

Title: PSI 2015/2016 Relativity - Neil Turok - Lecture 4

Date: Sep 11, 2015 09:00 AM

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Abstract:

We have seen how  $\eta_{\mu\nu}$  is used to define Lorentz-invariant distance

$$\text{If } x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \quad \text{then} \quad \eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \quad (*)$$
$$\eta_{\mu\nu} A'^{\mu} B'^{\nu} = \eta_{\mu\nu} A^{\mu} B^{\nu}$$

In fact, the most general coordinate transformations satisfying (\*) are

$$\text{Insist that} \quad \eta_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}}, \quad \text{allow } x'^{\mu} \text{ to be a}$$
$$= \eta_{\mu\nu} x'^{\mu}_{,\alpha} x'^{\nu}_{,\beta}$$



We have seen how  $\eta_{\mu\nu}$  is used to define  $L$

$$\text{If } x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \text{ then } \eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} dx^{\alpha} dx^{\beta}$$

In fact, the most general coordinate transformation

$$\text{I insist that } \eta_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}}$$

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} dx^{\alpha} \quad \quad \quad = \eta_{\mu\nu} x'^{\nu}$$

Jacobian matrix  
invertible if map  $x^{\alpha} \rightarrow x'^{\alpha}$  is 1-1.



is used to define Lorentz-invariant distances (and inner prod

$x^\mu, x^\nu$  then  $\eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu$  (\*)

$$\eta_{\mu\nu} A'^\mu B'^\nu = \eta_{\mu\nu} A^\mu B^\nu$$

coordinate transformations satisfying (\*) are Lorentz + translations =

that  $\eta_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta}$ , allow  $x'^\mu$  to be any function of  $x^\mu$ .

$$= \eta_{\mu\nu} X'^\mu_{,\alpha} X'^\nu_{,\beta}$$

is 1-1.

map  $X^\alpha \rightarrow X^\alpha$  is 1-1.

$$\partial_\gamma \left( X'^\mu_{,\alpha\gamma} X'^\nu_{,\beta} \eta_{\mu\nu} + X'^\mu_{,\alpha} X'^\nu_{,\beta\gamma} \eta_{\mu\nu} \right) = 0$$

$$= \left( X_{,\gamma\beta} X_{,\alpha} + X_{,\gamma} X_{,\alpha\beta} \right) = 0$$

$$X_{,\beta\alpha} X_{,\gamma} + X_{,\beta} X_{,\gamma\alpha}$$



1-1.

$$\begin{aligned}
 & \cancel{X'^{\mu}_{,\alpha\gamma} X'^{\nu}_{,\beta} \eta_{\mu\nu}} + \cancel{X'^{\mu}_{,\alpha} X'^{\nu}_{,\beta\gamma} \eta_{\mu\nu}} = 0 \\
 & - \left( \cancel{X'_{,\gamma\beta} X'_{,\alpha}} + \cancel{X'_{,\gamma} X'_{,\alpha\beta}} \right) = 0 \\
 & + \left( \cancel{X'_{,\beta\alpha} X'_{,\gamma}} + X'_{,\beta} X'_{,\gamma\alpha} \right) = 0
 \end{aligned}$$

$$\eta_{\mu\nu} X'^{\mu}_{,\alpha\gamma} X'^{\nu}_{,\beta} = 0$$

invertible      invertible

$$\eta_{\mu\nu} X'^{\mu}_{,\alpha} X'^{\nu}_{,\beta} = 0$$

$$\Rightarrow X'^{\mu}_{,\alpha} = 0$$

$\Rightarrow$   $x'$  must be  
linear in  $x$

$$\underline{x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}}$$



lorentz-invariant distances (and inner products of vectors)

$$ds'^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad (*)$$

$$= \eta_{\mu\nu} A^\mu B^\nu$$

transformations satisfying (\*) are lorentz + translations = Poincare group

$\frac{\partial x'^\mu}{\partial x^\beta}$ , allow  $x'^\mu$  to be any function of  $x^\mu$ .

