu} \sim \partial^2 h$
 $R_{\mu\nu} = R_{\rho\mu\sigma\nu} g^{\rho\sigma}$

$$\Rightarrow R_{ij} \approx \frac{1}{2} \delta_{ij} R$$

$$\Rightarrow R \approx -R_{00} + R_{kk} = -R_{00} + \frac{3}{2}R \Rightarrow \underline{R = 2R_{00}}$$

$R = g^{\mu\nu} R_{\mu\nu}$ at first order in h .
 can just use η

$g_{00} = \eta_{00} + h_{00}$
 depthwise
 $\Rightarrow G_{00} \approx R_{00} + \frac{1}{2}R = \underline{\underline{2R_{00}}}$

$\frac{dv^i}{dt} \ll \frac{dt}{dt} c$
 i.e. $\left| \frac{dx^i}{dt} \right| \ll c$

→ can just use η

$$g_{00} = \eta_{00} + h_{00} \Rightarrow G_{00} \approx R_{00} + \frac{1}{2}R \approx \underline{\underline{2R_{00}}} \quad (\neq)$$

↓
drop this

$$R_{\rho\mu\nu} = \frac{1}{2} (h_{\nu\sigma,\rho\mu} - h_{\rho\nu,\sigma\mu} - h_{\mu\sigma,\rho\nu} + h_{\rho\mu,\sigma\nu}) + O(h^2)$$

$$\Rightarrow R_{0000} = 0$$

↙ no ∂ 's

$$R_{0i0j} \approx -\frac{1}{2} h_{00,ij} \Rightarrow R_{00} \approx R_{0i0i} = -\frac{1}{2} \nabla^2 h_{00}$$

$$(\neq) \Rightarrow G_{00} = -\nabla^2 h_{00}$$

so the Einstein equations are:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

→ can just use η

$$J_{00} = \eta_{00} + h_{00} \Rightarrow G_{00} \approx R_{00} + \frac{1}{2}R = \underline{\underline{2R_{00}}} \quad (\neq)$$

↓
drop this

$$R_{\mu\nu} = \frac{1}{2} (h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma} - h_{\mu\sigma,\nu} + h_{\sigma\mu,\nu}) + O(h^2)$$

↙ no ∂ 's

$$\Rightarrow R_{0000} = 0$$

$$R_{i0j0} \approx -\frac{1}{2} h_{00,ij} \Rightarrow R_{00} \approx R_{0i0i} = -\frac{1}{2} \nabla^2 h_{00}$$

$$g^{\mu\nu} g_{\mu\nu} = 4$$

$$(\neq) \Rightarrow G_{00} = -\nabla^2 h_{00}$$

so the Einstein equations are:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

another form: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

$$G^\lambda{}_\lambda = R - 2R = -R$$

$$= \frac{8\pi G}{c^4} T^\lambda{}_\lambda$$

so Einstein eq's can also be written

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda \right)$$

$$G_{00} \approx R_{00} + \frac{1}{2}R = \underline{\underline{2R_{00}}} \quad (\neq)$$

$\downarrow$
 approx

$$\gamma_{\mu\nu} = h_{\mu\nu,0} + h_{\mu\nu,0} + o(h^2)$$

$$\Rightarrow R_{00} \approx R_{00,0} = -\frac{1}{2}\nabla^2 h_{00}$$

$$(\neq) \Rightarrow G_{00} = -\nabla^2 h_{00} \quad \text{so the Einstein equations are: } \boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad G^\lambda_\lambda = R - 2R = -R = \frac{8\pi G}{c^4} T^\lambda_\lambda$$

so Einstein eq's can also be written

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda_\lambda \right)$$

useful for gravitational radiation

extra term $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

extra term $G_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} + \Lambda g_{\mu\nu})$

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0$$

extra term $G_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} + \Lambda g_{\mu\nu})$ ← "cosmological constant"

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0 \quad \nabla_\mu g^{\mu\nu} = 0 \Rightarrow \text{consistent with energy-momentum conservation.}$$

extra term $G_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} + \Lambda g_{\mu\nu})$ ← "cosmological constant"

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0 \quad \nabla_\mu g^{\mu\nu} = 0 \Rightarrow \text{consistent with energy-momentum conservation.}$$

Solutions to the Einstein equations

V. complicated,

Solutions to the Einstein equations

V. complicated, so let's look at Maxwell first.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = -J^\nu$$

for given J^ν , the solution for A_μ is undetermined.

