

Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 6

Date: Sep 15, 2015 09:00 AM

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Abstract:

last time:

$$\nabla_\alpha g_{\beta\gamma} = 0 \Rightarrow (A) \partial_\alpha g_{\beta\gamma} - \Gamma^\lambda_{\alpha\beta} g_{\lambda\gamma} - \Gamma^\lambda_{\alpha\gamma} g_{\beta\lambda} = 0$$

$$(B) \partial_\beta g_{\gamma\alpha} -$$

$$(C) \partial_\gamma g_{\alpha\beta} -$$

$$(A) + (B) - (C) \quad \partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}$$

$$- \left(\underbrace{\Gamma_{\gamma\beta\alpha}}_A + \underbrace{\Gamma_{\gamma\alpha\beta}}_B \right) - \left(\underbrace{\Gamma_{\beta\gamma\alpha}}_A - \underbrace{\Gamma_{\beta\alpha\gamma}}_C \right) - \left(\underbrace{\Gamma_{\alpha\gamma\beta}}_B - \underbrace{\Gamma_{\alpha\beta\gamma}}_C \right) = 0$$

$$(\partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\gamma\alpha}) + \frac{1}{2} T_{\alpha\gamma\gamma} + \frac{1}{2} T_{\alpha\beta\beta}$$

"geometrical"
part of
connection

$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2} (\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) + \frac{1}{2} T_{\beta\alpha\gamma} + \frac{1}{2} T_{\alpha\beta\gamma}$$

$$\Rightarrow \Gamma_{\gamma\beta\alpha} = \frac{1}{2} (T_{\beta\alpha\gamma} - T_{\alpha\gamma\beta} + T_{\gamma\beta\alpha})$$

||

K_{β,αγ} CONTORSION

If $T_{\beta\alpha\gamma}$ is totally antisymmetric, then $K_{\beta\alpha\gamma} = T_{\beta\alpha\gamma}$

"geometrical"
part of
connection

$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2} (\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) + \frac{1}{2} T_{\beta\alpha\gamma} + \frac{1}{2} T_{\alpha\beta\gamma} = T_{\beta\alpha\gamma}$$

$$\Gamma_{\gamma\alpha\beta} = \frac{1}{2} \partial_\gamma g_{\alpha\beta}$$

$$\Rightarrow \Gamma_{\gamma\beta\gamma} =$$

CHRISTOFFEL

$$+ \frac{1}{2} (T_{\beta\alpha\gamma} - T_{\alpha\gamma\beta} + T_{\gamma\beta\alpha})$$

$K_{\beta\alpha\gamma}$ CONTORSION

If $T_{\beta\alpha\gamma}$ is totally antisymmetric, then $K_{\beta\alpha\gamma} = T_{\beta\alpha\gamma}$

$$-\left(\Gamma_{\gamma\beta\alpha} + \Gamma_{\alpha\gamma\beta}\right) - \left(\Gamma_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma}\right) - \left(\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma}\right) = 0$$

"geometrical"
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$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2} (\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) + \frac{1}{2} T_{\beta\alpha\gamma} + \frac{1}{2} T_{\alpha\beta\gamma}$$

$$\Rightarrow \Gamma_{\alpha\beta\gamma} = \underbrace{\frac{1}{2} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta})}_{\text{CHRISTOFFEL}} + \underbrace{\frac{1}{2} (T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta})}_{\substack{K_{\beta\alpha\gamma} \text{ CONTORSION} \\ = T_{\gamma\alpha\beta}}}$$

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} \partial_\alpha g_{\beta\gamma}$$

