Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 7

Date: Sep 16, 2015 09:00 AM

URL: http://pirsa.org/15090034

Abstract:

recall that $R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}_{\rho\mu} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\sigma}_{\alpha\nu} + \Gamma^{\alpha}_{\rho\nu}\Gamma^{\sigma}_{\alpha\mu}$ (*)

resulted from $[\nabla_\mu, \nabla_\nu] S_\rho \equiv -R^{\sigma}{}_{\rho\mu\nu} S_\sigma$ under the assumption of no torsion i.e. $T^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} = 0$

when we include torsion, adopt same definition of $R^{\sigma}{}_{\rho\mu\nu}$ as in (*), but this obeys $[\nabla_\mu, \nabla_\nu] S_\rho = -R^{\sigma}{}_{\rho\mu\nu} S_\sigma - T^{\beta}_{\mu\nu} \nabla_\beta S_\rho$

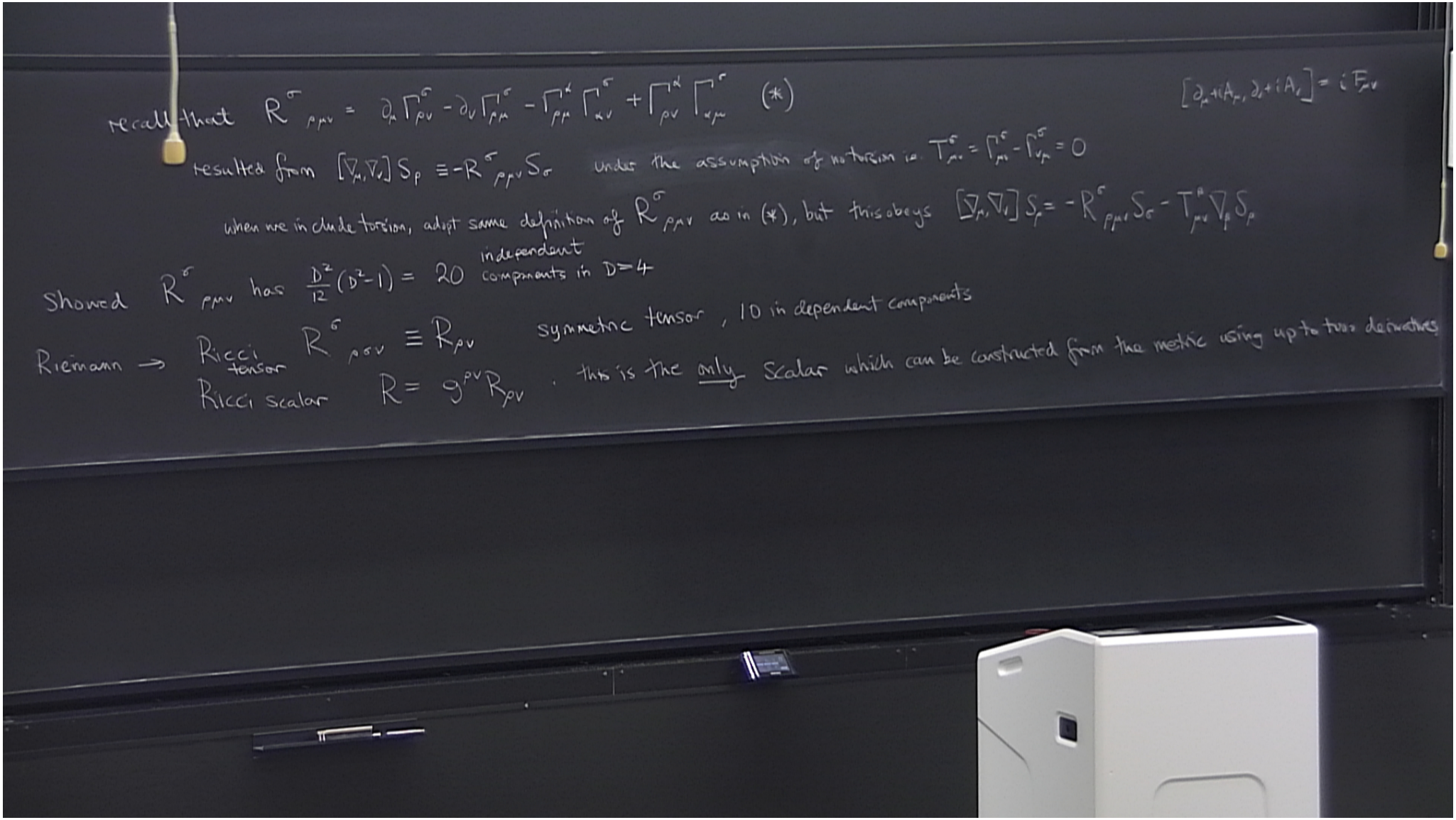
$[\partial_\mu + iA_\mu, \partial$

recall that $R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}_{\rho\mu} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\sigma}_{\alpha\nu} + \Gamma^{\alpha}_{\rho\nu}\Gamma^{\sigma}_{\alpha\mu}$ (*)

resulted from $[\nabla_\mu, \nabla_\nu] S_\rho \equiv -R^{\sigma}{}_{\rho\mu\nu} S_\sigma$ under the assumption of no torsion i.e. $T^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} = 0$

when we include torsion, adopt same definition of $R^{\sigma}{}_{\rho\mu\nu}$ as in (*), but this obeys $[\nabla_\mu, \nabla_\nu] S_\rho = -R^{\sigma}{}_{\rho\mu\nu} S_\sigma - T^{\alpha}_{\mu\nu}$