Solutions to the Einstein equations

V. complicated, so let's look at Maxwell first.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = -J^\nu$$

for given J^ν , the solution for A_μ is undetermined.

F.T. w.r.t spacetime

$$A_\mu = \sum_{k, \chi} a_\mu(k) e^{ik \cdot x}$$

$$\text{thus} \Rightarrow \underbrace{(-k^2 \delta_\mu^\nu + k_\mu k^\nu)}_{M_\mu^\nu} A^\mu = -J^\nu$$

M_μ^ν is not invertible: has a zero eigenvector M^μ_μ

Einstein equations

+ Maxwell first.

$$\partial^\mu A^\nu - \partial^\nu A^\mu = -J^\nu \quad \text{for given } J^\nu, \text{ the solution for } A_\mu \text{ is undetermined.}$$

$$\text{Ans} \Rightarrow \underbrace{(-k^2 \delta_\mu^\nu + k_\mu k^\nu)}_{M_\mu^\nu} A^\mu = -J^\nu$$

M_μ^ν is not invertible: has a zero eigenvector $M_\mu^\nu k_\nu = 0$ if $k^2 = 0$.

Einstein equations

+ Maxwell first.

$$\partial^\mu A^\nu - \partial^\nu A^\mu = -J^\nu \quad \text{for given } J^\nu, \text{ the solution for } A_\mu \text{ is undetermined.}$$

$$\text{Ans} \Rightarrow \underbrace{(-k^2 \delta_\mu^\nu + k_\mu k^\nu)}_{M_\mu^\nu} A^\mu = -J^\nu$$

M_μ^ν is not invertible: has a zero eigenvector $M_\mu^\nu k_\nu = 0$ if $k^2 = 0$.

$$k_\mu k_\nu \eta^{\mu\nu} = -k^2 + k^2 = 0$$

$$(k, 0, 0, k)$$

$$A_\mu = \sum_{\mathbf{k}} a_\mu(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}$$

$$\text{Lus} \Rightarrow \underbrace{(-k^2 \delta_\mu^\nu + k_\mu k^\nu)}_{M_\mu^\nu} A^\nu = -J$$

M_μ^ν is not invertible: has a zero eigenvector $M_\mu^\nu k_\nu = 0$ if $k^2 = 0$.

reason: Maxwell equations are gauge invariant under $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \epsilon$, where ϵ is any scalar function in spacetime

$$\begin{aligned} F_{\mu\nu} &\rightarrow F'_{\mu\nu} = F_{\mu\nu} \\ J_\nu &\rightarrow J'_\nu = J_\nu \end{aligned}$$

Solve M.E.'s uniquely, need to pick a gauge i.e. to fix the function ϵ .

$$A_\mu = \sum_{\mathbf{k}} a_\mu(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\text{Lus} \Rightarrow \underbrace{(-k^2 \delta_\mu^\nu + k_\mu k^\nu)}_{M_\mu^\nu} A^\nu = -J$$

M_μ^ν is not invertible: has a zero eigenvector $M_\mu^\nu k_\nu = 0$ if $k^2 = 0$.

reason: Maxwell equations are gauge invariant under $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \epsilon$, where ϵ is any scalar function in spacetime

$$\begin{aligned} F_{\mu\nu} &\rightarrow F'_{\mu\nu} = F_{\mu\nu} \\ J_\nu &\rightarrow J'_\nu = J_\nu \end{aligned}$$

solve M.E.'s uniquely, need to pick a gauge i.e. to fix the function ϵ .