$$\propto X_{\mu\beta}$$

Replacing  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$

4x4 symmetric, invertible matrix  
function of  $x^\alpha$ .

a transformation rule

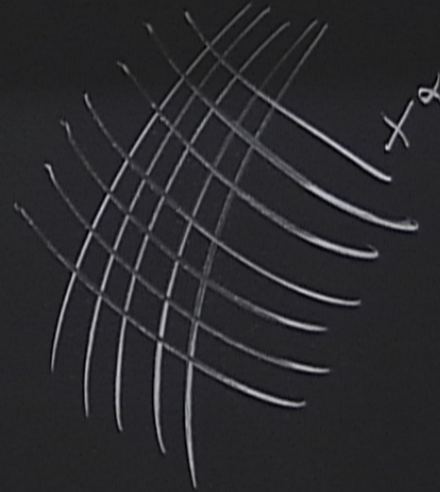
$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$



$g_{\mu\nu}(x)$

4x4 symmetric, invertible matrix  
function of  $x^\alpha$ .

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$



realise by replacing  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$

4x4 s  
func

with transformation rule  $g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$

coordinates have no intrinsic meaning.



Idea: generalise by replacing  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$

4x4 symmetric, invertible matrix  
function of  $x^\alpha$ .

with transformation rule 
$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$

coordinates have no intrinsic meaning.

Can show that it is always possible to choose coordinates in a sufficiently small neighbourhood of a point such that  $g_{\mu\nu} = \eta_{\mu\nu}$  at that point, and  $g_{\mu\nu,\alpha} = 0$  at that point.



Idea: generalise by replacing  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x^\alpha)$

with transformation rule  $g'_{\mu\nu}(x') = g_{\alpha\beta}(x)$

coordinates have no intrinsic meaning.

Can show that it is always possible to choose coordinates in a point such that  $g_{\mu\nu} = \eta_{\mu\nu}$  at that point, and



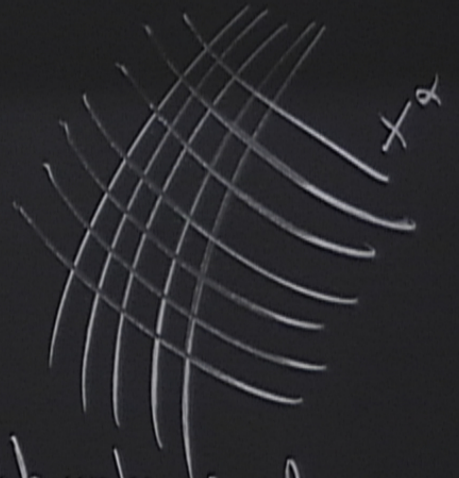
$$g_{\mu\nu} \rightarrow g_{\mu\nu}(x^\alpha)$$

4x4 symmetric, invertible matrix  
function of  $x^\alpha$ .