Showed $R^{\sigma}{}_{\rho\mu\nu}$ has $\frac{D^2}{12}(D^2-1) = 20$ independent components in $D=4$


recall that $R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}_{\rho\mu} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\sigma}_{\alpha\nu} + \Gamma^{\alpha}_{\rho\nu}\Gamma^{\sigma}_{\alpha\mu}$ (*)
resulted from $[\nabla_\mu, \nabla_\nu] S_\rho \equiv -R^{\sigma}{}_{\rho\mu\nu} S_\sigma$ under the assumption of no torsion i.e. $T^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} = 0$
$[\partial_\mu + iA_\mu, \partial_\nu + iA_\nu] = iF_{\mu\nu}$
when we include torsion, adopt same definition of $R^{\sigma}{}_{\rho\mu\nu}$ as in (*), but this obeys $[\nabla_\mu, \nabla_\nu] S_\rho = -R^{\sigma}{}_{\rho\mu\nu} S_\sigma - T^{\alpha}_{\mu\nu} \nabla_\alpha S_\rho$
Showed $R^{\sigma}{}_{\rho\mu\nu}$ has $\frac{D^2}{12}(D^2-1) = 20$ independent components in $D=4$
Riemann → Ricci tensor $R^{\sigma}{}_{\rho\sigma\nu} \equiv R_{\rho\nu}$ symmetric tensor, 10 independent components
Ricci scalar $R = g^{\rho\nu} R_{\rho\nu}$. this is the only scalar which can be constructed from the metric using up to two derivatives

when we include torsion, adopt same definition of $R^{\sigma}{}_{\rho\mu\nu}$ as in (*), but this obeys $[\nabla_\mu, \nabla_\nu] S_\rho = -R^{\sigma}{}_{\rho\mu\nu} S_\sigma - T^{\lambda}{}_{\mu\nu} \nabla_\lambda S_\rho$

$R^{\sigma}{}_{\rho\mu\nu}$ has $\frac{D^2}{12}(D^2-1) = 20$ independent components in $D=4$

→ Ricci tensor $R^{\sigma}{}_{\rho\sigma\nu} \equiv R_{\rho\nu}$ symmetric tensor, 10 independent components

Ricci scalar $R = g^{\rho\nu} R_{\rho\nu}$. this is the only scalar which can be constructed from the metric using up to

⇒ unique variational principle for gravity. $\int \underbrace{d^4x\sqrt{-g}}_{\text{Scalar}} R$

g/tensor

showed

$\rho\mu\nu$ has $\frac{1}{12}$

Riemann →

Ricci tensor $R^{\sigma}{}_{\rho\sigma\nu} \equiv R_{\rho\nu}$ symmetric tensor, 10 in dependent components

Ricci scalar $R = g^{\rho\nu} R_{\rho\nu}$. this is the only scalar which can be constructed from the metric ($g_{\mu\nu}$)

$\Rightarrow$ unique variational principle for gravity. $\int \underbrace{d^4x\sqrt{-g}}_{\text{scalar}} R$

Weyl tensor $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)}\left(g_{\mu\sigma}R_{\nu\rho} + g_{\nu\rho}R_{\mu\sigma} - g_{\nu\sigma}R_{\rho\mu} - g_{\mu\rho}R_{\nu\sigma}\right) + \frac{1}{(D-1)(D-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right)R$

$\Rightarrow$ traceless on all pairs of indices e.g. $C^{\mu}{}_{\nu\mu\sigma} = 0$

showed

$\rho\mu\nu$ has $\frac{1}{12}$

iemann → Ricci tensor $R^{\sigma}{}_{\rho\sigma\nu} \equiv R_{\rho\nu}$ symmetric tensor, 10 in dependent components

Ricci scalar $R = g^{\rho\nu} R_{\rho\nu}$. this is the only scalar which can be constructed from the metric us

$\Rightarrow$ unique variational principle for gravity. $\int \underbrace{d^4x\sqrt{-g}}_{\text{scalar}} R$

Weyl tensor $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{(D-2)}\left(g_{\mu\sigma}R_{\nu\rho} + g_{\nu\rho}R_{\mu\sigma} - g_{\nu\sigma}R_{\rho\mu} - g_{\mu\rho}R_{\nu\sigma}\right) + \frac{1}{(D-1)(D-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right)R$

