

Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 5

Date: Sep 14, 2015 09:00 AM

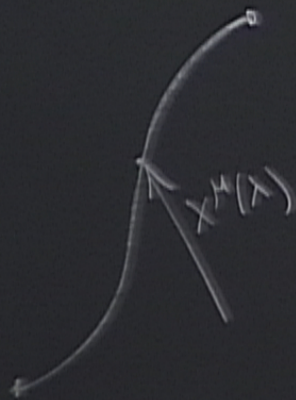
URL: <http://pirsa.org/15090032>

Abstract:

We need to understand how to do calc
 as motivation, let us start from a variational principal

$$S' = -mc \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$\uparrow$
 $g_{\mu\nu}(x(\lambda))$



$$\delta S' = -mc \int \frac{1}{2} \left(\sqrt{} \right)^{-1} \left(-g_{\mu\nu,\lambda} \delta x^\lambda \right)$$

$\uparrow$
 $S'(x+\delta x) - S'(x)$

stand how to do calculus on curved spaces i.e. differ

on a variational principal

$$\frac{dx^\nu}{d\lambda} d\lambda$$

$$\left(\frac{d}{d\lambda} \right)^{-1} \left(-g_{\mu\nu,\lambda} \delta x^\lambda \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - 2g_{\mu\nu} \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)$$

stand how to do calculus on curved spaces i.e. differ

on a variational principal

$$\frac{dx^\nu}{d\lambda} d\lambda$$

Note:

$$\frac{d}{d\tau} = \frac{1}{\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} \frac{d}{d\lambda}$$

$$\left(\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \right)^{-1} \left(-g_{\mu\nu, \lambda} \delta x^\lambda \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - 2g_{\mu\nu} \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) d\lambda$$

calculus on curved spaces i.e. differential geometry

principal

Note:

$$\frac{d}{d\tau} = \frac{1}{\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} \frac{d}{d\lambda}$$

$$\frac{d(\cdot)}{d\tau} \equiv (\cdot)^{\cdot}$$

$$\left(\frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - 2g_{\mu\nu} \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) d\lambda = -mc \int \frac{1}{2} d\tau \left(-g_{\mu\nu\lambda} \delta x^\mu \dot{x}^\nu \dot{x}^\lambda - 2g_{\mu\nu} \dot{x}^\mu \delta \dot{x}^\nu \right)$$

$$\frac{d}{d\tau} \equiv \left(\frac{d}{d\tau} \right)$$

$$-mc \int \frac{1}{2} d\tau \left(-g_{\mu\nu} \delta x^\mu \ddot{x}^\nu - 2g_{\mu\nu} \dot{x}^\mu \delta \dot{x}^\nu \right)$$

integrate
by parts

$$+ 2g_{\mu\nu} \ddot{x}^\mu \delta x^\nu + 2g_{\mu\nu} \dot{x}^\mu \delta x^\nu$$

ϵ
 $\left(\begin{array}{c} \cdot \\ \cdot \end{array} \right)$

$$\epsilon \int \frac{1}{2} d\tau \left(-g_{\mu\nu, \lambda} \delta X^\lambda \dot{X}^\mu \dot{X}^\nu - 2 g_{\mu\nu} \dot{X}^\mu \delta \dot{X}^\nu \right)$$

integrate
by parts

$$+ 2 g_{\mu\nu, \alpha} \dot{X}^\alpha \dot{X}^\mu \delta X^\nu$$

$$= 0 \neq \delta X^\mu(\tau)$$

$$\Rightarrow g_{\mu\nu} \ddot{X}^\mu + \underbrace{g_{\mu\nu,\alpha} \dot{X}^\mu \dot{X}^\alpha}_{\frac{1}{2}(g_{\mu\nu,\alpha} + g_{\alpha\nu,\mu}) \dot{X}^\mu \dot{X}^\alpha} - \frac{1}{2} g_{\mu\lambda,\nu} \dot{X}^\mu \dot{X}^\lambda = 0$$

$$\Rightarrow \ddot{X}^\mu + \left\{ \begin{matrix} \mu \\ \nu\alpha \end{matrix} \right\} \dot{X}^\nu \dot{X}^\alpha = 0$$

where the Christoffel symbol $\left\{ \begin{matrix} \mu \\ \nu\alpha \end{matrix} \right\} \equiv \frac{1}{2} g^{\mu\beta} (g_{\nu\beta,\alpha} + g_{\alpha\beta,\nu} - g_{\nu\alpha,\beta})$

$$\ddot{x}^\mu \ddot{x}^\mu = 0$$

Geodesic equation: gives curve which extremises the invariant length.