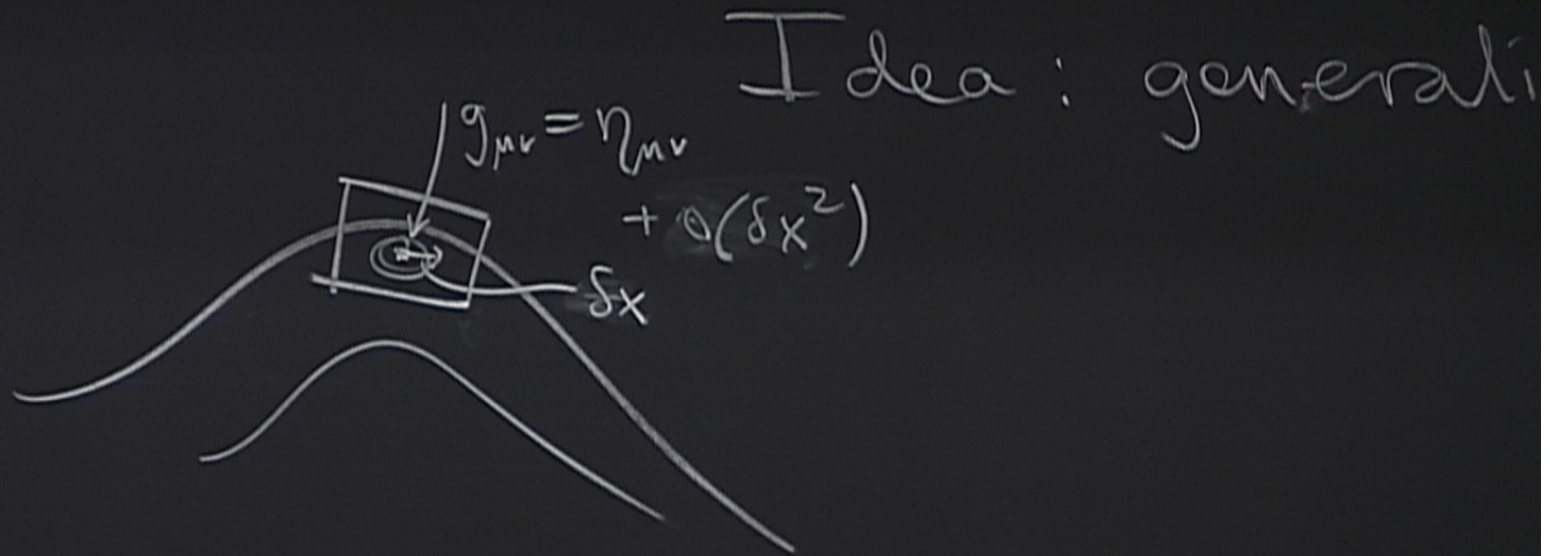
transformation rule

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$

no intrinsic meaning.



choose coordinates in a sufficiently small neighbourhood of  
 $g_{\mu\nu}$  at that point, and  $g_{\mu\nu,\alpha} = 0$  at that point.



Can show that it is

a metric tensor



$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \quad \text{Taylor series}$$

$x \leftrightarrow x'$

Proof: assume  $g_{\alpha\beta}(x) = g_{\alpha\beta}(0) + A_{\alpha\beta,\lambda} x^\lambda + \frac{1}{2} B_{\alpha\beta,\lambda\sigma} x^\lambda x^\sigma + \dots$

$$x^\alpha = K^\alpha_\gamma x'^\gamma + \frac{1}{2} L^\alpha_{\gamma\delta} x'^\gamma x'^\delta + \frac{1}{6} M^\alpha_{\gamma\delta\epsilon} x'^\gamma x'^\delta x'^\epsilon + \dots$$

$$g_{\alpha\beta}(x) = g_{\alpha\beta}(0) + A_{\alpha\beta,\lambda} x^\lambda + \frac{1}{2} B_{\alpha\beta,\lambda\sigma} x^\lambda x^\sigma + \dots$$

Taylor series

Proof: assume  $g_{\alpha\beta}(x) = g_{\alpha\beta}(0) + A_{\alpha\beta,\lambda} x^\lambda + \frac{1}{2} B_{\alpha\beta,\lambda\sigma} x^\lambda x^\sigma + \dots$

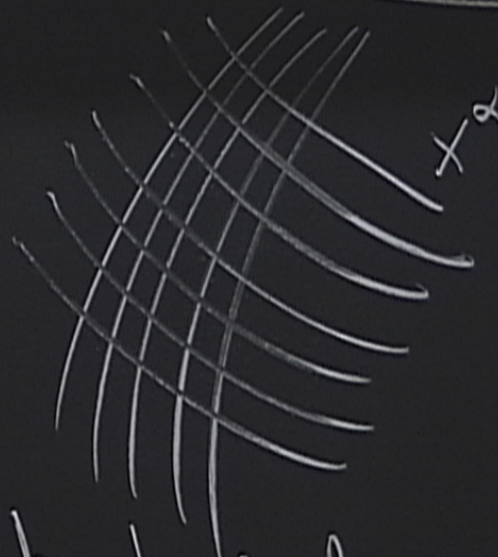
$$x^\mu = K^\mu_\nu x'^\nu + \frac{1}{2} L^\mu_{\nu\sigma} x'^\nu x'^\sigma + \frac{1}{6} M^\mu_{\nu\sigma\epsilon} x'^\nu x'^\sigma x'^\epsilon + \dots$$

$\Rightarrow$  can choose  $K, L$  to set  $g'_{\mu\nu}(0) = \eta_{\mu\nu}$  and  $g'_{\mu\nu,\alpha}(0) = 0$ .