Lorentz gauge

if $\partial_\mu A^\mu \neq 0$, choose ϵ st. $\partial_\mu A'^\mu = 0$.

examples: impose $\partial_\mu A^\mu = 0$

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \epsilon = 0 \Rightarrow \partial^2 \epsilon = -\partial_\mu A^\mu$$

can always be solved for ϵ , but solution is still ambiguous, because g-condition only fixes ϵ up to functions satisfying $\partial^2 \epsilon = 0$.

reason: Maxwell equations are gauge invariant under $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \epsilon$
 $F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu}$
 $J_\nu \rightarrow J'_\nu = J_\nu$

to solve M.E.'s uniquely, need to pick a gauge i.e. to fix the function ϵ .

examples: impose $\partial_\mu A^\mu = 0$ (Lorentz gauge) if $\partial_\mu A^\mu \neq 0$, choose ϵ st. $\partial_\mu A'^\mu = 0$.
 $\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \epsilon = 0 \Rightarrow \partial^2 \epsilon = -\partial_\mu A^\mu$, can a
 g. cond

int under $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \epsilon$, where ϵ is any scalar function in spacetime

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu}.$$

$$J_\nu \rightarrow J'_\nu = J_\nu$$

to fix the function ϵ .

ϵ st. $\partial_\mu A'^\mu = 0$.

$$= 0 \Rightarrow \partial^2 \epsilon = -\partial_\mu A'^\mu,$$

can always be solved for ϵ , but solution is still ambiguous, because
g.condition only fixes ϵ up to functions satisfying $\partial^2 \epsilon = 0$.

$$R_{\mu\nu} = R_{\mu\sigma\nu} g^{\sigma\rho}$$

$$R = g^{\mu\nu} R_{\mu\nu} \text{ at first order in } h$$

→ can just use η

$$\Rightarrow R \approx -R_{00} + R_{kk} = -R_{00} + \frac{3}{2}R \Rightarrow \underline{R = 2R_{00}}$$

$$g_{00} = \eta_{00} + h_{00} \Rightarrow G_{00} \approx R_{00} + \frac{1}{2}R = \underline{\underline{2R_{00}}} \quad (\neq)$$

↓
depression

$$\left| \frac{d\mathbf{x}}{dt} \right| \ll c$$

• f Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ completely fixes gauge freedom

$$\nabla \mathbf{A}' = \nabla \mathbf{A} + \nabla^2 \mathbf{E} = 0 \Rightarrow \text{uniquely fixes } \mathbf{E}$$

$\nexists$ solutions to $\nabla^2 \mathbf{E} = 0$ with $\mathbf{E} \rightarrow 0$ at ∞ except $\mathbf{E} = 0$.

G.R. $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$: looks like 10 equations for 10 unknowns, $g_{\mu\nu}$.
but: there are four identities $\nabla_\mu G^{\mu\nu} = 0 \Rightarrow$ really only six independent eqns.

10 unknowns, $g_{\mu\nu}$.

only six independent eqns.

$$\partial_\mu F^{\mu\nu} = -J^\nu$$

$$\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu J^\nu = 0$$

$\Rightarrow \exists$ 1 differential identity

$\rightarrow A_\mu$ not completely determined.

G.R. $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$: looks like 10 equations for 10 unknowns, $g_{\mu\nu}$.

but: there are four identities $\nabla_\mu G^{\mu\nu} = 0 \Rightarrow$ really only six independent eqs.

reason for ambiguity in solution is gauge invariance:

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x)$$

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

$$\delta g_{\mu\nu} = -(\epsilon_{\nu;\mu} + \epsilon_{\mu;\nu})$$

Lie derivative $\mathcal{L}_\epsilon g_{\mu\nu}$.

$$g'_{\mu\nu}(x) - g_{\mu\nu}(x)$$

$$A_\mu = \sum_{\mathbf{k}} a_\mu(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

M_μ^ν is not invertible : has a zero eigenvector $M_\mu^\nu k_\nu$

Need to choose a gauge : one choice is "harmonic gauge"

M_{μ}^{ν} is not invertible : has a zero eigenvector

Need to choose a gauge : one choice is "harmonic gauge"

$$\text{e.g. } \partial_{\mu} g^{\mu\nu} = 0$$

c.f.

$$\Gamma_{\mu\nu}^{\lambda} g^{\mu\nu} \equiv \Gamma^{\lambda} = 0$$

particularly nice because $\underbrace{g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \phi}_{\partial_{\lambda} \phi} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi)$