$$\{^M_{V\alpha}\} \equiv \frac{1}{2} g^{\mu\beta} (g_{V\beta,\alpha} + g_{\alpha\beta,V} - g_{V\alpha,\beta})$$

$$= 0 \neq \delta x^\mu(\tau)$$

$$\Rightarrow g_{\mu\nu} \ddot{x}^\mu + \underbrace{g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\alpha}_{\frac{1}{2}(g_{\mu\nu,\alpha} + g_{\alpha\nu,\mu}) \dot{x}^\mu \dot{x}^\alpha} - \frac{1}{2} g_{\mu\lambda,\nu} \dot{x}^\mu \dot{x}^\lambda = 0$$

$$\Rightarrow \boxed{\ddot{x}^\mu + \left\{ \begin{matrix} \mu \\ \nu\alpha \end{matrix} \right\} \dot{x}^\nu \dot{x}^\alpha = 0}$$

Geodesic equation: gives curve which ex

where the Christoffel symbol

$$\left\{ \begin{matrix} \mu \\ \nu\alpha \end{matrix} \right\} \equiv \frac{1}{2} g^{\mu\beta} (g_{\nu\beta,\alpha} + g_{\alpha\beta,\nu} - g_{\alpha\beta,\mu})$$

Equation is invariant under coord transformations, but neither

$$x^\mu(\tau)$$

$$\dot{x}^\mu(\tau)$$

$$= x'^\mu(x^\alpha(\tau))$$

$$\frac{dx'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{dx^\alpha}{d\tau}$$

does transform as a contravariant vector

$$\frac{d^2 x'^\mu}{d\tau^2} = \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} + \frac{\partial x'^\mu}{\partial x^\alpha} \frac{d^2 x^\alpha}{d\tau^2}$$

$$\frac{dx'^{\mu}}{d\tau} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{d\tau}$$

$$\frac{d^2 x'^{\mu}}{d\tau^2}$$

$$= \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau}}_{\text{wrong transfm. law.}} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{d^2 x^{\alpha}}{d\tau^2}$$

does transform as a contravariant

transform as a contravariant vector

$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{dt^2}$, however $\begin{Bmatrix} M \\ v_{\alpha} \end{Bmatrix}$ not a tensor either,

$$\begin{Bmatrix} M \\ v_{\alpha} \end{Bmatrix}'(x') =$$

whether the acceleration $\frac{d^2 x^\lambda}{d\tau^2}$ nor $\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\}$ are tensors.

it vector

$\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\}$ not a tensor either,

$$\left(\frac{d^2 x^\lambda}{d\tau^2} \right)' = \frac{\partial x'^\mu}{\partial x^\delta} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\alpha} \left\{ \begin{smallmatrix} \delta \\ \beta \gamma \end{smallmatrix} \right\} + \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\nu \partial x'^\alpha}$$

invariant under coord transformations, but neither the

$$\frac{\partial X'^{\mu}}{\partial X^{\alpha}} \frac{dx^{\alpha}}{d\tau}$$

does transform as a contravariant vector

however $\left\{ \begin{matrix} M \\ v_{\alpha} \end{matrix} \right\}$ not a

$$\frac{\partial^2 X'^{\mu}}{\partial X^{\alpha} \partial X^{\beta}} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} + \frac{\partial X'^{\mu}}{\partial X^{\alpha}} \frac{d^2 X^{\alpha}}{d\tau^2}$$

,

$$\left\{ \begin{matrix} M \\ v_{\alpha} \end{matrix} \right\}'(x') =$$

wrong transformⁿ
law.