4x4 symmetric, invertible matrix (3 pos. evals,  
1 neg. eval)  
function of  $x^\alpha$ .

$$g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$



a sufficiently small neighbourhood of  
and  $g_{\mu\nu, \kappa} = 0$  at that point.



$$\Rightarrow \text{set } K = Ol, \text{ where } l = \begin{pmatrix} \frac{1}{e_0} & 0 & 0 & 0 \\ 0 & \frac{1}{e_1} & 0 & 0 \\ 0 & 0 & \frac{1}{e_2} & 0 \\ 0 & 0 & 0 & \frac{1}{e_3} \end{pmatrix} \Rightarrow g' =$$

Similarly,

$$g'_{\mu\nu,\lambda}(0) = A_{\alpha\beta,\delta} K_{\lambda}^{\delta} K_{\mu}^{\alpha} K_{\nu}^{\beta} + g'_{\alpha\beta}(0) (L_{\mu\lambda}^{\alpha} K_{\nu}^{\beta} + K_{\mu}^{\alpha} L_{\nu\lambda}^{\beta})$$



$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$L_{\mu\lambda}^{\alpha} (L_{\mu\lambda}^{\beta} K_{\nu}^{\gamma} + K_{\mu}^{\alpha} L_{\nu\lambda}^{\beta})$$

$$\Rightarrow g' = K^T g K = \lambda O^T g O \lambda = \lambda D \lambda = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \eta.$$



$\Rightarrow$  set  $K = O l$ , where  $l = \begin{pmatrix} \frac{1}{e_0} & 0 & 0 & 0 \\ 0 & \frac{1}{e_1} & 0 & 0 \\ 0 & 0 & \frac{1}{e_2} & 0 \\ 0 & 0 & 0 & \frac{1}{e_3} \end{pmatrix} \Rightarrow g' = K^T g K =$

$\Downarrow$  given  
 $g_{\alpha\beta}(0) = A_{\alpha\beta\gamma\delta} K^\gamma_\lambda K^\lambda_\mu K^\mu_\nu + g_{\alpha\beta}(0) (L^\alpha_{\mu\lambda} K^\beta_\nu + K^\alpha_\mu L^\beta_{\nu\lambda})$

$\stackrel{?}{=} 0$  can we solve for  $L$ 's YES, just as above.



set  $K = O l$ , where  $l = \begin{pmatrix} \frac{1}{e_0} & 0 & 0 & 0 \\ 0 & \frac{1}{e_1} & 0 & 0 \\ 0 & 0 & \frac{1}{e_2} & 0 \\ 0 & 0 & 0 & \frac{1}{e_3} \end{pmatrix} \Rightarrow g' = K^T g K = l$

given

$$o) = A_{\alpha\beta\gamma} K_{\lambda}^{\alpha} K_{\mu}^{\beta} K_{\nu}^{\gamma} + g_{\alpha\beta}(0) (L_{\mu\lambda}^{\alpha} K_{\nu}^{\beta} + K_{\mu}^{\alpha} L_{\nu\lambda}^{\beta})$$

$\stackrel{?}{=} 0$

can we solve for  $L$ 's

YES, just as above. (\*)



$$\partial_\gamma \left( \cancel{x'^\mu_{,\alpha\gamma} x'^\nu_{,\beta}} \eta_{\mu\nu} + \cancel{x'^\mu_{,\alpha} x'^\nu_{,\beta\gamma}} \eta_{\mu\nu} \right) = 0$$

$$- \left( \cancel{x_{,\gamma\beta} x_{,\alpha}} + \cancel{x_{,\alpha\beta} x_{,\gamma}} \right) = 0$$

$$+ \left( \cancel{x_{,\beta\alpha} x_{,\gamma}} + \cancel{x_{,\beta} x_{,\gamma\alpha}} \right) = 0$$