M_{μ}^{ν} is not invertible : has a zero eigenvector

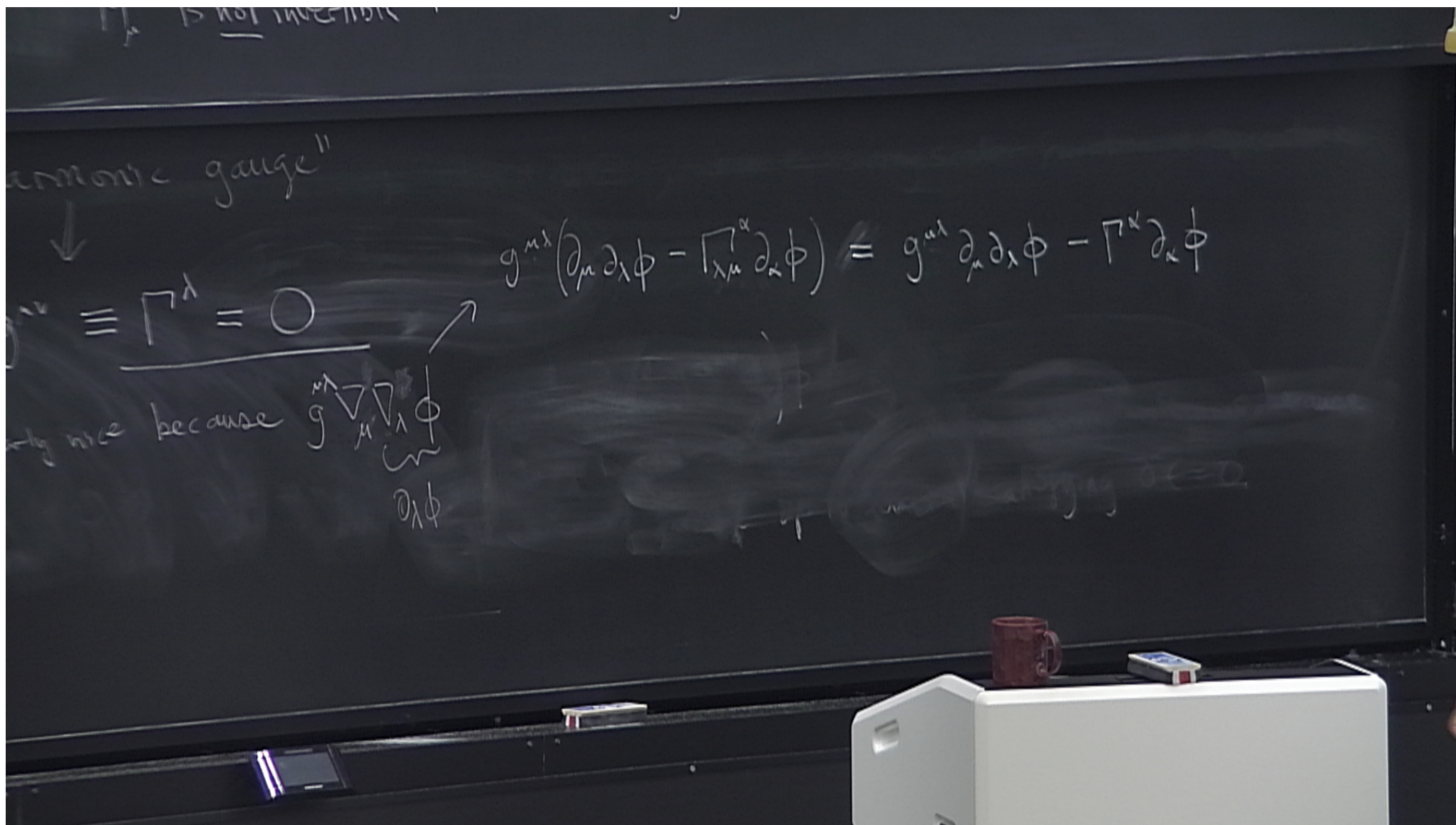
Need to choose a gauge : one choice is "harmonic gauge"