If $T_{\beta\alpha\gamma}$ is totally antisymmetric, then $K_{\beta\alpha\gamma} = T_{\beta\alpha\gamma}$

eg. $N=1$ Supergravity

Einstein + Rarita-Schwinger

eg. Einstein + Fermion

$\frac{R}{16G}$

$\bar{\psi} i \not{\partial} \psi$
(Kibble, 1960)

$$\left. \begin{array}{l} \\ \end{array} \right\} T_{\mu\nu\rho} \propto \bar{\psi} \underbrace{\gamma_\mu \gamma_\nu \gamma_\rho}_{\text{Spin Current}} \psi$$

spinor
vector

$$0 \propto g_{\beta\gamma} + g_{\gamma\alpha} - g_{\alpha\beta}$$

$$-\left(\underbrace{\Gamma_{\gamma\beta\alpha}}_A + \underbrace{\Gamma_{\alpha\gamma\beta}}_B\right) - \left(\underbrace{\Gamma_{\beta\gamma\alpha}}_A - \underbrace{\Gamma_{\beta\alpha\gamma}}_C\right) - \left(\underbrace{\Gamma_{\alpha\gamma\beta}}_B - \underbrace{\Gamma_{\alpha\beta\gamma}}_C\right) = 0$$

Geometrical part of connection $\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2} (\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) + \frac{1}{2} T_{\beta\alpha\gamma} + \frac{1}{2} T_{\alpha\beta\gamma}$

$\Gamma_{\gamma\beta\alpha} = \underbrace{\frac{1}{2} (\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma})}_{\text{CHRISTOFFEL}} + \underbrace{\frac{1}{2} (T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma} + T_{\gamma\beta\alpha})}_{\substack{K_{\beta\alpha\gamma} \text{ CONTORSION}}}$

$\Rightarrow \Gamma_{\beta\alpha\gamma} = \dots$

If $T_{\beta\alpha\gamma}$ is totally antisymmetric, then $K_{\beta\alpha\gamma} = T_{\beta\alpha\gamma}$

e.g. $N=1$ Supergravity

Einstein + Rarita-Schwinger

$\Rightarrow T_{\mu\nu\rho} \propto \bar{\psi}_\mu \gamma_\nu \psi_\rho$ not totally antisymmetric.

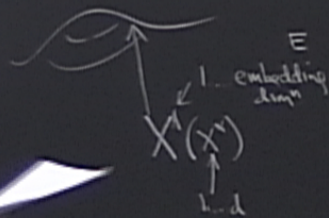
e.g. Einstein + Fermion
 $\frac{R}{16\pi G}$
 $\bar{\psi}\psi$
 (Kibble, 1960)

$T_{\mu\nu\rho} \propto \bar{\psi} \underbrace{\gamma_\mu \gamma_\nu \gamma_\rho}_{\text{Spin Current}} \psi$

spin
 $\downarrow$
 K vector

Geometrical meaning of covariant derivative.

embed curved space in a flat Euclidean space
(Minkowskian)

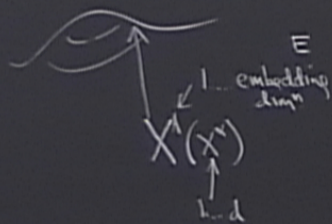


tangent vectors $\vec{e}_\alpha = \vec{X}_{,\alpha}$

[Th^m locally, any smooth manifold M^d can be embedded in $\mathbb{R}^{\frac{d(d+1)}{2}}$
globally, any compact manifold M^d " " " in $\mathbb{R}^{\frac{d}{2}(3d+1)}$
John Nash]

Geometrical meaning of covariant derivative.

embed curved space in a flat Euclidean space
(Minkowskian)



tangent vectors $\vec{e}_\alpha = \vec{X}_{,\alpha} \quad \alpha=1\dots d$

$$\begin{aligned} \text{metric } dl^2 &= d\vec{X} \cdot d\vec{X} = \underbrace{X^A_{,\alpha} X^A_{,\beta}}_{g_{\alpha\beta}} dx^\alpha dx^\beta \\ &= e^A_\alpha e^A_\beta \end{aligned}$$

"induced metric on M^d "