(\*)

}  $\Rightarrow$

$x'$  must be linear in  $x$

invertible  
 $\eta_{\mu\nu} x'^\mu_{,\alpha\gamma} x'^\nu_{,\beta\delta}$

$\Rightarrow x'^\mu_{,\alpha}$

$x'^\mu_{,\beta}$

4x4 symmetric invertible



$$K = O l, \text{ where } l = \begin{pmatrix} \frac{1}{e_0} & 0 & 0 & 0 \\ 0 & \frac{1}{e_1} & 0 & 0 \\ 0 & 0 & \frac{1}{e_2} & 0 \\ 0 & 0 & 0 & \frac{1}{e_3} \end{pmatrix} \Rightarrow g' = K^T g K = l O^T$$

given

$$g_{\alpha\beta} K^\alpha_\lambda K^\lambda_\mu K^\mu_\nu + g_{\alpha\beta}(0) (L^\alpha_{\mu\lambda} K^\beta_\nu + K^\alpha_\mu L^\beta_{\nu\lambda})$$

Can we solve for  $L$ 's

YES, just as above. (\*)

$$\frac{(D+1)}{2}$$

$$\frac{D(D+1)}{2} \cdot D$$

$$D \mid \frac{D(D+1)}{2}$$

And deñus:

$$B_{\alpha\beta, \lambda\sigma}$$

$$\frac{D(D+1)}{2}$$

$$\frac{D(D+1)}{2}$$

$$\left. \begin{array}{l} B_{\alpha\beta, \lambda\sigma} \\ \frac{D(D+1)}{2} \quad \frac{D(D+1)}{2} \end{array} \right\} \frac{1}{4} D^2 (D+1)^2 \text{ parameters.}$$



$$\frac{D(D+1)}{2}$$

$$\frac{D(D+1) \cdot D}{2}$$

$$D \mid \frac{D(D+1)}{2}$$

2nd derivs:

$$B_{\alpha\beta, \lambda\sigma}$$

$$\frac{D(D+1)}{2}$$

$$\frac{D(D+1)}{2}$$

$$\left. \begin{array}{l} B_{\alpha\beta, \lambda\sigma} \\ \frac{D(D+1)}{2} \quad \frac{D(D+1)}{2} \end{array} \right\} \frac{1}{4} D^2 (D+1)^2 \text{ parameters.}$$

f.  $M^{\alpha}$   $\gamma \delta \epsilon$   
          
 Symm.  $\Rightarrow \frac{1}{6} D(D+1)(D+2)$   
 Order  
 3 indices



Young tableaux

# boxes = # indices

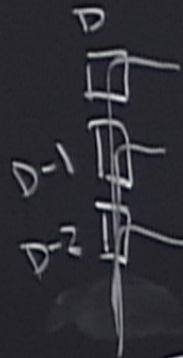
$D, D+1, D+2$



symmetric

given

$T_{ijklm}$   
 $\underbrace{\quad\quad\quad}_{\text{sym}}$   
 $\underbrace{\quad\quad\quad}_{\text{antiSym}}$



antisymmetric

$$T_{\mu\nu\lambda} \Rightarrow \frac{1}{6} D(D-1)(D-2).$$

$M^{\alpha}_{\beta}$   
 $\nwarrow \nearrow$   
 $D$   
 $\delta \delta \epsilon$   
Symm.

