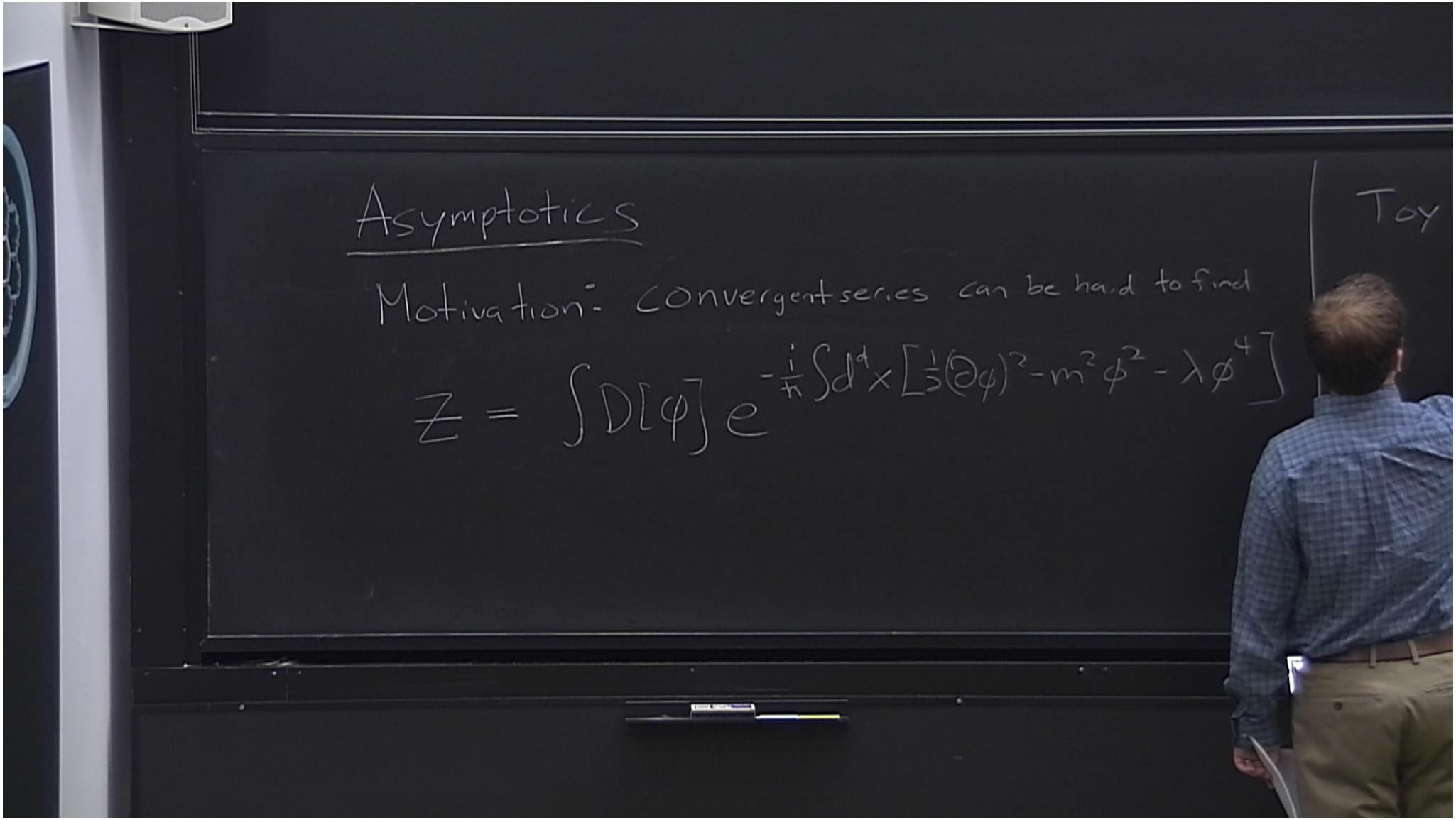
Title: Special Functions

Date: Sep 03, 2015 10:30 AM

URL: http://pirsa.org/15090016

Abstract:

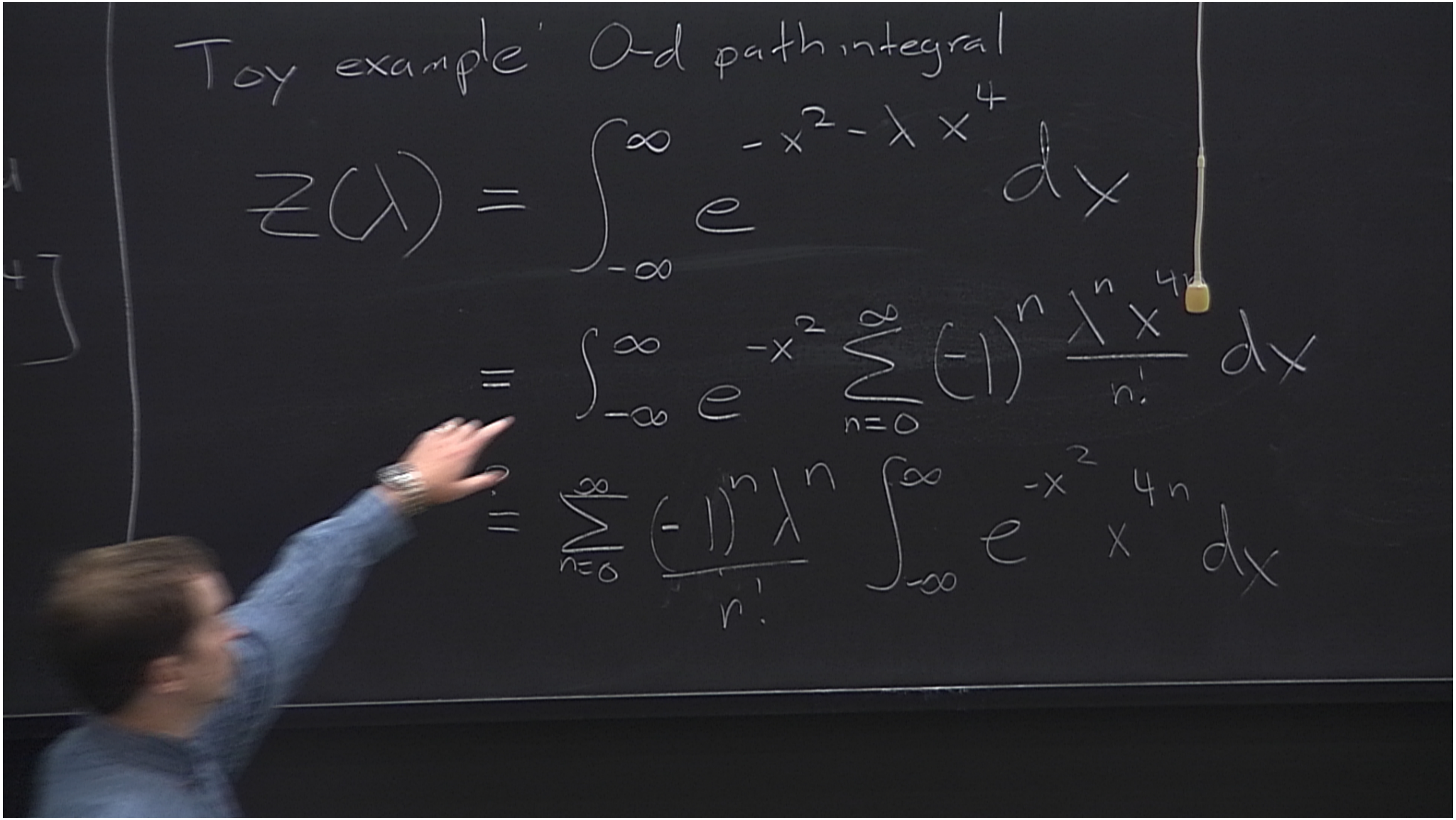

Asymptotics
Motivation: convergent series can be hard to find
$Z = \int D[\varphi]\, e^{-\frac{i}{\hbar}\int d^4x \left[\frac{1}{2}(\partial\varphi)^2 - m^2\varphi^2 - \lambda\varphi^4\right]}$
Toy