[Th^m locally, any smooth manifold M^d can be embedded in $\mathbb{R}^{\frac{d(d+1)}{2}}$
globally, any compact manifold M^d " " " in $\mathbb{R}^{\frac{d}{2}(d+1)}$
John Nash]

$$X^A(x^\mu)$$

metric $dl^2 = dX \cdot dX = \underbrace{X_{,\alpha} X_{,\beta}}_{g_{\alpha\beta}} dx^\alpha dx^\beta$ "induced metric on M^d "

$$= e_\alpha^A e_\beta^A$$

any vector in tangent space can be expressed as a sum of the $\vec{e}_\mu$, e.g. $\vec{V} = V^\mu \vec{e}_\mu$
 $\uparrow$ d-dimensional components

differentiate w.r.t. x^α ,

change in tangent vector basis

$$\partial_\alpha \vec{V} = (\partial_\alpha V^\mu) \vec{e}_\mu + V^\mu \partial_\alpha \vec{e}_\mu$$

covariant derivative: just project out the component normal to the tangent space of M^d

$$\partial_\alpha \vec{e}_\mu = \underbrace{\Gamma_{\mu\alpha}^\lambda \vec{e}_\lambda}_{\text{within tangent space to } M^d} + \underbrace{K_{\alpha\mu} \vec{n}}_{\substack{\text{unit normal to } M^d \\ \text{"extrinsic curvature" / "second fundamental form"}}}$$

$\vec{n}$ is the unit normal to M^d

$$D_\alpha \vec{V} = (\partial_\alpha V^\mu + \Gamma_{\lambda\alpha}^\mu V^\lambda) \vec{e}_\mu$$

$\Rightarrow$ automatically symmetric under $\mu \leftrightarrow \alpha$.

(Kibble, 1960)

Spin Current

last time:

$$\nabla_\alpha g_{\beta\gamma} = 0 \Rightarrow (A) \partial_\alpha g_{\beta\gamma} - \Gamma^\gamma_{\alpha\beta} g - \Gamma^\gamma_{\alpha\gamma} g = 0$$

$$(D'_\mu \Psi)' = \Psi' = e^{i\theta(x)} \Psi(x) \quad (B) \partial_\mu g_{\nu\alpha} -$$

$$(D_\mu + iA'_\mu) \Psi' = e^{i\theta} (i\partial_\mu \theta + iA'_\mu + \partial_\mu) \Psi \quad (C) \partial_\gamma g_{\alpha\beta} -$$

$$\boxed{A'_\mu + \partial_\mu \theta \equiv A_\mu}$$

$$= e^{i\theta} D_\mu \Psi$$

(A)+(B)-(C)

$$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}$$

$$- \left(\underbrace{\Gamma_{\gamma\beta\alpha}}_A + \underbrace{\Gamma_{\alpha\beta\gamma}}_B \right) - \left(\underbrace{\Gamma_{\beta\gamma\alpha}}_A - \underbrace{\Gamma_{\beta\alpha\gamma}}_C \right) - \left(\underbrace{\Gamma_{\alpha\gamma\beta}}_B - \underbrace{\Gamma_{\alpha\beta\gamma}}_C \right) = 0$$

(Kibble, 1960)

Spin Current

Upshot of above: to define derivatives, we need a connection.

Important exceptions:

- ① totally antisymmetric tensors
(differential forms)

e.g. $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - \Gamma_{\mu\nu}^\alpha A_\alpha + \Gamma_{\nu\mu}^\alpha A_\alpha$

Upshot of above: to define derivatives, we need a connection.

Important exceptions:

- ① totally antisymmetric tensors
(differential forms)

$$e.g. \quad F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - \underbrace{\Gamma_{\mu\alpha}^{\alpha} A_{\nu} + \Gamma_{\nu\alpha}^{\alpha} A_{\mu}}_{=0 \text{ if } T=0} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}.$$

$$\text{likewise } \nabla_{[\mu} F_{\nu\lambda]} = \partial_{[\mu} F_{\nu\lambda]}$$

Upshot of above: to define derivatives, we need a connection.