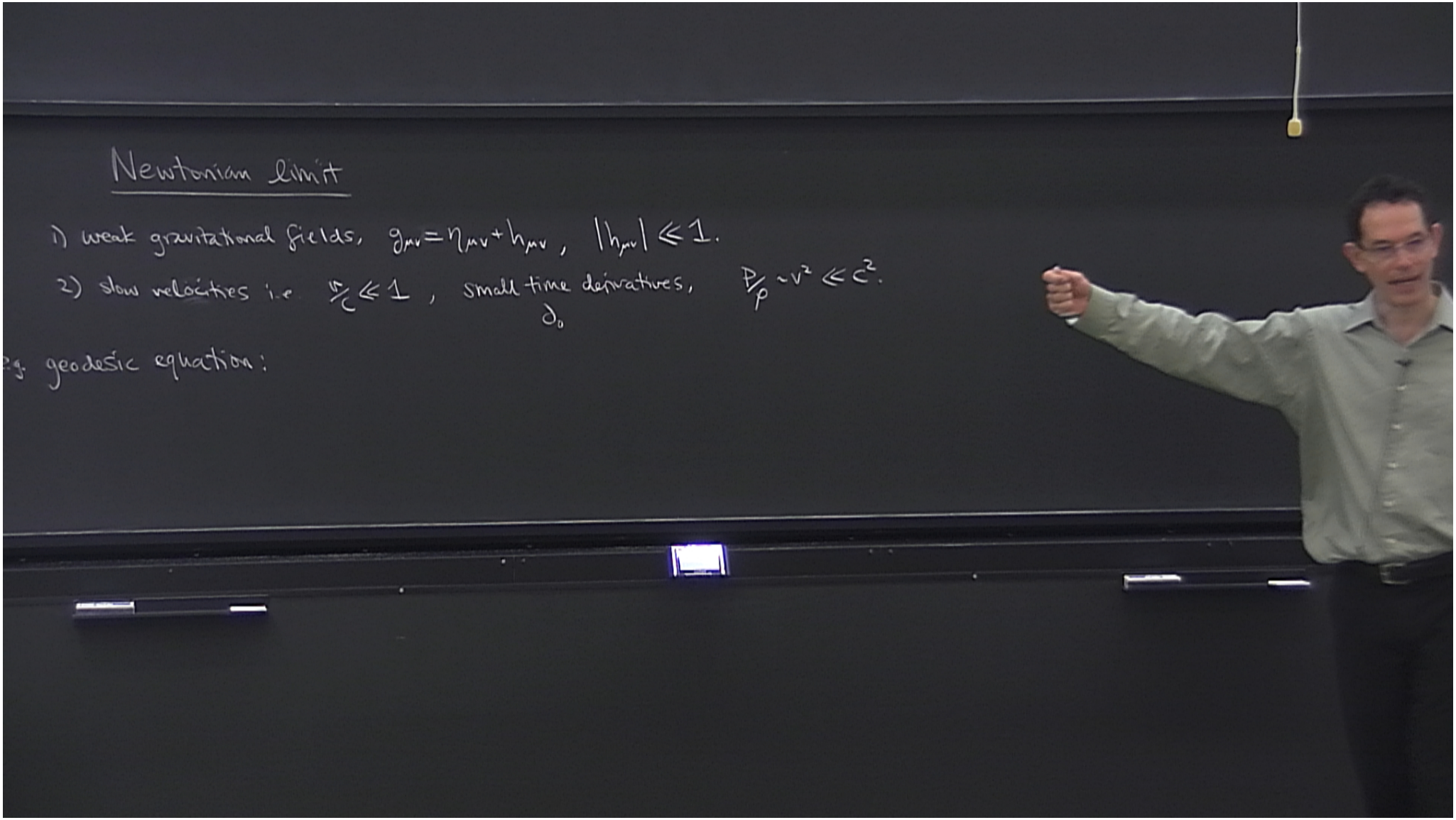
$\Rightarrow$ traceless on all pairs of indices e.g. $C^{\alpha}{}_{\nu\mu\sigma} = 0$

$\| \; g^{\alpha\lambda} C_{\lambda\nu\mu\sigma}$

Two theorems:

1) If $R^{\mu}{}_{\nu\rho\sigma} = 0 \Rightarrow \exists$ a set of coordinate systems in which $g_{\mu\nu} = \eta_{\mu\nu}$.

2) If $C_{\mu\nu\rho\sigma} = 0 \Rightarrow$ " " " $g_{\mu\nu} = \Omega^2(x)\,\eta_{\mu\nu}$ (conformal factor) — metric is conformally flat.