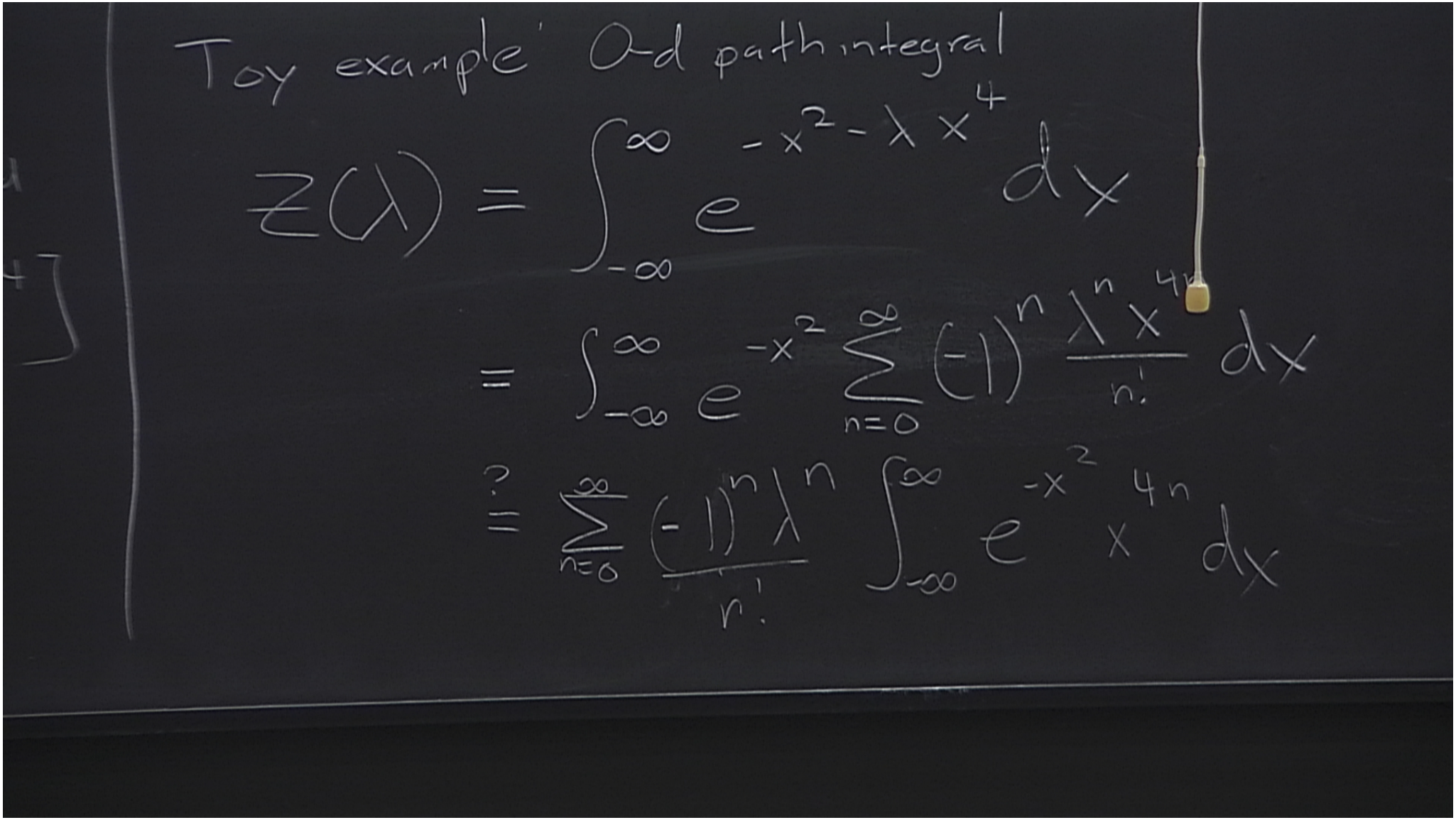


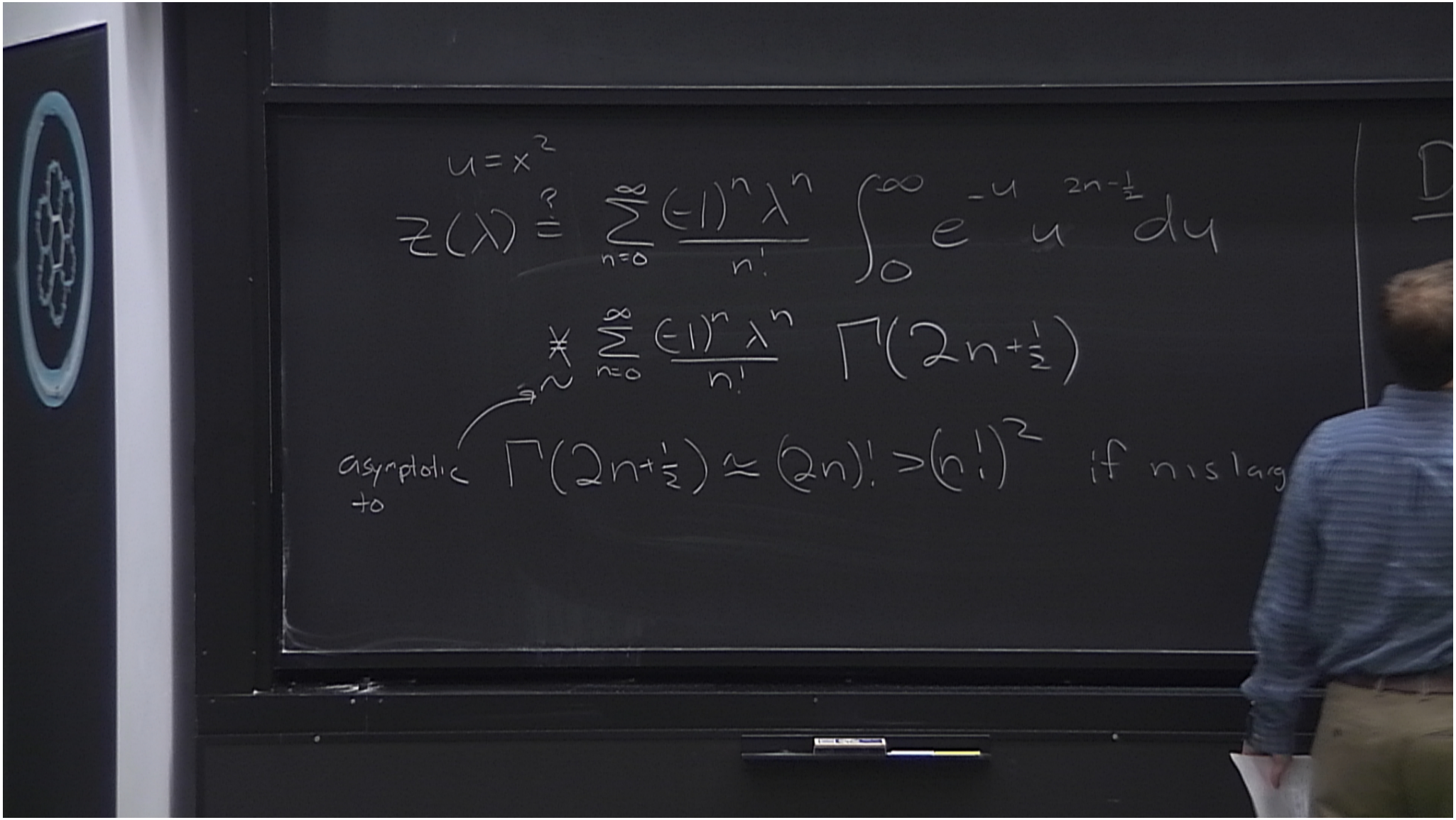
Asymptotics

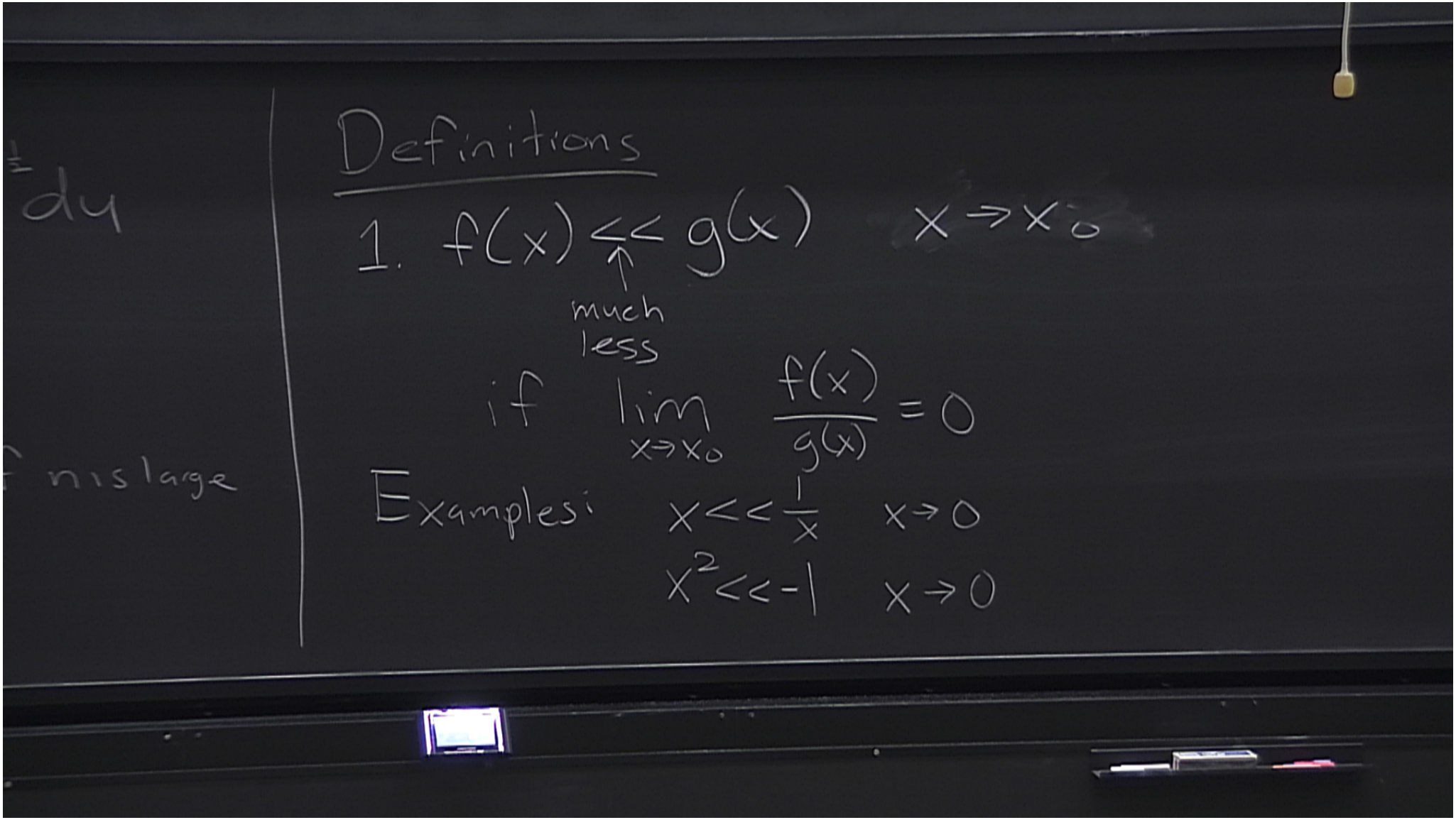
Motivation: convergent series can be hard to find