Important exceptions:

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$$e.g. F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - \underbrace{\Gamma_{\mu\alpha}^\alpha A_\nu + \Gamma_{\nu\alpha}^\alpha A_\mu}_{=0 \text{ if } T=0} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\text{likewise } \nabla_{[\alpha} F_{\mu\nu]} = \partial_{[\alpha} F_{\mu\nu]}$$

② Lie derivatives, given $P^\mu(x)$, $Q^\nu(x)$

$$\mathcal{L}_P Q^\nu \equiv \overset{\text{naive}}{P^\mu \nabla_\mu Q^\nu} - \overset{\text{correction term}}{Q^\mu \nabla_\mu P^\nu} = P^\mu \partial_\mu Q^\nu - Q^\mu \partial_\mu P^\nu$$

Connection cancels up to torsion terms

"geometrical" part of connection

$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2} (\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) + \frac{1}{2} T_{\beta\alpha\gamma} + \frac{1}{2} T_{\alpha\beta\gamma}$$

$$\Rightarrow \Gamma_{\alpha\beta\gamma} = \underbrace{\frac{1}{2} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta})}_{\text{CHRISTOFFEL}} + \underbrace{\frac{1}{2} (T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta})}_{K_{\beta\alpha\gamma} \text{ CONTORSION}} = T_{\alpha\beta\gamma}$$

$$\text{likewise } \mathcal{L}_p Q_\nu = p^\mu \nabla_\mu Q_\nu + Q_\mu \nabla_\nu p^\mu$$

$$= p^\mu \partial_\mu Q_\nu + Q_\mu \partial_\nu p^\mu \quad \text{if } \Gamma_{\alpha\beta}^\mu = \Gamma_{\beta\alpha}^\mu \text{ i.e. } T=0.$$

$$\mathcal{L}_p \phi = p^\mu \nabla_\mu \phi = p^\mu \partial_\mu \phi$$

$$\mathcal{L}_p S_{\alpha\beta} = p^\mu \nabla_\mu S_{\alpha\beta} + S_{\mu\beta} \nabla_\alpha p^\mu + S_{\alpha\mu} \nabla_\beta p^\mu \text{ etc.}$$

differentiate w.r.t. x^α ,

$$\partial_\alpha \vec{V} = (\partial_\alpha V^\mu) \vec{e}_\mu + V^\mu \partial_\alpha \vec{e}_\mu$$

change in tangent

basis

to the tangent

$$L_P \phi = P^\mu \nabla_\mu \phi = P^\mu \partial_\mu \phi$$

$$L_P S_{\alpha\beta} = P^\mu \nabla_\mu S_{\alpha\beta} + S_{\mu\beta} \nabla_\alpha P^\mu + S_{\alpha\mu} \nabla_\beta P^\mu \text{ etc.}$$

$$\nabla_\alpha (g_{\mu\beta} P^\mu) = \nabla_\alpha P_\beta$$

$$\text{eg } L_P g_{\alpha\beta} = \cancel{P^\mu \nabla_\mu g_{\alpha\beta}} + \nabla_\alpha P_\beta + \nabla_\beta P_\alpha \quad \text{using metricity } \nabla_\mu g_{\alpha\beta} = 0$$

$= 0$

Meaning of Lie derivative: change in the functional form of metric (or a tensor, in general) under the corresponding coordinate transformation: $x'^\mu = x^\mu + \epsilon P^\mu$: $L_P g_{\alpha\beta} \equiv \epsilon L_P g_{\alpha\beta} = g'_{\alpha\beta}(x) - g_{\alpha\beta}(x)$.

From connection to curvature
in flat space, $[\partial_\mu, \partial_\nu] = 0$, but $\left\{ \begin{array}{l} \text{in curved spacetime} \\ [\nabla_\mu, \nabla_\nu] \neq 0. \end{array} \right.$

From connection to curvature
in flat space, $[\partial_\mu, \partial_\nu] = 0$, but $\int$ in curved spacetime $[\nabla_\mu, \nabla_\nu] \neq 0$.

$$\text{e.g., } \nabla_\mu \nabla_\nu S_\rho - \nabla_\nu \nabla_\mu S_\rho \equiv -R^\sigma{}_{\rho\mu\nu} S_\sigma \quad \forall S_\mu.$$

$$\nabla_\mu Q^\nu - Q^\mu \nabla_\mu P^\nu = P^\mu \partial_\mu Q^\nu - Q^\mu \partial_\mu P^\nu$$

c.f. Yang-Mills $D_\mu = \partial_\mu + iA_\mu$

$$[D_\mu, D_\nu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu) = iF_{\mu\nu}.$$

0.