Equation is invariant under coord transformations

$$\begin{aligned} & x'^{\mu}(\tau) \\ & x'^{\mu}(\tau) \\ & = x'^{\mu}(x^{\alpha}(\tau)) \end{aligned}$$

$$\frac{dx'^{\mu}}{d\tau} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{d\tau}$$

does transform

$$\frac{d^2 x'^{\mu}}{d\tau^2} = \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau}}_{\text{wrong transfm. law.}} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{d^2 x^{\alpha}}{d\tau^2}$$

but neither the acceleration $\frac{d^2 x^\lambda}{d\tau^2}$ nor $\left\{ \begin{smallmatrix} \lambda \\ \mu\alpha \end{smallmatrix} \right\}$ are tensors.

covariant vector

$\left\{ \begin{smallmatrix} \mu \\ \nu\alpha \end{smallmatrix} \right\}$ not a tensor either,

$$\left\{ \begin{smallmatrix} \mu \\ \nu\alpha \end{smallmatrix} \right\}'(x') = \frac{\partial x'^\mu}{\partial x^\delta} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\alpha} \left\{ \begin{smallmatrix} \delta \\ \beta\gamma \end{smallmatrix} \right\} + \underbrace{\frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\nu \partial x'^\alpha}}_{\text{bad term}}$$

curved spaces

$$\frac{\partial x^\lambda}{\partial x'^\rho} + \frac{\partial x^\lambda}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\alpha} \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda} = 0$$

bad term is $-\frac{\partial x^\lambda}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\alpha} \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}$

note $\frac{\partial X'^{\mu}}{\partial X^{\lambda}} \frac{\partial X^{\lambda}}{\partial X'^{\rho}} = \delta_{\rho}^{\mu}$

$$\frac{\partial}{\partial X'^{\alpha}} \Rightarrow \frac{\partial X'^{\mu}}{\partial X^{\lambda}} \frac{\partial^2 X^{\lambda}}{\partial X'^{\rho} \partial X'^{\alpha}} + \frac{\partial X^{\lambda}}{\partial X'^{\rho}} \frac{\partial X'^{\nu}}{\partial X'^{\alpha}} \frac{\partial^2 X'^{\mu}}{\partial X'^{\nu} \partial X^{\lambda}} = 0 \quad ; \text{ bad term}$$

this precisely cancels the bad term coming from

but neither the acceleration $\frac{d^2 X^\lambda}{d\tau^2}$ nor $\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\}$ are tensors.

contravariant vector

over $\left\{ \begin{smallmatrix} M \\ \nu \alpha \end{smallmatrix} \right\}$ not a tensor either,

$$\left\{ \begin{smallmatrix} M \\ \nu \alpha \end{smallmatrix} \right\}'(x') = \frac{\partial x'^\mu}{\partial x^\delta} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\alpha} \left\{ \begin{smallmatrix} \delta \\ \beta \gamma \end{smallmatrix} \right\} + \underbrace{\frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\nu \partial x'^\alpha}}_{\text{bad term}}$$

transformation
law
for $\left\{ \begin{smallmatrix} M \\ \nu \alpha \end{smallmatrix} \right\}$

This is a very general theme in modern physics: Conn

are not tensors but they allow us to construct tensorial

In fact, $\{\Gamma^{\mu}_{\nu\alpha}\}$ is only a special case of the general connection in

the same transformation law i.e.

$$\Gamma'^{\mu}_{\nu\alpha} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \Gamma^{\rho}_{\sigma\lambda} + \frac{\partial^2 x^{\rho}}{\partial x'^{\nu} \partial x'^{\alpha}} \frac{\partial x'^{\mu}}{\partial x^{\rho}}$$

in modern physics: Connections like $\{\Gamma^M_{\nu\alpha}\}$ or A_μ

used to construct tensorial derivatives.