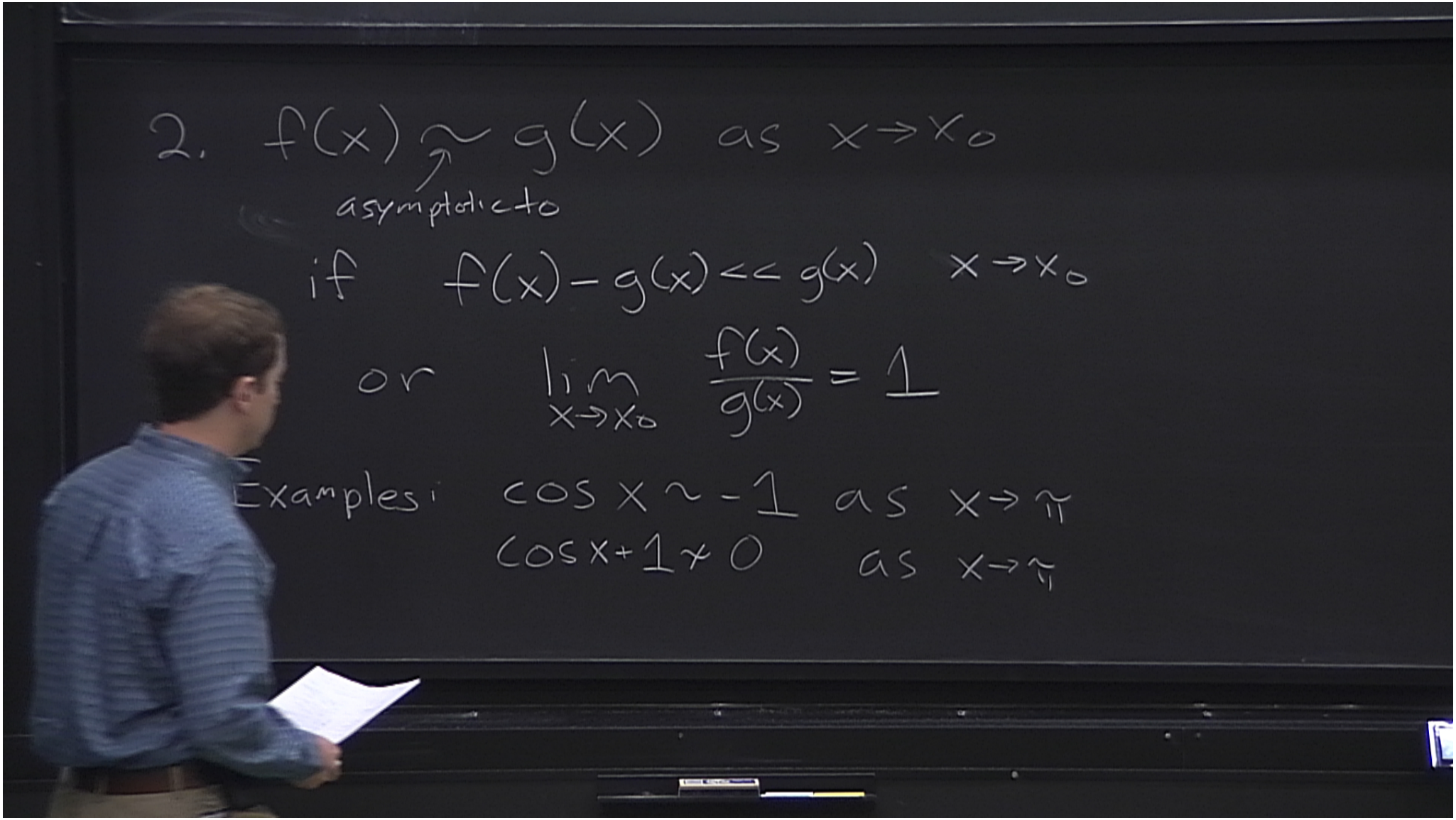
$$Z = \int D[\phi]\, e^{-\frac{i}{\hbar}\int d^4x \left[\frac{1}{2}(\partial\phi)^2 - m^2\phi^2 - \lambda\phi^4\right]}$$