$\Rightarrow \frac{1}{6} D(D+1)(D+2)$   
 Order  
 3 indices

$\begin{matrix} 111 \\ 112 \\ 122 \\ 222 \end{matrix}$

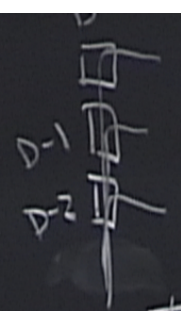
given  $T_{ijk\ell m}$   
Symm antisymm

$$\frac{D^2(D+1)(D+2)}{6}$$

$D$   
 $D-1$   
 $D-2$   
 $T_{\mu\nu}$



given  $1234m$   
 $\underbrace{\quad\quad\quad}_{\text{sym}}$   
 $\underbrace{\quad\quad\quad}_{\text{antisym}}$



antisymmetric

$\left. \begin{array}{l} 111 \\ 112 \\ 122 \\ 222 \end{array} \right\}$   
 $D(D+1)(D+2)$

$$\frac{D^2(D+1)(D+2)}{6}$$

$$T_{\text{MVA}} \Rightarrow \frac{1}{6} D(D-1)(D-2)$$

$$D(D+1) \left( \frac{\frac{1}{4} D(D+1) - \frac{1}{6} D(D+2)}{\frac{1}{12} D(D-1)} \right) = \frac{D^2}{12} (D^2 - 1)$$

| # 2nd<br>deriv<br>parameters | D |
|------------------------------|---|
| 0                            | 1 |
| 1                            | 2 |
| 6                            | 3 |
| 20                           | 4 |



$D \times D$  matrix  $\rightarrow \frac{D(D+1)}{2}$  parameters

$$\left. \begin{array}{l} 0 \rightarrow \frac{D(D-1)}{2} \\ 1 \rightarrow D \end{array} \right\} \frac{D(D+1)}{2}$$

2nd derivs:

$$\left. \begin{array}{l} B_{\alpha\beta, \gamma} \\ \frac{D(D+1)}{2} \quad \frac{D(D+1)}{2} \end{array} \right\} \frac{1}{4} D^2(D+1)^2 \text{ parameters.}$$

# of second derivatives of the metric which cannot be set zero by any coordinate transformation =  $D(D+1) \left( \frac{1}{4} D(D+1) \right)$

$$\underbrace{A_{\alpha\beta, \gamma}}_{\frac{D(D+1)}{2} \cdot D} \leftrightarrow \underbrace{L_{\gamma\delta}^{\alpha}}_{D \mid \frac{D(D+1)}{2}}$$

cf.  $M_{\gamma\delta\epsilon}^{\alpha}$   $\downarrow$   $D$

Symm.  $\Rightarrow \frac{1}{6} D(D+1)(D+2)$

Order 3 indices

$$\left. \begin{array}{l} 111 \\ 112 \\ 122 \\ 222 \end{array} \right\}$$



second derivatives which cannot be set zero by any coordinate transformation

It will turn out that  $\exists$  coordinates in nhd of any point, called Riemann n

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} X^\alpha X^\beta + O(x^3)$$

$\uparrow$  Riemann tensor.



hd of any point, called Riemann normal coordinates, sit.

$$\frac{1}{6} R_{\mu\alpha\nu\beta} X^\alpha X^\beta + O(X^3)$$

↑  
Riemann tensor.



$(x')$

$4 \times 4$  symmetric, invertible matrix  
function of  $x^\alpha$ .

(3 pos. evals,  
1 neg. eval)

$$g_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$

$$\Rightarrow g_{\mu\nu} dx^\mu dx^\nu = \overbrace{g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}}^{g'_{\alpha\beta}} dx'^\alpha dx'^\beta$$



exists in a sufficiently small neighbourhood of  
point, and  $g_{\mu\nu,x} = 0$  at that point.