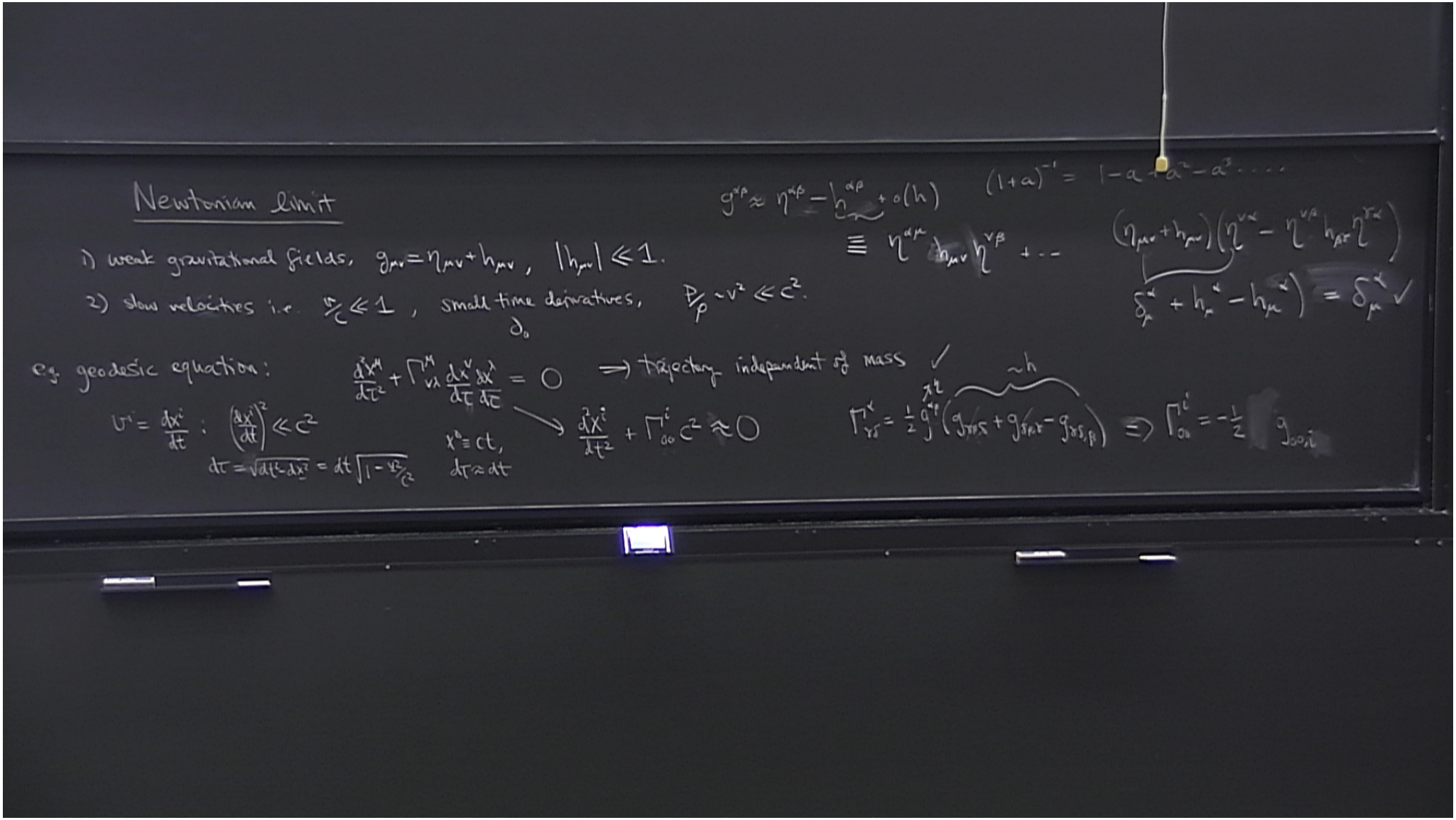

Newtonian limit
1) weak gravitational fields, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$.
2) slow velocities i.e $\frac{v}{c} \ll 1$, small time derivatives, $\frac{P}{\rho} \sim v^2 \ll c^2$.
∂_0
e.g. geodesic equation:

Newtonian limit

1) weak gravitational fields, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$.

2) slow velocities i.e. $\frac{v}{c} \ll 1$, small time derivatives, ∂_0 $\frac{P}{\rho} \sim v^2 \ll c^2$.

e.g. geodesic equation: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$


Newtonian limit
1) weak gravitational fields, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$.
2) slow velocities i.e. $\frac{v}{c} \ll 1$, small time derivatives, $\frac{\partial_0}{\partial_0}$, $\frac{p}{\rho} \sim v^2 \ll c^2$.
$g^{\alpha\beta} \approx \eta^{\alpha\beta} - h^{\alpha\beta} + o(h)$
$\equiv \eta^{\alpha\mu} h_{\mu\nu} \eta^{\nu\beta} + \dots$
$(1+a)^{-1} = 1 - a + a^2 - a^3 \dots$
$(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\alpha} - \eta^{\nu\rho} h_{\rho\sigma} \eta^{\sigma\alpha})$
$\delta^\alpha_\mu + h^\alpha_\mu - h^\alpha_\mu) = \delta^\alpha_\mu$ ✓
e.g. geodesic equation:
$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$ ⇒ trajectory independent of mass ✓
$v^i = \frac{dx^i}{dt}$; $\left(\frac{dx^i}{dt}\right)^2 \ll c^2$
$d\tau = \sqrt{dt^2 - dx^2} = dt\sqrt{1 - v^2/c^2}$
$x^0 \equiv ct$, $d\tau \approx dt$
$\rightarrow \frac{d^2 x^i}{dt^2} + \Gamma^i_{00} c^2 \approx 0$
$\Gamma^\alpha_{\gamma\delta} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\gamma,\delta} + g_{\beta\delta,\gamma} - g_{\gamma\delta,\beta})$ ~h
$\Rightarrow \Gamma^i_{00} = -\frac{1}{2} g_{00,i}$

Newtonian limit

$g^{\alpha\beta} \approx \eta^{\alpha\beta} - h^{\alpha\beta} + o(h)$ $\quad (1+a)^{-1} = 1 - a + a^2 - a^3 \dots$

$\equiv \eta^{\alpha\mu} h_{\mu\nu} \eta^{\nu\beta} + \dots$ $\quad (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\alpha} - \eta^{\nu\rho} h_{\rho\sigma} \eta^{\sigma\alpha})$

$\delta^\alpha_\mu + h_\mu{}^\alpha - h_\mu{}^\alpha) = \delta^\alpha_\mu$ ✓

1) weak gravitational fields, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$.

2) slow velocities i.e. $v/c \ll 1$, small time derivatives, ∂_0 $\quad P/\rho \sim v^2 \ll c^2$.