Toy example: 0-d path integral
$Z(\lambda) = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4} dx$
$= \int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n x^{4n}}{n!} dx$
$= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \int_{-\infty}^{\infty} e^{-x^2} x^{4n} dx$


Toy example: 0-d path integral
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$\stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \int_{-\infty}^{\infty} e^{-x^2} x^{4n} dx$


$u = x^2$
$Z(\lambda) \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \int_0^{\infty} e^{-u} u^{2n-\frac{1}{2}} du$
$\stackrel{?}{\sim} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \Gamma(2n+\frac{1}{2})$
asymptotic to
$\Gamma(2n+\frac{1}{2}) \simeq (2n)! > (n!)^2$ if n is larg


dy
nislage
Definitions
1. $f(x) \ll g(x)$ $x \to x_0$
much less
if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$
Examples: $x \ll \frac{1}{x}$ $x \to 0$
$x^2 \ll -1$ $x \to 0$


2. $f(x) \sim g(x)$ as $x \to x_0$
asymptotic to
if $f(x) - g(x) << g(x)$ $x \to x_0$
or $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$
Examples: $\cos x \sim -1$ as $x \to \pi$
$\cos x + 1 \neq 0$ as $x \to \pi$

2. $f(x) \sim g(x)$ as $x \to x_0$

asymptotic to

if $f(x) - g(x) << g(x)$ $x \to x_0$

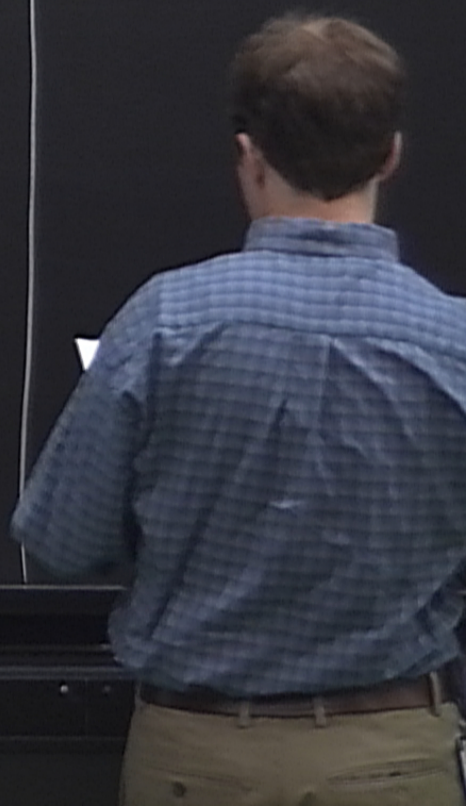
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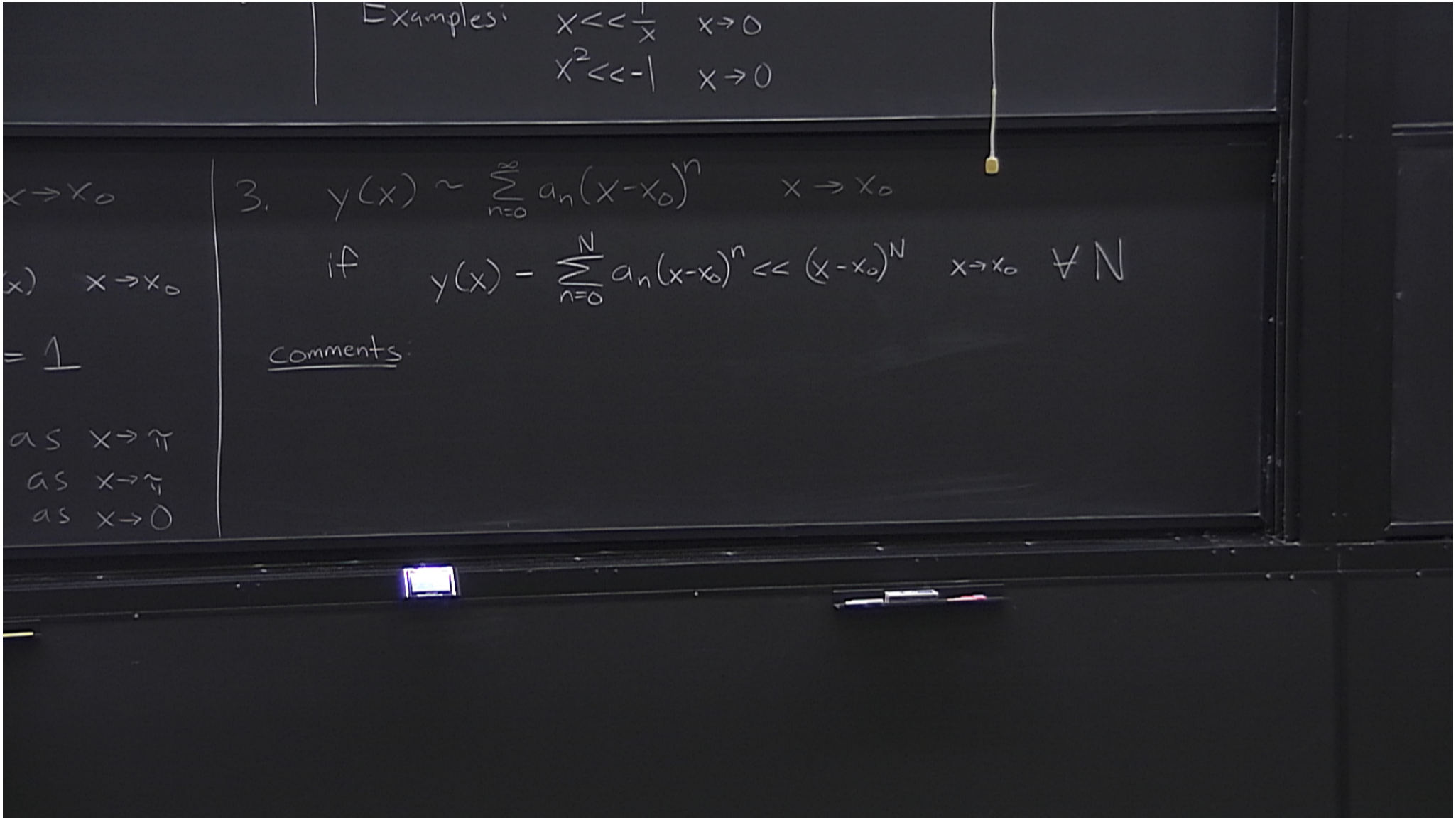
Examples: $\cos x \sim -1$ as $x \to \pi$