of the general connection in GR, $\Gamma^M_{\nu\alpha}$ which, by definition, obeys

$$\frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^\mu}{\partial x'^\nu} \right) + \Gamma^M_{\nu\alpha} \frac{\partial x^\mu}{\partial x'^M} = \frac{\partial}{\partial x'^\alpha} \left(\frac{\partial x^\mu}{\partial x'^\nu} \right)$$

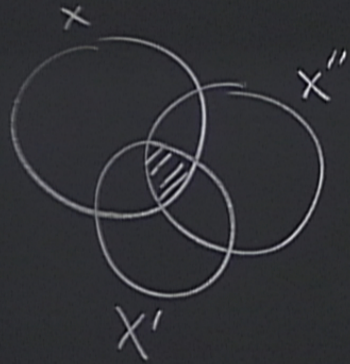
this allows to ensure derivatives are tensors:

$$\begin{aligned}
 \text{covariant derivative} \quad \nabla_\mu A^{\alpha \dots}_{\beta \dots} &= \partial_\mu A^{\alpha \dots}_{\beta \dots} + \underbrace{\Gamma^\alpha_{\lambda\mu} A^\lambda}_{\text{for each contravariant index}} + \dots - \underbrace{\Gamma^\lambda_{\beta\mu} A_\lambda}_{\text{for every covariant index}} - \dots
 \end{aligned}$$

(can check that this is a tensor.)

15 & 15.5

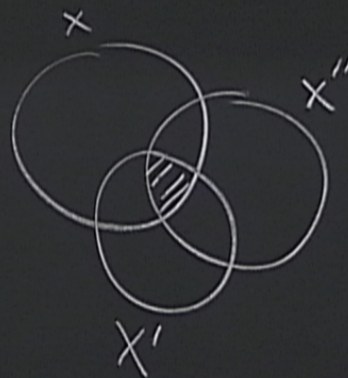
very important that Γ is consistently defined across all coordinate systems



: need to check that $x \rightarrow x' \rightarrow x''$ gives same Γ'' as $x \rightarrow x''$.

Can verify this. (Hughston + Tod intro to GR)

Note: it is fundamentally important that Γ is consistently defined



: need to check that

as

Can verify this. (Hence)

$\{v^\alpha\}$
for Γ is symmetrical under lower two indices.

$$\nabla_\mu g_{\alpha\beta} = 0$$

"metric is covariantly constant"

Note that the "bad" term in the transformation law for Γ is

$$\Rightarrow T_{\nu\lambda}^{\mu} = \Gamma_{\nu\lambda}^{\mu} - \Gamma_{\lambda\nu}^{\mu} \text{ is a tensor.}$$

- called the torsion tensor.

How do we determine $\Gamma_{\nu\lambda}^{\mu}$?

Basic postulate of ^{differential} Riemannian geometry is that $\nabla_{\mu} g_{\alpha\beta} = 0$

information seen for Γ is symmetrical under lower two indices.

is a tensor.

such that $\nabla_\mu g_{\alpha\beta} = 0$ "metric is covariantly constant"

at $\nabla_\mu g_{\alpha\beta} = 0$ "metric is covariantly constant"
(METRICITY)

can set $\partial_\mu g_{\alpha\beta} = 0$ at any particular spacetime point,

can guarantee that $\underline{\Gamma^\lambda_{(\mu\nu)}} = 0$ in any coord syst where $\partial_\mu g_{\alpha\beta} = 0$.

recall: with appropriate coord trans, can set

If we impose $\nabla_\mu g_{\alpha\beta} = 0$ (in all coords), then guar

Proof: $\nabla_\alpha g_{\beta\gamma} = 0 \Rightarrow \partial_\alpha g_{\beta\gamma} - \Gamma_\alpha^\beta \gamma - \Gamma_\alpha^\gamma \beta$

recall: with appropriate coord trans, can set

If we impose $\nabla_\mu g_{\alpha\beta} = 0$ (in all coords), then guarant

$$\nabla_\alpha g_{\beta\gamma} = 0 \Rightarrow (A) \partial_\alpha g_{\beta\gamma} - \Gamma_{\beta\alpha}^\lambda g_{\lambda\gamma} - \Gamma_{\gamma\alpha}^\lambda g_{\beta\lambda} = 0$$