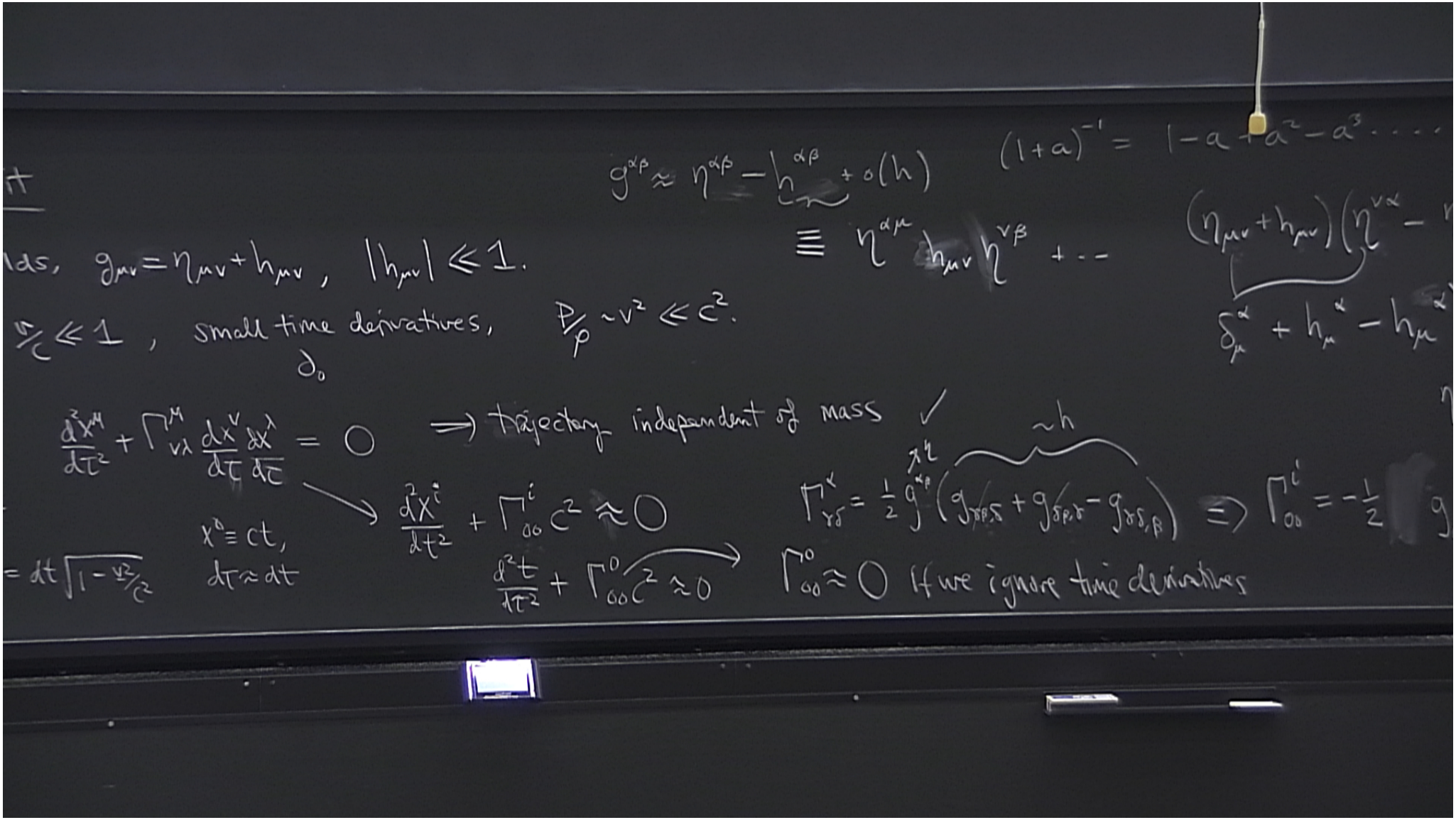
e.g. geodesic equation: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$ $\Rightarrow$ trajectory independent of mass ✓

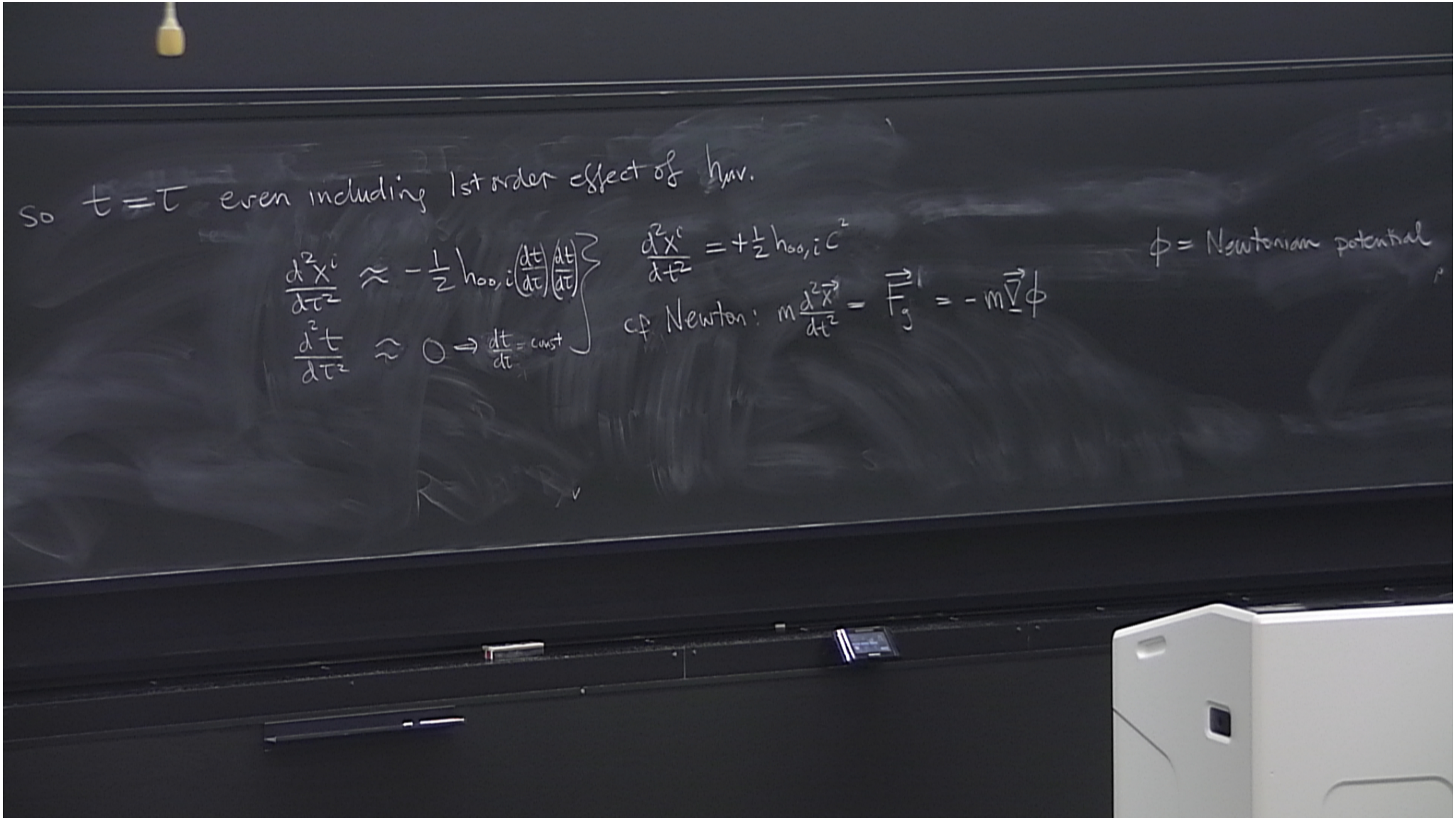
$v^i = \frac{dx^i}{dt}$; $\left(\frac{dx^i}{dt}\right)^2 \ll c^2$