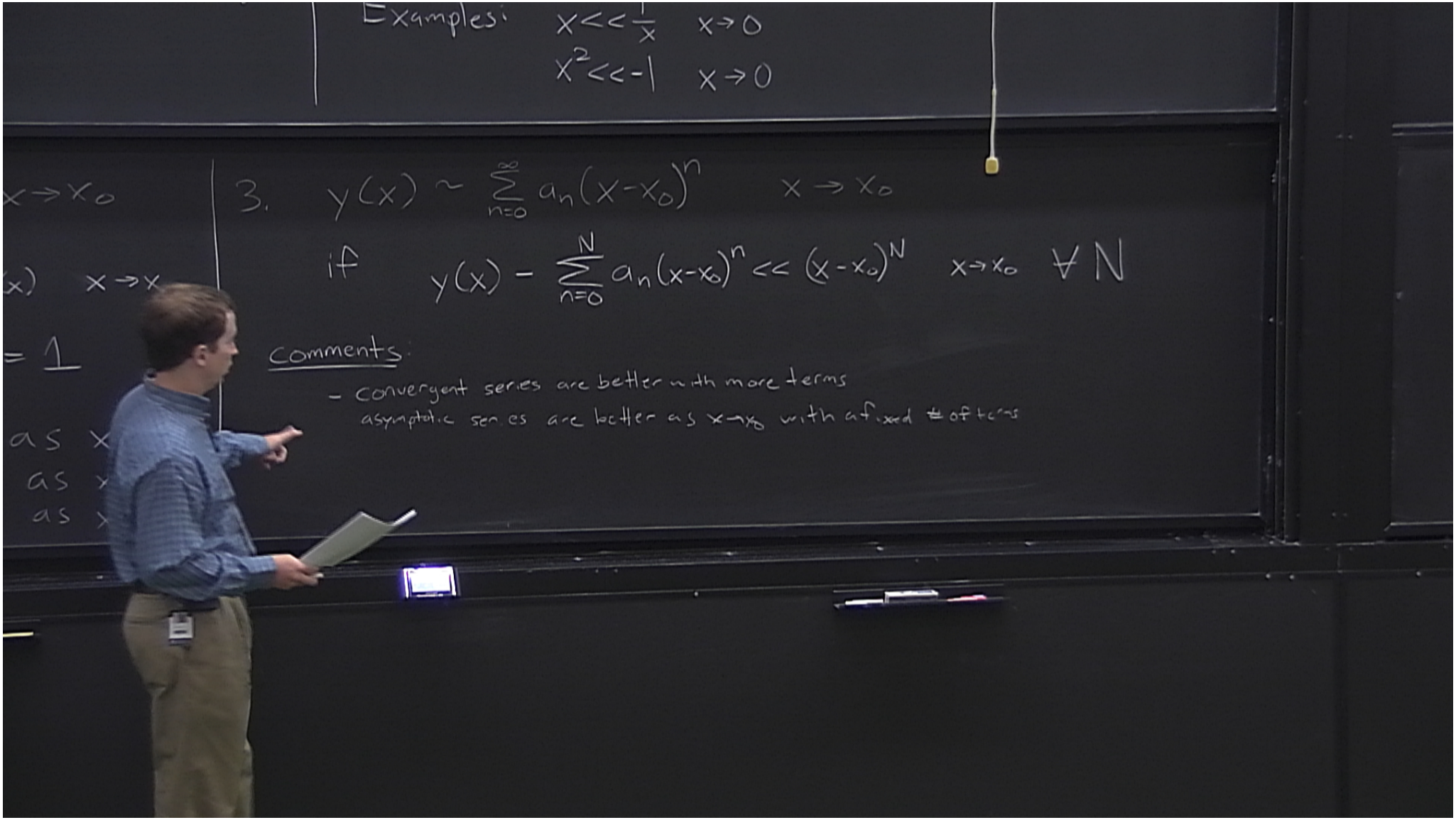
$\cos x + 1 \not\sim 0$ as $x \to \pi$

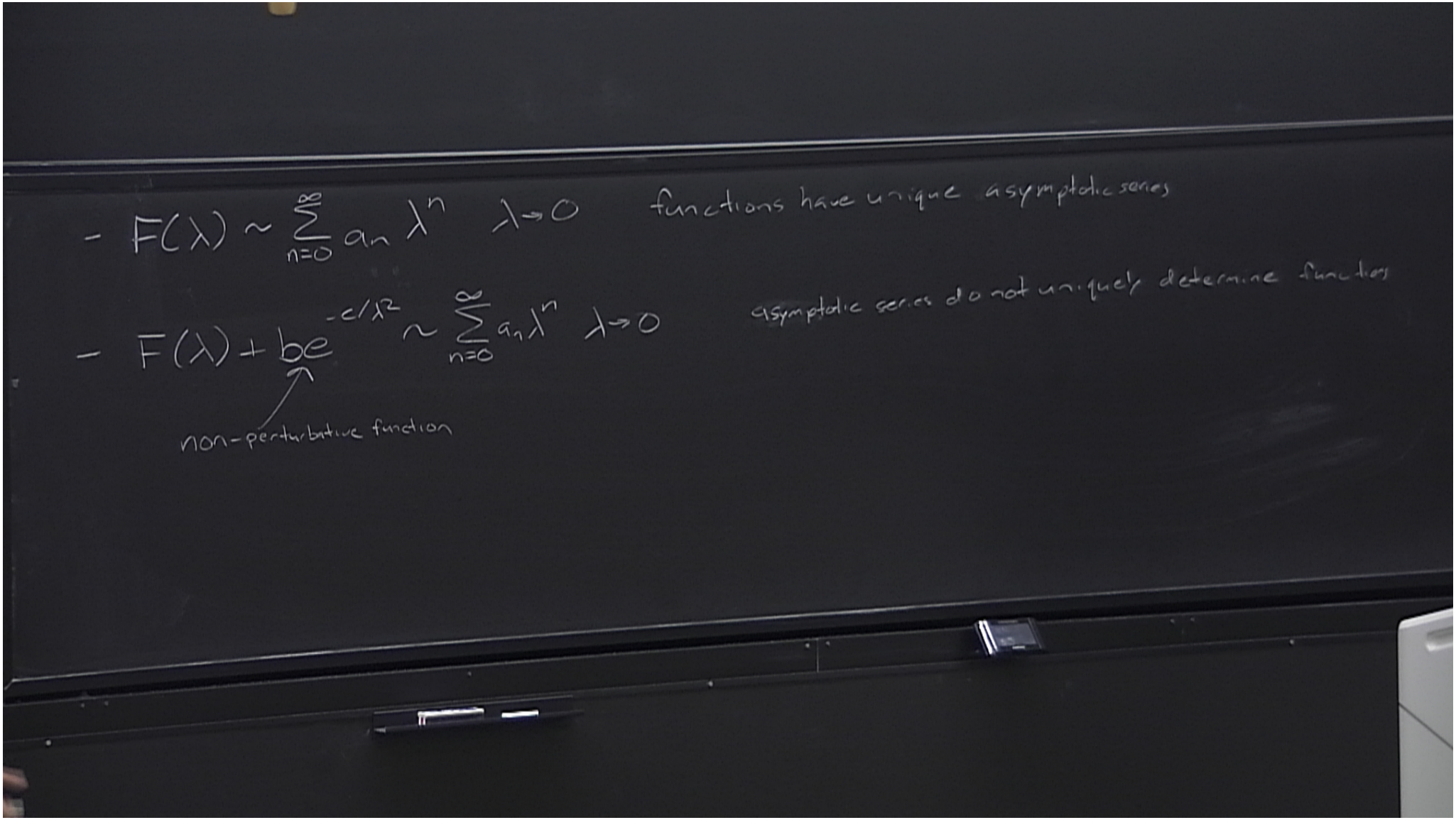
$x^2 \not\sim x$ as $x \to 0$

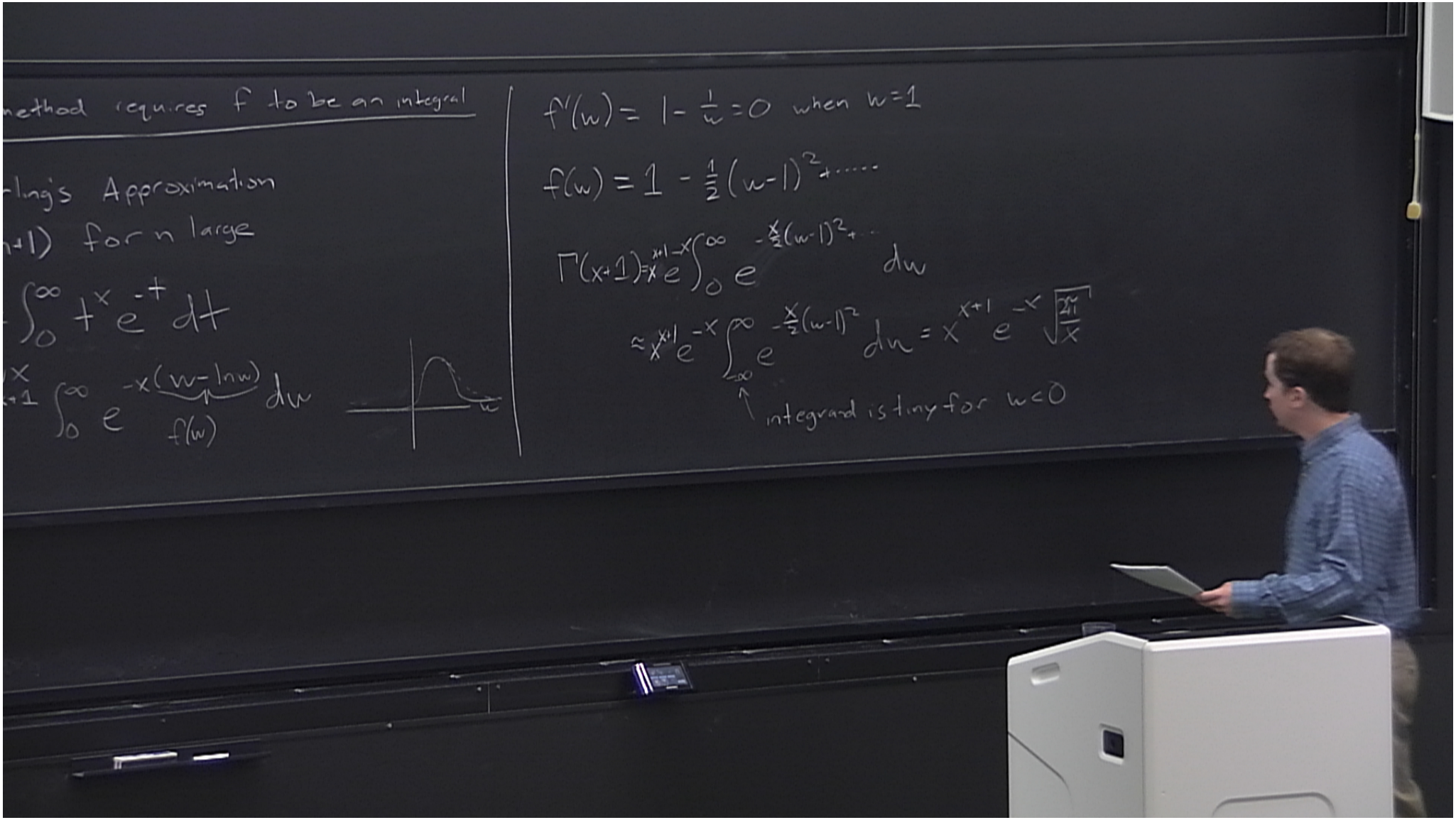
3.

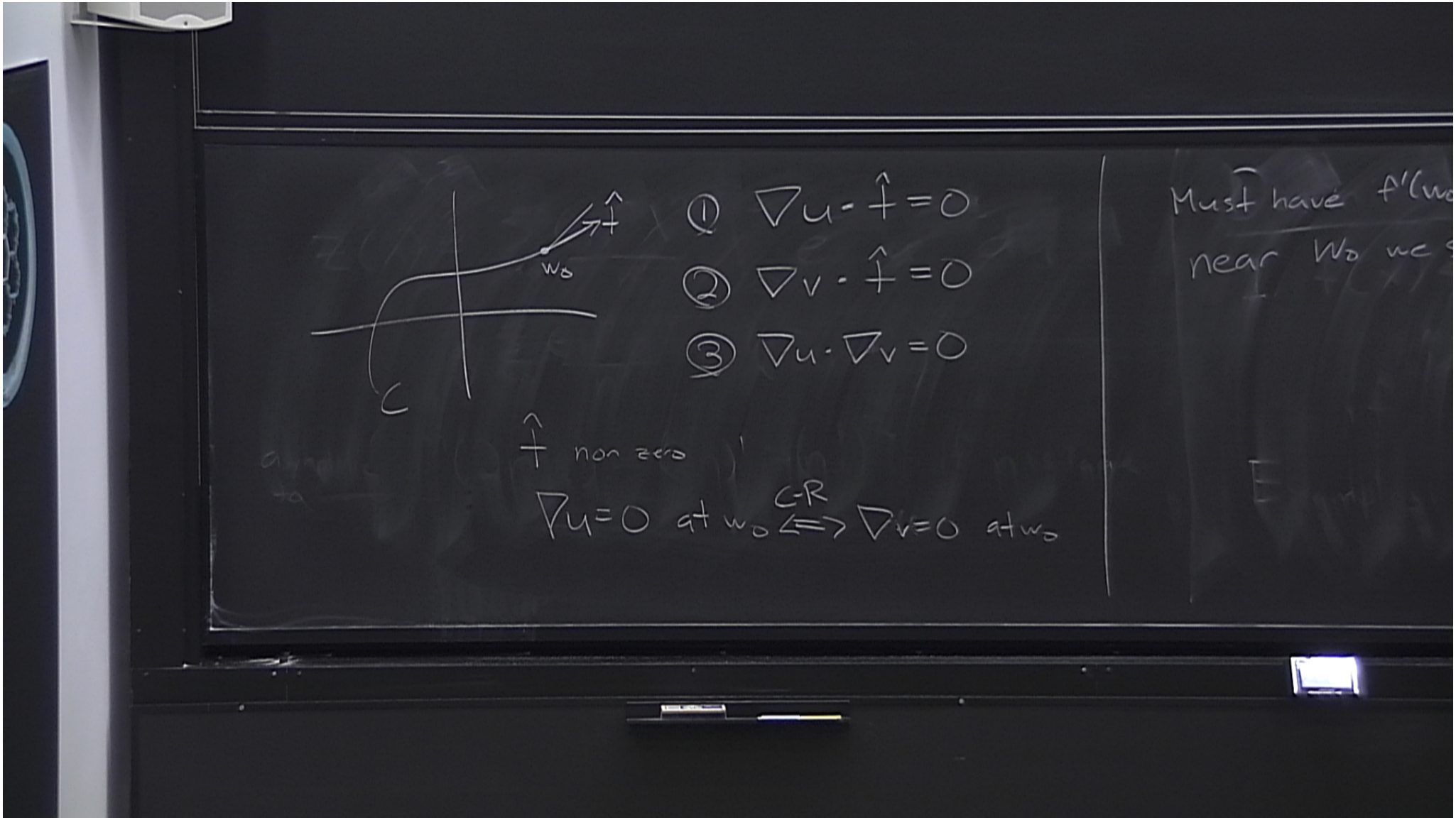



Examples:
$x \ll \frac{1}{x}$ $x \to 0$
$x^2 \ll -1$ $x \to 0$
$x \to x_0$
$x) \quad x \to x_0$
$= 1$
as $x \to \pi$
as $x \to \pi$
as $x \to 0$
3. $y(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad x \to x_0$
if $y(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \ll (x-x_0)^N \quad x \to x_0 \quad \forall N$
comments:


Examples:
$x \ll \frac{1}{x} \quad x \to 0$
$x^2 \ll -1 \quad x \to 0$
$x \to x_0$
3. $y(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad x \to x_0$
if $y(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \ll (x-x_0)^N \quad x \to x_0 \quad \forall N$
Comments:
- convergent series are better with more terms
asymptotic series are better as $x \to x_0$ with a fixed # of terms


- $F(\lambda) \sim \sum_{n=0}^{\infty} a_n \lambda^n \quad \lambda \to 0$ functions have unique asymptotic series
- $F(\lambda) + be^{-c/\lambda^2} \sim \sum_{n=0}^{\infty} a_n \lambda^n \quad \lambda \to 0$ asymptotic series do not uniquely determine functions
non-perturbative function


method requires f to be an integral
-ling's Approximation
n+1) for n large
$\int_0^\infty t^x e^{-t}\,dt$
$\int_0^\infty e^{-x(w-\ln w)}\,dw$
f(w)
$f'(w) = 1 - \frac{1}{w} = 0$ when $w=1$
$f(w) = 1 - \frac{1}{2}(w-1)^2 + \cdots$
$\Gamma(x+1) = x^{x+1}e^{-x}\int_0^\infty e^{-\frac{x}{2}(w-1)^2+\cdots}\,dw$
$\approx x^{x+1}e^{-x}\int_{-\infty}^\infty e^{-\frac{x}{2}(w-1)^2}\,dw = x^{x+1}e^{-x}\sqrt{\frac{2\pi}{x}}$
integrand is tiny for $w<0$