$$(B) \partial_\beta g_{\gamma\alpha} - \Gamma_{\gamma\beta}^\delta g_{\delta\alpha} - \Gamma_{\alpha\beta}^\delta g_{\gamma\delta} = 0$$

$$(C) \partial_\gamma g_{\alpha\beta} - \Gamma_{\alpha\gamma}^\delta g_{\delta\beta} - \Gamma_{\beta\gamma}^\delta g_{\alpha\delta} = 0$$

$$\Gamma_{[\mu\nu]}^\lambda \equiv \frac{1}{2}(\Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda) = \frac{1}{2}T_{\mu\nu}^\lambda$$

$$\Gamma_{(\mu\nu)}^\lambda \equiv \frac{1}{2}(\Gamma_{\mu\nu}^\lambda + \Gamma_{\nu\mu}^\lambda)$$

$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}$
 $= (\Gamma_{\gamma\beta\alpha} + \Gamma_{\gamma\alpha\beta}) - (\Gamma_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma}) - (\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma})$
 $\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})$

$$(c) \partial_\gamma g_{\alpha\beta} - \Gamma_{\alpha\gamma} g_{\delta\beta} - \Gamma_{\beta\gamma} g_{\alpha\delta} = 0$$

$$(a) + (b) - (c)$$

$$\begin{aligned} & \partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \\ & - (\Gamma_{\gamma\beta\alpha} + \Gamma_{\gamma\alpha\beta}) - (\Gamma_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma}) - (\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma}) = 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} & \partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \\ & - (\Gamma_{\gamma\beta\alpha} + \Gamma_{\gamma\alpha\beta}) - (\Gamma_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma}) - (\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma}) = 0 \end{aligned}} \right\} \Gamma_{\gamma\beta\alpha} =$$

$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \underbrace{\frac{1}{2} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})}_{g_{\mu\nu} \left\{ \begin{matrix} \mu \\ \beta\gamma \end{matrix} \right\}} + \frac{1}{2} T_{\beta\alpha\gamma} + \frac{1}{2} T_{\alpha\beta\gamma}$$

(8) - (C)

$$\begin{aligned}
 & \partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \\
 & - (\Gamma_{\gamma\beta\alpha} + \Gamma_{\gamma\alpha\beta}) - (\Gamma_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma}) + (\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma}) = 0 \\
 \Rightarrow \Gamma_{\gamma(\beta\alpha)} &= \underbrace{\frac{1}{2} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})}_{g_{\mu\nu} \left\{ \begin{smallmatrix} \mu \\ \beta\gamma \end{smallmatrix} \right\}} + \frac{1}{2} T_{\beta\alpha\gamma} - \frac{1}{2} T_{\alpha\beta\gamma}
 \end{aligned}$$

$$(A) + (B) - (C)$$

$$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}$$

$$- (\Gamma_{\gamma\beta\alpha} + \Gamma_{\gamma\alpha\beta}) - (\Gamma_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma}) + (\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma}) = 0$$

$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \underbrace{\frac{1}{2} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})}_{g_{\gamma\mu} \left\{ \begin{matrix} \mu \\ \beta\alpha \end{matrix} \right\}} + \frac{1}{2} T_{\beta\alpha\gamma} - \frac{1}{2} T_{\alpha\beta\gamma}$$

$$\Gamma_{\gamma\beta\alpha} = g_{\gamma\mu} \left\{ \begin{matrix} \mu \\ \beta\alpha \end{matrix} \right\} + \underbrace{\frac{1}{2} (T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma} + T_{\gamma\beta\alpha})}_{\text{CONTORSION } K_{\beta\alpha\gamma}}$$

$$(A) + (B) - (C)$$

$$\Gamma_{\alpha\beta\gamma} = g_{\alpha\delta} \Gamma_{\beta\gamma}^{\delta}$$

$$\gamma^{\mu}$$

$$g_{\mu\nu} = e_{\mu}^a e_{\nu}^b \eta_{ab}$$

$$\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta} - (\Gamma_{\gamma\beta\alpha} + \Gamma_{\gamma\alpha\beta}) - (\Gamma_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma}) + (\Gamma_{\alpha\beta\gamma} - \Gamma_{\alpha\gamma\beta})$$