$$t' = t$$

$$x' = x - vt - \frac{1}{2}gt^2$$

$$m\ddot{x} = gm + F_{elec}$$

$$m\ddot{x}' = m\ddot{x} - gm = F_{elec}$$

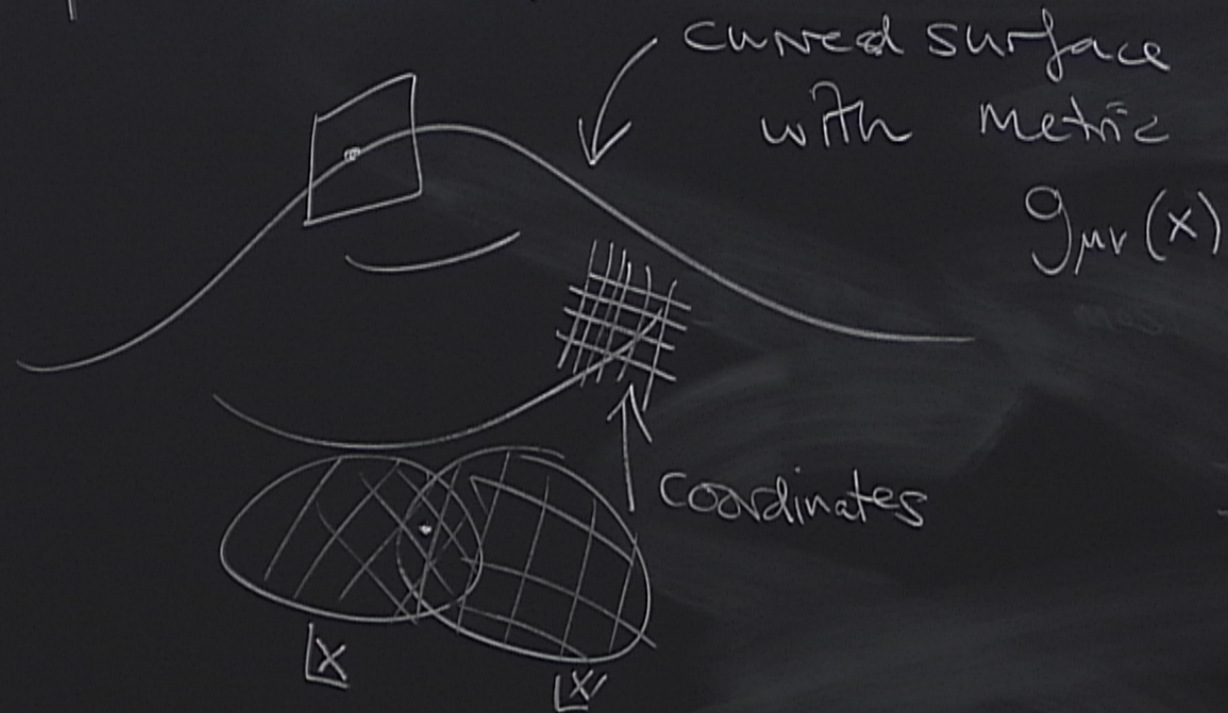
turn out that  $\exists$  coordinates in nhd of any point, called Riemann normal

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} X^\alpha X^\beta + O(x^3)$$

$\uparrow$   
 Riemann tensor.



picture of spacetime



IVNC.  $x \rightarrow x' \alpha$  is 1-

# Tensors on Manifolds

Simplest:  $\phi'(x') = \phi(x)$  Scalars

$$V'^{\mu}(x') \frac{\partial}{\partial x'^{\mu}} = V^{\mu}(x) \frac{\partial}{\partial x^{\mu}}$$

contravariant vector :  $V'^{\mu}(x')$

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial}{\partial x'^{\alpha}}$$



on Manifolds

$$\phi(x') = \phi(x) \quad \text{Scalars}$$

$$\frac{\partial}{\partial x'^\mu} = V^\mu(x) \frac{\partial}{\partial x^\mu} \quad \text{contravariant vector} \quad : \quad V'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\mu} V^\mu(x)$$

$\left( \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial}{\partial x'^\alpha} \right) \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}$



# Tensors on Manifolds

st:  $\phi'(x') = \phi(x)$  Scalars

eg.  $V'^{\mu}(x') \frac{\partial}{\partial x'^{\mu}} = V^{\mu}(x) \frac{\partial}{\partial x^{\mu}}$  contravariant vector

$$V'^{\alpha}(x') = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} V^{\mu}(x)$$

eg.  $W'_{\mu} = \frac{\partial}{\partial x'^{\mu}} \phi'(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}} \phi(x)$  covariant vector components

$$W'_{\mu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} W_{\alpha}(x)$$



$$\text{Sim}^2 \quad T'^{\alpha\beta\gamma\dots}_{\mu\nu\dots}(x') = \left( \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \dots \right) \left( \frac{\partial x^{\beta}}{\partial x'^{\nu}} \dots \right) T^{\mu\nu\dots}_{\alpha\beta\gamma\dots}(x)$$