S_μ

$\forall S_\mu$

$$\nabla_\mu Q^\nu - Q^\mu \nabla_\mu P^\nu = P^\mu \partial_\mu Q^\nu - Q^\mu \partial_\mu P^\nu$$

c.f. Yang-Mills $D_\mu = \partial_\mu + iA_\mu$

$$[D_\mu, D_\nu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu) = iF_{\mu\nu}$$

$$S = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$\forall S_\mu$

From connection to curvature

in flat space, $[\partial_\mu, \partial_\nu] = 0$, but $\uparrow$ in curved spacetime $[\nabla_\mu, \nabla_\nu] \neq 0$.

e.g. $\nabla_\mu \nabla_\nu S_\rho - \nabla_\nu \nabla_\mu S_\rho \equiv -R^\sigma_{\rho\mu\nu} S_\sigma \quad \forall S_\mu.$

explicitly,

$$\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma) - \Gamma_{\nu\mu}^\rho (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\nu S_\alpha - \Gamma_{\alpha\nu}^\sigma S_\sigma)$$

in curved spacetime

$$[\nabla_\mu, \nabla_\nu] \neq 0.$$

$$\equiv -R^\sigma_{\rho\mu\nu} S_\sigma.$$

$$\forall S_\mu.$$

$$\Gamma^\rho_{\nu\mu} (\partial_\alpha S_\rho - \Gamma^\sigma_{\rho\alpha} S_\sigma) - \Gamma^\alpha_{\rho\mu} (\partial_\nu S_\alpha - \Gamma^\sigma_{\alpha\nu} S_\sigma) = -(\mu \leftrightarrow \nu)$$

c.f. Yang-Mills

$$D_\mu = \partial_\mu + iA_\mu$$

$$[D_\mu, D_\nu] = i(\partial_\mu A_\nu - \partial_\nu A_\mu) = iF_{\mu\nu}$$

$$S = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

From connection to curvature

in flat space, $[\partial_\mu, \partial_\nu] = 0$, but $\left[\nabla_\mu, \nabla_\nu \right] \neq 0$ in curved spacetime.

c.f. Yang-Mills $D_\mu = \partial_\mu + A_\mu$
 $[D_\mu, D_\nu] = F_{\mu\nu}$

e.g. $\nabla_\mu \nabla_\nu S_\rho - \nabla_\nu \nabla_\mu S_\rho \equiv -R^\sigma{}_{\rho\mu\nu} S_\sigma \quad \forall S_\mu$

explicitly,

assume $T_{\mu\nu} = 0$.

$$\underbrace{\partial_\mu (\partial_\nu S_\rho - \Gamma_{\rho\nu}^\sigma S_\sigma)}_{\text{cancels}} - \underbrace{\Gamma_{\nu\mu}^\rho (\partial_\alpha S_\rho - \Gamma_{\rho\alpha}^\sigma S_\sigma)}_{\text{cancels}} - \Gamma_{\rho\mu}^\alpha (\partial_\nu S_\alpha - \Gamma_{\alpha\nu}^\sigma S_\sigma) = (\mu \leftrightarrow \nu)$$

∂S term is symm $\mu \leftrightarrow \nu$, so cancels

define derivatives, we need a connection.

= 0 if $T = 0$

0.  Cancels.

Cancels:

∂ term is symm $\mu \leftrightarrow \nu$, so cancels.

$$= -(\partial_\mu \Gamma_{\rho\nu}^\sigma) S_\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma S_\sigma - (\mu \leftrightarrow \nu) \equiv -R_{\rho\mu\nu}^\sigma S_\sigma$$

$$\Rightarrow R_{\rho\mu\nu}^\sigma = \partial_\mu \Gamma_{\rho\nu}^\sigma - \partial_\nu \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma$$

0. Cancels.

Cancels:

∂ term is symm $\mu \leftrightarrow \nu$, so cancels.

$$= -(\partial_\mu \Gamma_{\rho\nu}^\sigma) S_\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma S_\sigma - (\mu \leftrightarrow \nu) \equiv -R^\sigma_{\rho\mu\nu} S_\sigma$$

$$\Rightarrow R^\sigma_{\rho\mu\nu} = \partial_\mu \Gamma_{\rho\nu}^\sigma - \partial_\nu \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma \quad \text{is a tensor.}$$

interesting to calculate in a locally flat coordinate system, in which Γ vanishes.