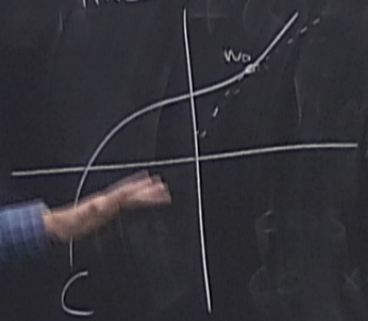

$\hat{t}$
w_0
C
① $\nabla u \cdot \hat{t} = 0$
② $\nabla v \cdot \hat{t} = 0$
③ $\nabla u \cdot \nabla v = 0$
$\hat{t}$ non zero
$\nabla u = 0$ at $w_0 \overset{CR}{\Longleftrightarrow} \nabla v = 0$ at w_0
Must have $f'(w_0)$
near w_0 we

$\nabla u = 0$ at $w_0 \overset{C\text{-}R}{\Longleftrightarrow} \nabla v = 0$ at w_0

$$f(w) = f(w_0) + \tfrac{1}{2}(w - w_0)^2 f''(w_0)$$

$$f''(w_0) = e^{i\delta_0}|f''(w_0)|$$

linearize contour



$$w = w_0 + e^{i\phi} t$$

want $f(w) = f(w_0) - \tfrac{1}{2} t^2 |f''(w_0)| + \cdots$

$$(w - w_0)^2 e^{i\delta_0} = -t^2$$

$$e^{2i\phi} t^2 e^{i\delta_0} = -t^2$$

$$w - w_0 = e^{-i\delta_0/2} e^{\pm i\pi/2} t$$

global considerations

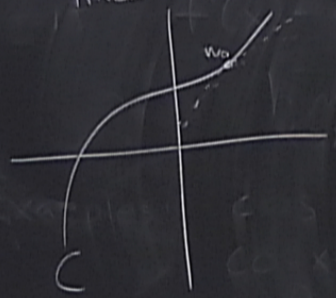
t non zero

$\nabla u = 0$ at $w_0 \overset{GR}{\Longleftrightarrow} \nabla v = 0$ at w_0

$$f(w) = f(w_0) + \frac{1}{2}(w - w_0)^2 f''(w_0)$$

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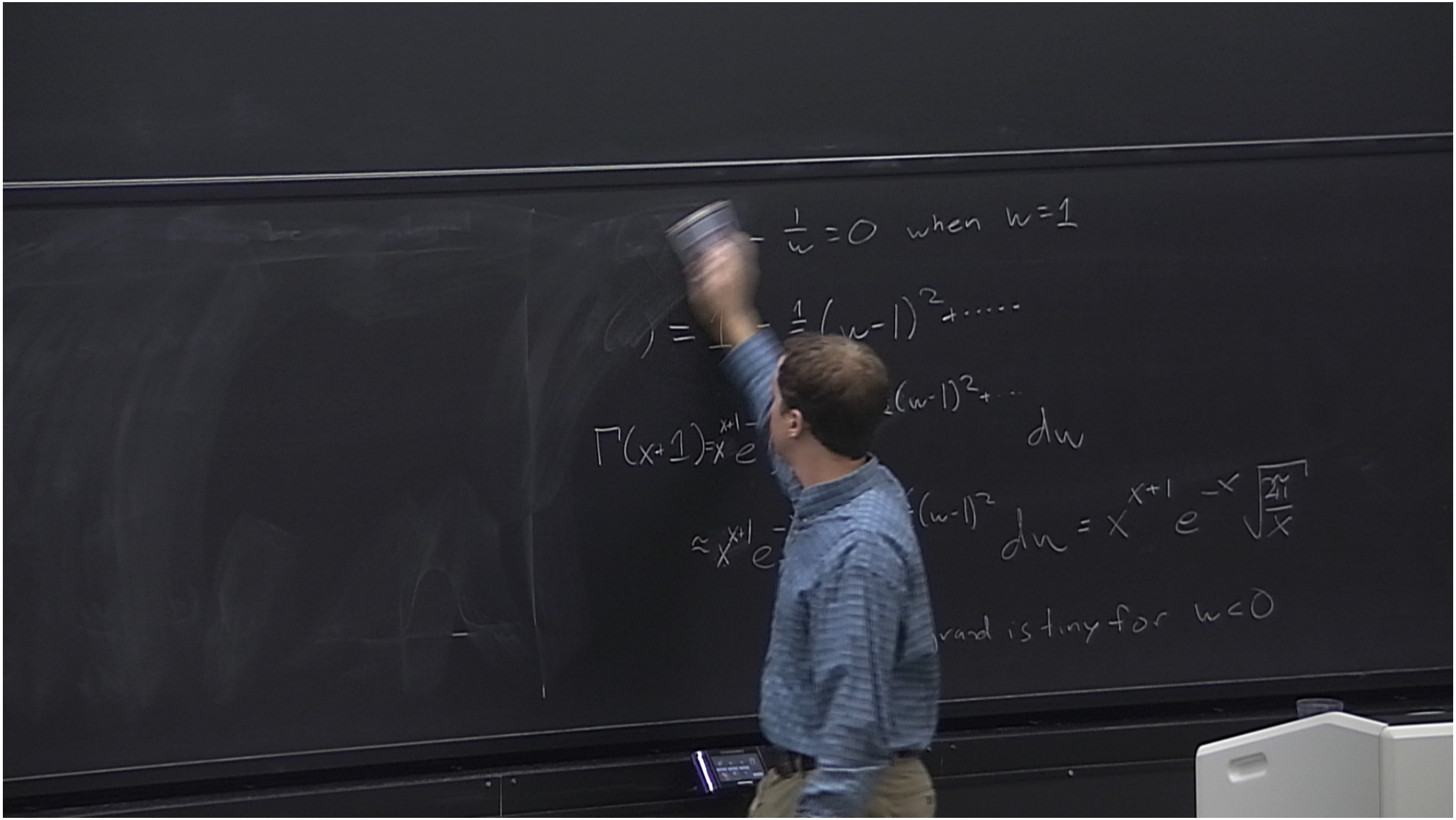
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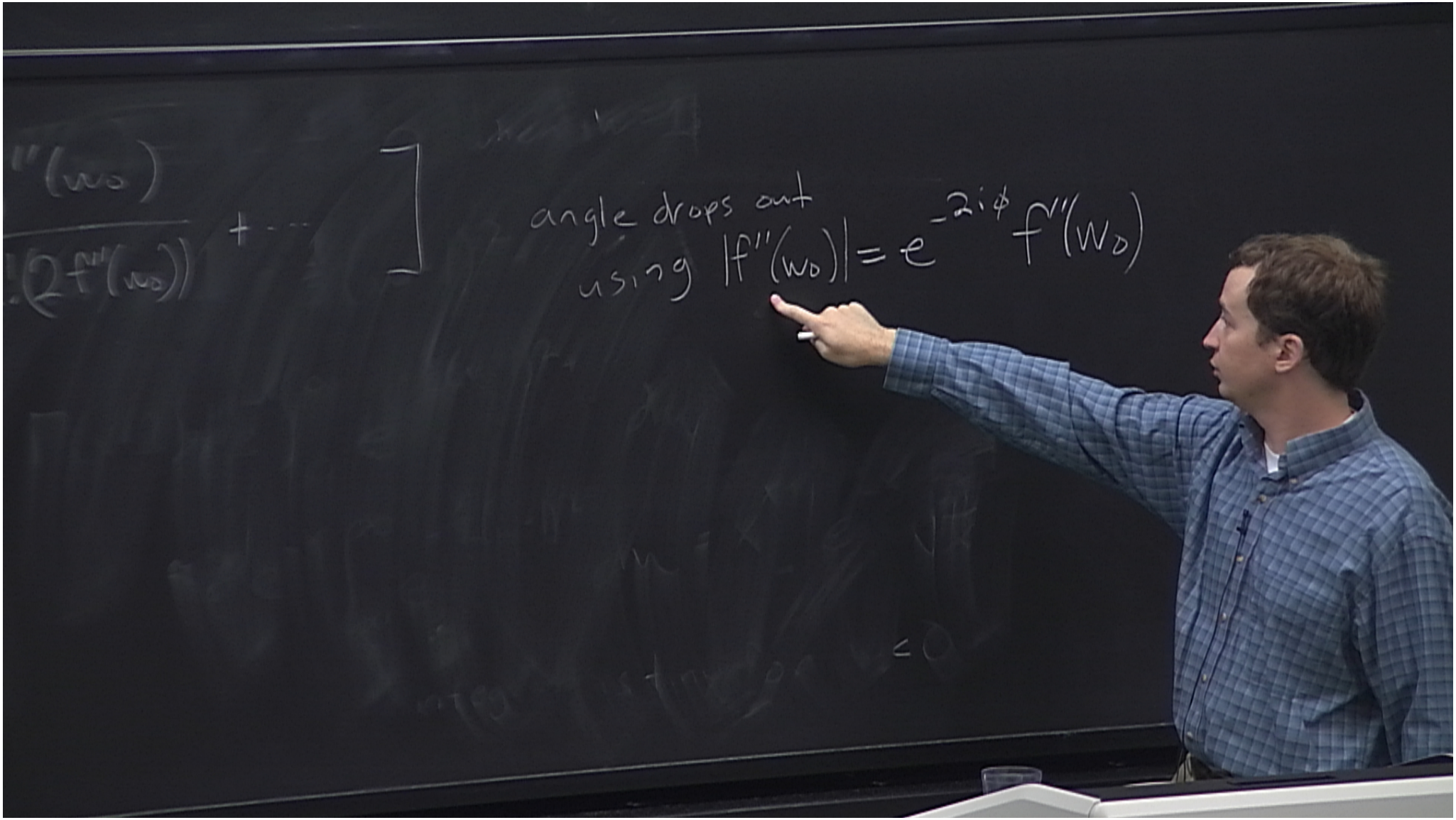
global considerations