$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2} \underbrace{(\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta})}_{g_{\gamma\mu} \left\{ \begin{matrix} \mu \\ \beta\alpha \end{matrix} \right\}}$$

$$\Gamma_{\alpha\beta\gamma} = g_{\alpha\delta} \Gamma_{\beta\gamma}^{\delta}$$

$$\Gamma_{\mu}^{\alpha} = \gamma^{\alpha}_{\mu} e^{\mu}_{\nu} \eta^{\nu}_{ab}$$

$$g_{\mu\nu} = e^{\alpha}_{\mu} e^{\beta}_{\nu} \eta_{ab}$$

 $\Rightarrow$

$$\Gamma_{\gamma(\beta\alpha)} =$$

$$- (\Gamma_{\gamma\beta\alpha} +$$

$$\partial_{\alpha} g_{\beta\gamma} +$$

but neither the acceleration $\frac{d^2x^\mu}{d\tau^2}$ nor $\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\}$ are tensors.

as a contravariant vector

we can set $\partial_\mu g_{\alpha\beta} = 0$ at any particular spacetime point.

we then guarantee that $\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\} = 0$ in any coord syst where $\partial_\mu g_{\alpha\beta} = 0$.

$\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\} = 0$ (if torsion is totally antisymmetric (or zero))

$$\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\} \equiv \frac{1}{2} \left(\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} \lambda \\ \nu \mu \end{smallmatrix} \right\} \right) = \frac{1}{2} T_{\mu\nu}^\lambda$$

$$\frac{1}{2}(\Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}) = \frac{1}{2}l_{\mu\nu}^{\lambda}$$

$$\frac{1}{2}(\Gamma_{\mu\nu}^{\lambda} + \Gamma_{\nu\mu}^{\lambda})$$

CONTORSION $K_{\beta-\alpha\gamma}$

vanishes if $\underbrace{T_{\beta\gamma\alpha}}_{\text{(all three)}}$ is totally antisymm

○ then $\Gamma_{\beta-\mu\alpha} = 0$.

$$T_{\alpha\mu\gamma} = \bar{\Psi} i \gamma^5 \gamma_\alpha \Sigma_{\mu\gamma} \Psi$$

$$= g_{\alpha\mu} \sum_{\beta\gamma}^M + \frac{1}{2} \left(T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma} + T_{\gamma\beta\alpha} \right)$$

CONTORSION $K_{\beta\alpha\gamma} = T_{\beta\alpha\gamma}$

if $\underbrace{T_{\beta\gamma\alpha}}$ is totally antisymm
(all three)

$$(A) + (B) - (C)$$

$$\Gamma_{\alpha\beta\gamma} = g_{\alpha\delta} \Gamma_{\beta\gamma}^{\delta}$$

$$\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta}$$

$$- (\Gamma_{\gamma\beta\alpha} + \Gamma_{\gamma\alpha\beta}) - (\Gamma_{\beta\gamma\alpha} - \Gamma_{\beta\alpha\gamma}) + (\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma}) = 0$$

$$\eta_{\mu\nu} = \gamma^a e_{\mu}^a \eta_{ab}$$

$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \underbrace{\frac{1}{2} (\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta})}_{g_{\gamma\mu} \left\{ \begin{matrix} \mu \\ \beta\alpha \end{matrix} \right\}} + \frac{1}{2} T_{\beta\alpha\gamma} - \frac{1}{2}$$

$$g_{\mu\nu} = e_{\mu}^a e_{\nu}^b \eta_{ab}$$

$$\Gamma, g \rightarrow e, \omega$$