SUM
 $\Sigma = 0$ $\Rightarrow$ Cancels.
 Cancels.
 ∂S term is symm $\mu \leftrightarrow \nu$, so cancels

$$= -(\partial_\mu \Gamma_{\rho\nu}^\sigma) S_\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma S_\sigma - (\mu \leftrightarrow \nu) \equiv -R^\sigma_{\rho\mu\nu} S_\sigma$$

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interesting to calculate in a locally flat coordinate system, in which Γ vanishes.

$$\partial_\mu \Gamma_{\rho\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda} \partial_\mu (g_{\rho\lambda,\nu} + g_{\lambda\rho,\nu} - g_{\rho\nu,\lambda})$$

$$= -(\partial_\mu \Gamma_{\rho\nu}^\sigma) S_\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma S_\sigma - (\mu \leftrightarrow \nu) \equiv -R^\sigma_{\rho\mu\nu} S_\sigma$$

$$\Rightarrow R^\sigma_{\rho\mu\nu} = \partial_\mu \Gamma_{\rho\nu}^\sigma - \partial_\nu \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma \quad \text{is a tensor.}$$

interesting to calculate in a locally flat coordinate system, in which Γ vanishes.

$$\partial_\mu \Gamma_{\rho\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda} \partial_\mu (g_{\rho\lambda,\nu} + g_{\lambda\rho,\nu} - g_{\rho\nu,\lambda}) \Rightarrow R_{\sigma\rho\mu\nu} = \frac{1}{2} (g_{\nu\sigma,\rho\mu} - g_{\rho\nu,\sigma\mu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu})$$

from connection to curvature
in flat space, $[\partial_\mu, \partial_\nu] = 0$, but in curved space $[\nabla_\mu, \nabla_\nu] \neq 0$.

c.f. Yang-Mills

$$[D_\mu, D_\nu] = i g F_{\mu\nu}$$

$$S^\dagger =$$

Symmetries:

① $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$ symmetric exchange first and second pair of indices

② antisymmetric in each pair

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \quad \text{since} \quad R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\nu\mu}$$

③ cyclicity under last 3 indices

$$R_{\rho\sigma\mu\nu} + R_{\sigma\nu\mu\rho} + R_{\sigma\mu\nu\rho} = 0$$

(follows from $[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] = 0$)

eg $L_P g_{\alpha\beta} = P^\mu \nabla_\mu g_{\alpha\beta} + \nabla_\alpha P_\beta + \nabla_\beta P_\alpha$ using metricity $\nabla_\mu g_{\alpha\beta} = 0$

$$R_{\rho\mu\nu} + R_{\sigma\nu\mu} + R_{\sigma\mu\nu} = 0$$

(follows from $[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] = 0$)

of components taking ① and ② into account:

$$R_{\rho\sigma\mu\nu} \Rightarrow \frac{1}{2} \frac{D(D-1)}{2} \left(\frac{D(D-1)}{2} + 1 \right) = \frac{1}{8} D(D-1) (D^2 - D + 2)$$

$\frac{D(D-1)}{2}$ but $\left. \begin{array}{l} \text{lhs of} \\ \text{but} \end{array} \right\}$ ③ is totally antisymmetric (using ① and ②) under exchange of all four indices

$\Rightarrow$ ③ represents $\frac{1}{4!} D(D-1)(D-2)(D-3)$ additional constraints,

$$\rightarrow \# \text{ of independent components of } R_{\mu\nu\rho\sigma} \text{ is } \frac{1}{8} D(D-1) (D^2 - D + 2) - \frac{1}{24} D(D-1)(D-2)(D-3) = \frac{D}{12} (D-1)$$

EXACTLY the
of second derivatives
of $g_{\mu\nu}$ which cannot
be set zero by coordinate
transform