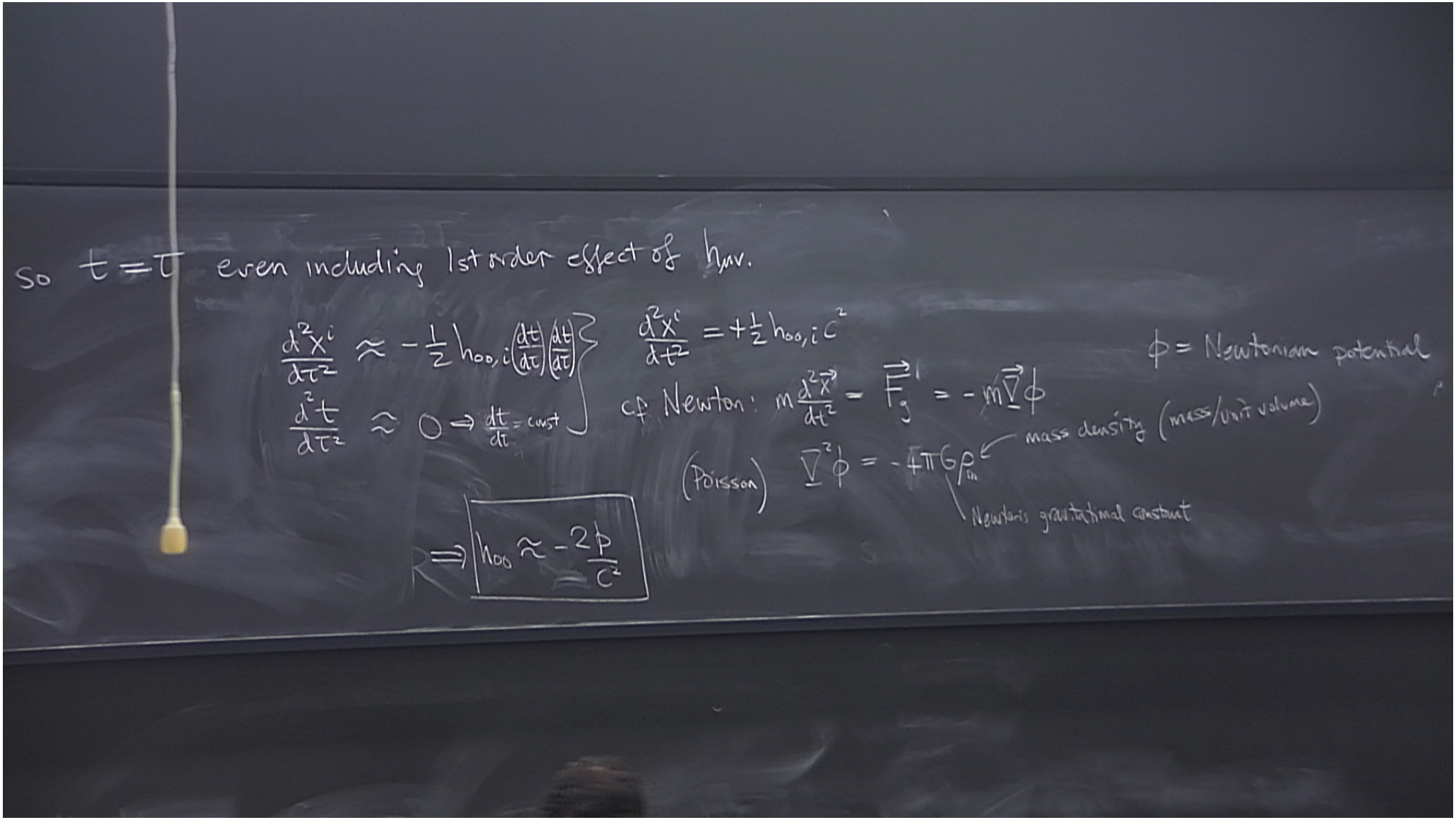
$d\tau = \sqrt{dt^2 - dx^2} = dt\sqrt{1 - v^2/c^2}$ $\quad x^0 \equiv ct$, $d\tau \approx dt$