Tensor algebra: if  $T, S$  are tensors

- 1) if they have same number of contravariant and covariant indices
- 2) outer multiplication  $T^{\mu\nu\dots}_{\alpha\beta\dots} S^{\gamma\delta\dots}_{\epsilon\zeta\dots}$  is a tensor
- 3) inner multiplication (contraction)  $T^{\mu\nu\dots}_{\alpha\beta\dots} S^{\alpha\gamma\dots}_{\mu\delta\dots}$  is a tensor w/



$$\left( \frac{\partial x^\alpha}{\partial x'^\mu} \dots \right) T^{\rho \dots}{}_{\sigma \dots}(x)$$

invariant and covariant indices, then  $T+S$  is a tensor

$S_{\alpha\beta\dots}$  is a tensor

$S_{\mu\rho\dots}$  is a tensor w/ indices  $\nu\dots$  etc.

$$\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x'^\beta} = \delta_\mu^\nu$$



indices, then  $TFS$  is a tensor

$$\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} = \delta_\mu^\nu$$

$\nu \dots$

$\mu$  indices  $\beta \dots$  etc.



1) if they have same number of contravariant and covariant indices, then  $T \cdot S$  is a tensor

2) outer multiplication  $T^{\mu\nu\dots} S_{\alpha\beta\dots}$  is a tensor

3) inner multiplication (contraction)  $T^{\mu\nu\dots} S_{\mu\beta\dots}$  is a tensor w/ indices  $\nu\dots_{\beta\dots}$  etc.

$$\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} = \delta_\mu^\nu$$

Useful facts:

1) if a tensor is zero in any one coordinate system, it is zero in all coordinate systems.

2) quotient rule: if  $T^{\mu\nu} S_\nu = 0 \quad \forall S_\nu$ , then  $T^{\mu\nu} = 0$ .



Useful facts:

1) if a tensor is zero in any one system.

2) quotient rule: if  $T^{\mu\nu} S_\nu = 0 \quad \forall$

Volumes: use fact that we can choose coordinates  $x_f^\mu$  in which  $g_{\mu\nu} = \eta_{\mu\nu}$  and  
$$dV = dx_f^0 dx_f^1 dx_f^2 dx_f^3$$

1) if a tensor is zero in any one coordinate system.

2) quotient rule: if  $T^{\mu\nu} S_\nu = 0 \quad \forall S_\nu$

we can choose coordinates  $x_f^\mu$  in which  $g_{\mu\nu} = \eta_{\mu\nu}$  at a

$$dV = dx_f^0 dx_f^1 dx_f^2 dx_f^3 = \det\left(\frac{\partial x_f^a}{\partial x^\mu}\right) d^4 x^\mu$$



any one coordinate system, it is zero in all coord

$$\circ \quad \forall S_v, \text{ then } T^{\mu\nu} = 0.$$

$g_{\mu\nu} = \eta_{\mu\nu}$  at a point

$$g_{\mu\nu} = \frac{\partial X^a}{\partial x^\mu} \frac{\partial X^b}{\partial x^\nu} \eta_{ab}$$



any one coordinate system, it is zero in all coordinate

○  $\forall S_\nu$ , then  $T^{\mu\nu} = 0$ .

$$g \equiv \det g_{\mu\nu}$$

$= \eta_{\mu\nu}$  at a point

$$g_{\mu\nu}(x) = \frac{\partial X_f^a}{\partial x^\mu} \frac{\partial X_f^b}{\partial x^\nu} \eta_{ab}$$

$$\sqrt{-g} = \left( \det \frac{\partial X_f^a}{\partial x^\mu} \right)$$

$$\Rightarrow \det \frac{\partial X_f^a}{\partial x^\mu} = \sqrt{-g}$$

$$\Rightarrow \boxed{dV = \sqrt{-g} d^4x}$$