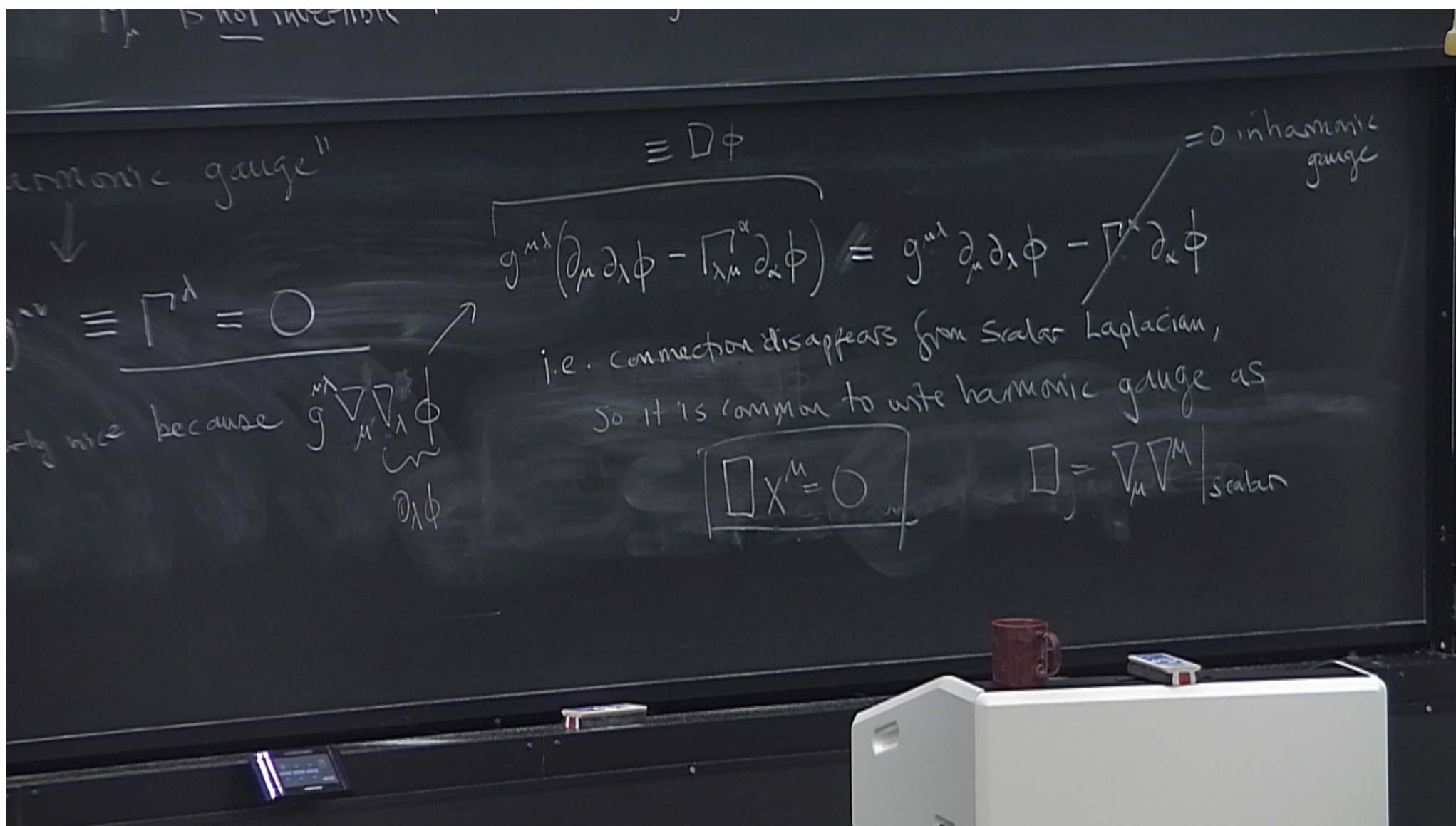
e.g. $\partial_{\mu} g^{\mu\nu} = 0$

c.f.

$$\Gamma_{\mu\nu}^{\lambda} g^{\mu\nu} \equiv \Gamma^{\lambda} = 0$$

particularly nice because $\underbrace{\bar{g}^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \phi}_{\partial_{\lambda} \phi} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi)$





Need to choose a gauge : one choice is "harmonic"

$$\text{e.g. } \partial_\mu g^{\mu\nu} = 0$$

$$\text{cf } \Gamma_{\mu\nu}^\lambda g^{\mu\nu} \equiv$$

particularly nice

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\int \partial_\mu A_\nu F^{\mu\nu} + c F^{\mu\nu} F_{\mu\nu}$$