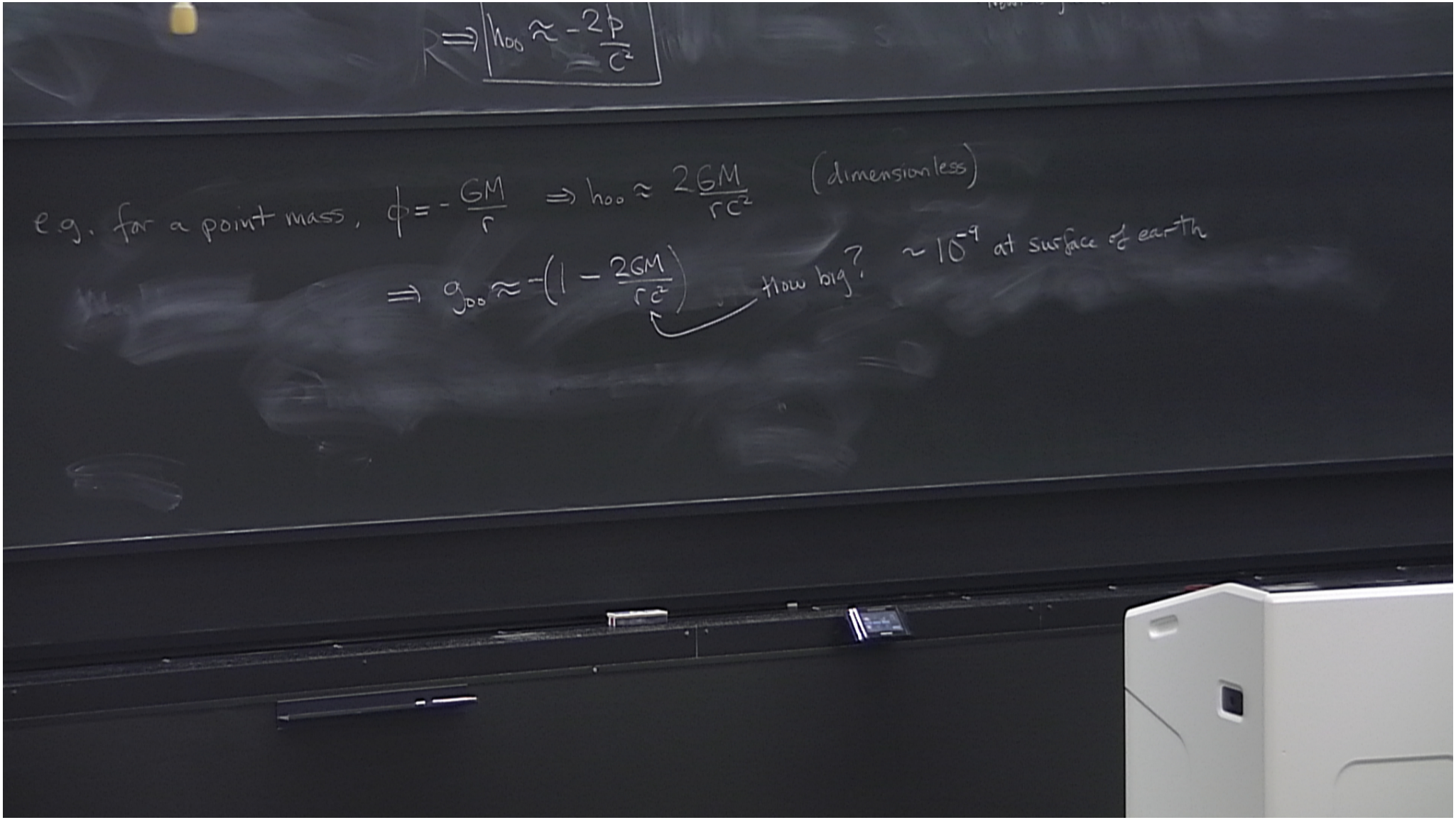
$\Rightarrow \frac{d^2x^i}{dt^2} + \Gamma^i_{00} c^2 \approx 0$

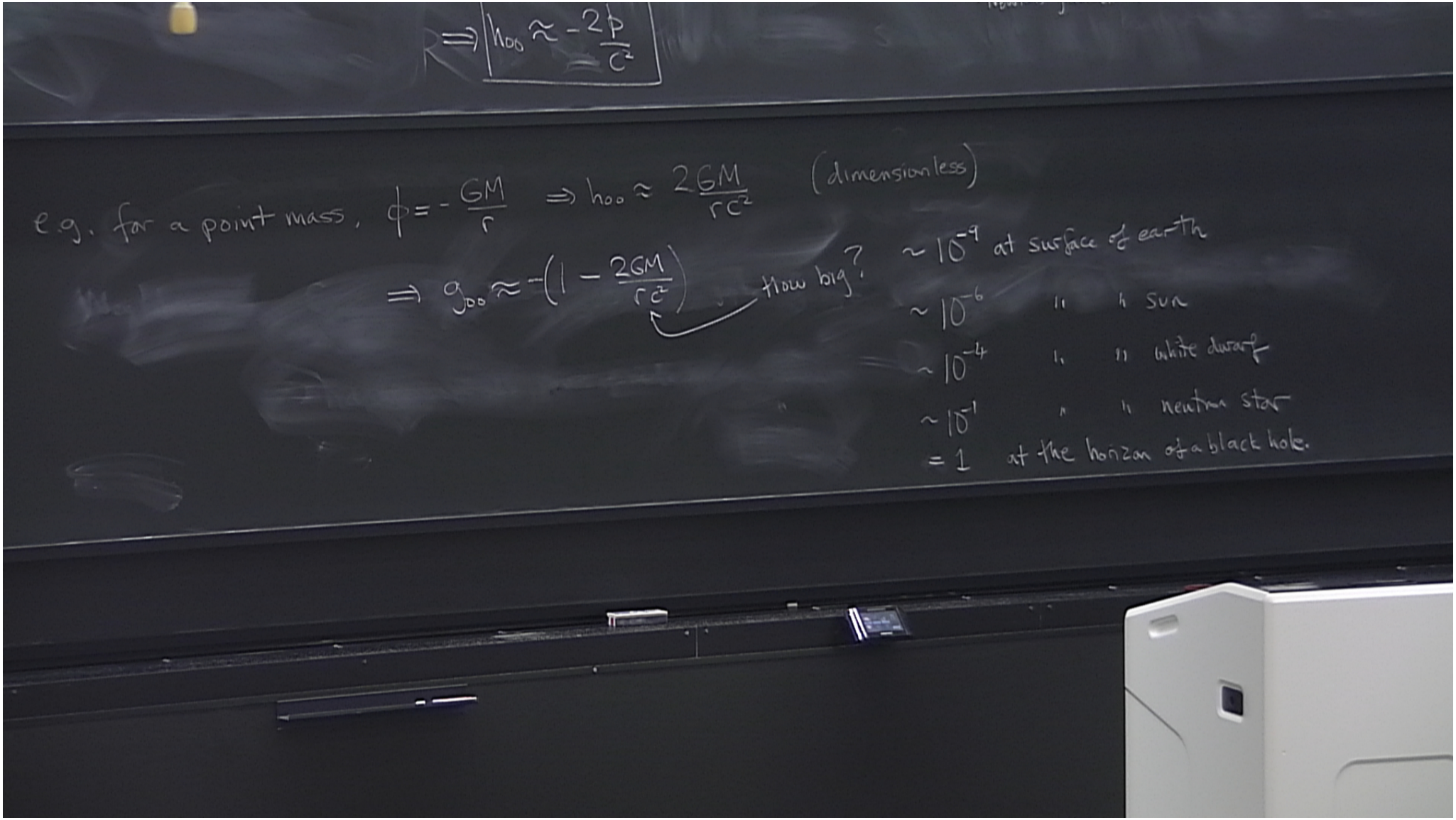
$\Gamma^\alpha_{\gamma\delta} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\gamma,\delta} + g_{\beta\delta,\gamma} - g_{\gamma\delta,\beta} \right)$ ($\sim h$) $\Rightarrow \Gamma^i_{00} = -\frac{1}{2} g_{00,i} = -\frac{1}{2} h_{00,i}$