$-\frac{1}{w}=0$ when $w=1$
$=1-\frac{1}{2}(w-1)^2+\cdots$
$\Gamma(x+1)=x^{x+1}e^{-x}$
dw
$\approx x^{x+1}e^{-x}$
$(w-1)^2$
$dw=x^{x+1}e^{-x}\sqrt{\frac{2\pi}{x}}$
grand is tiny for $w<0$

do Gaussian integrals

$$I(z) \sim \sqrt{\frac{2\pi}{z}} \frac{e^{z f(w_0)}}{\sqrt{-f''(w_0)}} \left[g(w_0) - \frac{1}{z} \frac{g''(w_0)}{1!(2 f''(w_0))} + \cdots \right]$$

+ ... breaks down at some point due to f'''

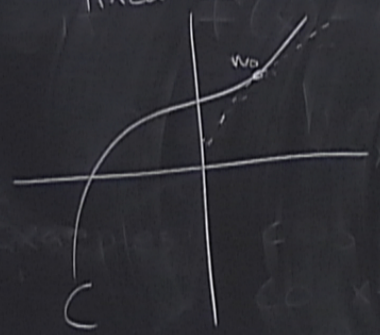

"(w0)
(2f"(w0))
+ ···
]
angle drops out
using |f"(w0)| = e^{-2iφ} f"(w0)

$\nabla\mu = 0$ at $w_0 \overset{GR}{\Longleftrightarrow} \nabla v = 0$ at w_0

$$f(w) = f(w_0) + \tfrac{1}{2}(w - w_0)^2 f''(w_0)$$

$$e^{i\delta_0}|f''(w_0)|$$

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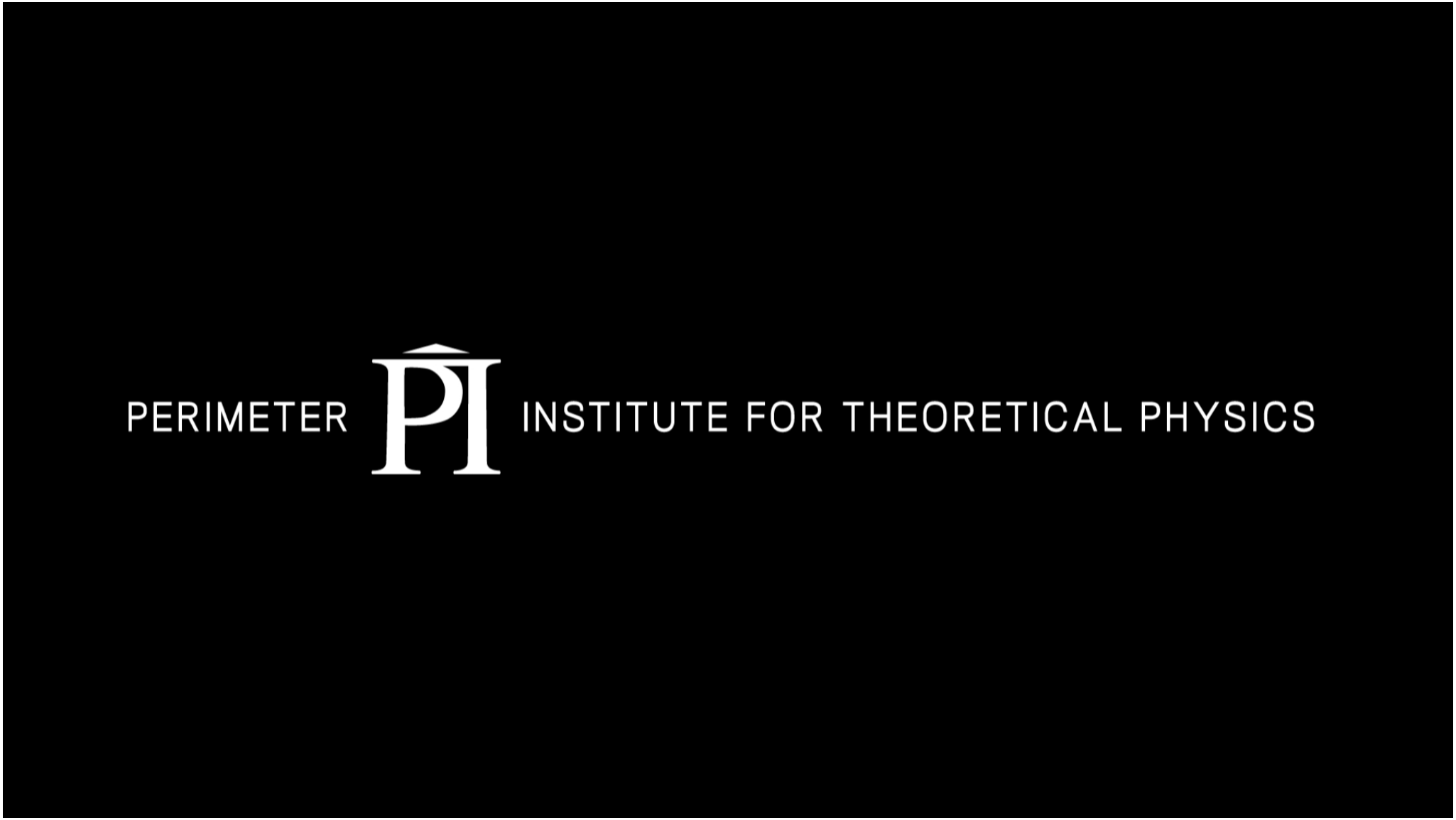
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global consider

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