$g^{\alpha\beta} \approx \eta^{\alpha\beta} - h^{\alpha\beta} + o(h)$
$\equiv \eta^{\alpha\mu} h_{\mu\nu} \eta^{\nu\beta} + \cdots$
$(1+a)^{-1} = 1 - a + a^2 - a^3 \ldots$
$(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\alpha} - \ldots$
$\delta^\alpha_\mu + h_\mu{}^\alpha - h_\mu{}^\alpha$
$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1.$
$v/c \ll 1$, small time derivatives, ∂_0
$P/\rho \sim v^2 \ll c^2.$
$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \Rightarrow$ trajectory independent of mass ✓
$\frac{d^2x^i}{dt^2} + \Gamma^i_{00} c^2 \approx 0$
$x^0 \equiv ct, \quad d\tau \approx dt$
$= dt\sqrt{1 - v^2/c^2}$
$\frac{d^2t}{d\tau^2} + \Gamma^0_{00} c^2 \approx 0$
$\Gamma^\nu_{\gamma\delta} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\gamma,\delta} + g_{\beta\delta,\gamma} - g_{\gamma\delta,\beta})$
$\sim h$
$\Rightarrow \Gamma^i_{00} = -\frac{1}{2}$
$\Gamma^0_{00} \approx 0$ if we ignore time derivatives


So $t = \tau$ even including 1st order effect of $h_{\mu\nu}$.
$\frac{d^2x^i}{d\tau^2} \approx -\frac{1}{2} h_{00,i} \left(\frac{dt}{d\tau}\right)\left(\frac{dt}{d\tau}\right)$
$\frac{d^2t}{d\tau^2} \approx 0 \Rightarrow \frac{dt}{d\tau} = \text{const}$
$\frac{d^2x^i}{dt^2} = +\frac{1}{2} h_{00,i} c^2$
cf Newton: $m\frac{d^2\vec{x}}{dt^2} = \vec{F}_g = -m\vec{\nabla}\phi$
ϕ = Newtonian potential


So $t = \tau$ even including 1st order effect of $h_{\mu\nu}$.
$\frac{d^2x^i}{d\tau^2} \approx -\frac{1}{2} h_{00,i} \left(\frac{dt}{d\tau}\right)\left(\frac{dt}{d\tau}\right)$
$\frac{d^2t}{d\tau^2} \approx 0 \Rightarrow \frac{dt}{d\tau} = \text{const}$
$\frac{d^2x^i}{dt^2} = +\frac{1}{2} h_{00,i} c^2$
cf Newton: $m\frac{d^2\vec{x}}{dt^2} = \vec{F}_g = -m\vec{\nabla}\phi$
ϕ = Newtonian potential
(Poisson) $\nabla^2\phi = -4\pi G\rho_m$
ρ_m ← mass density (mass/unit volume)
G ← Newton's gravitational constant
$\Rightarrow \boxed{h_{00} \approx -\frac{2\phi}{c^2}}$


$\Rightarrow \boxed{h_{00} \approx -\frac{2\phi}{c^2}}$
e.g. for a point mass, $\phi = -\frac{GM}{r} \Rightarrow h_{00} \approx \frac{2GM}{rc^2}$ (dimensionless)
$\Rightarrow g_{00} \approx -\left(1 - \frac{2GM}{rc^2}\right)$
how big? $\sim 10^{-9}$ at surface of earth


$\Rightarrow \boxed{h_{00} \approx -\frac{2\phi}{c^2}}$
e.g. for a point mass, $\phi = -\frac{GM}{r} \Rightarrow h_{00} \approx \frac{2GM}{rc^2}$ (dimensionless)
$\Rightarrow g_{00} \approx -\left(1 - \frac{2GM}{rc^2}\right)$
how big?
$\sim 10^{-9}$ at surface of earth
$\sim 10^{-6}$ " " sun
$\sim 10^{-4}$ " " white dwarf
$\sim 10^{-1}$ " " neutron star
$= 1$ at the horizon of a black hole.