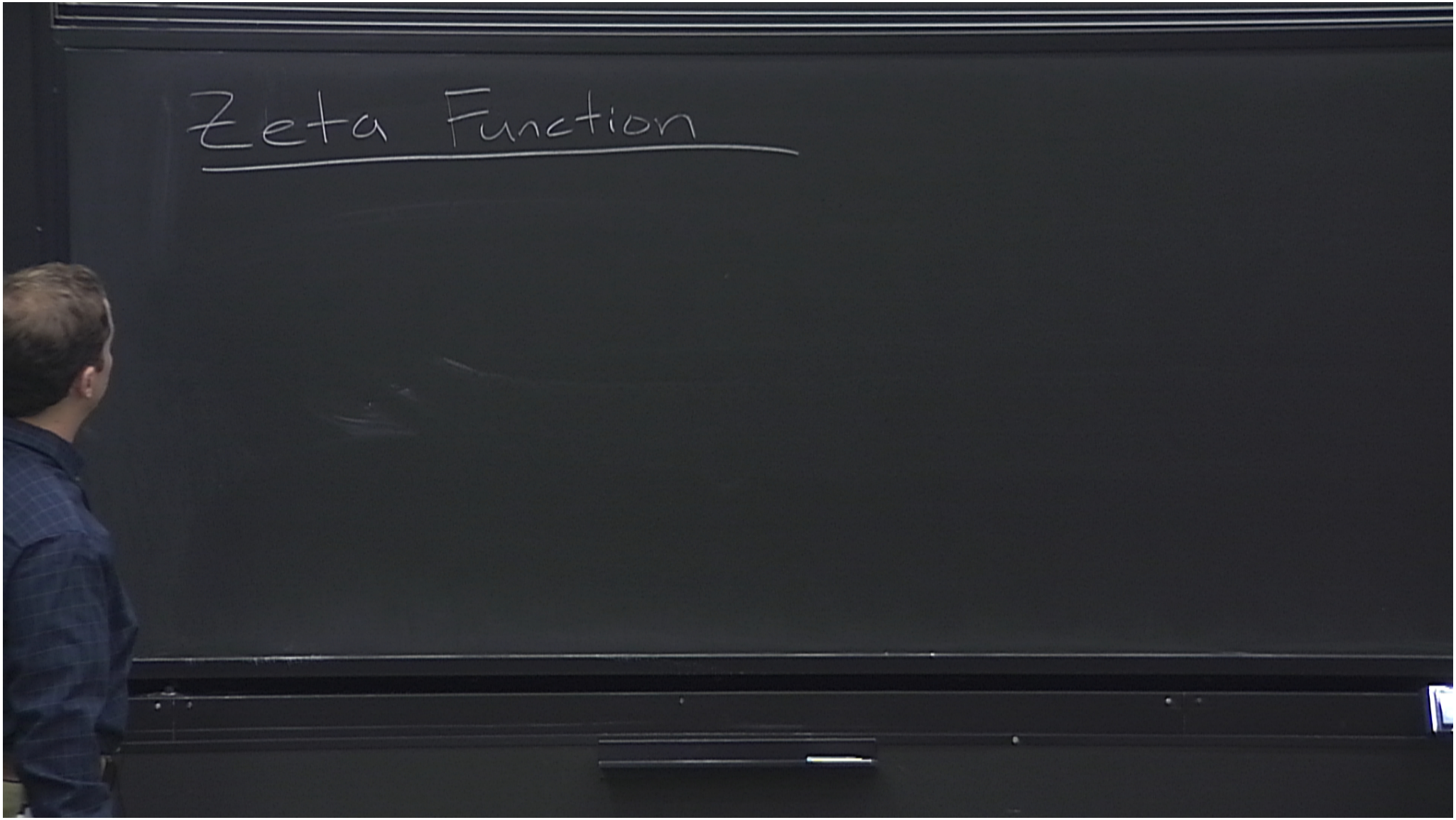
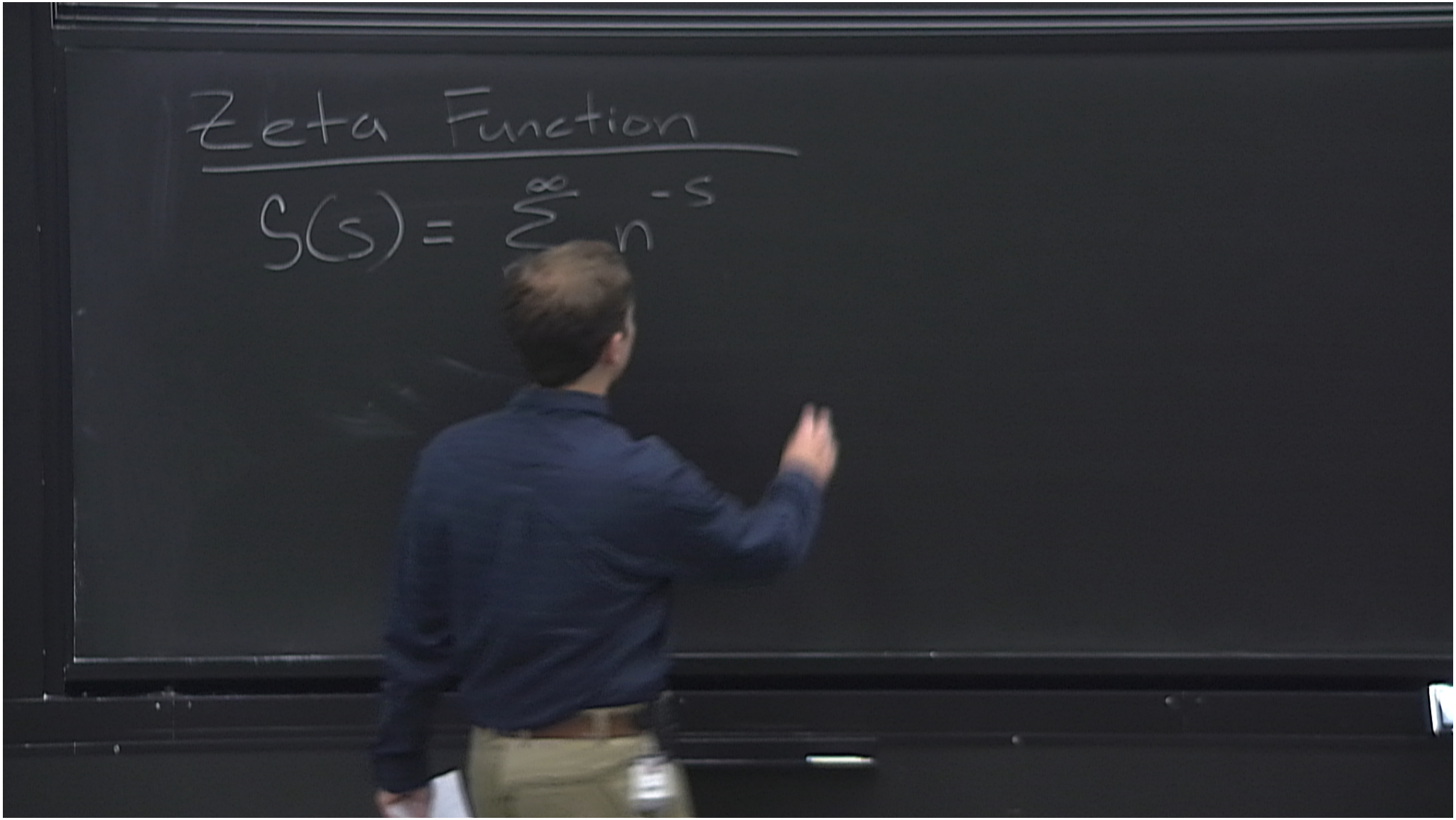
Title: PSI 2015/2016 Special Functions - Dan Wohns -3

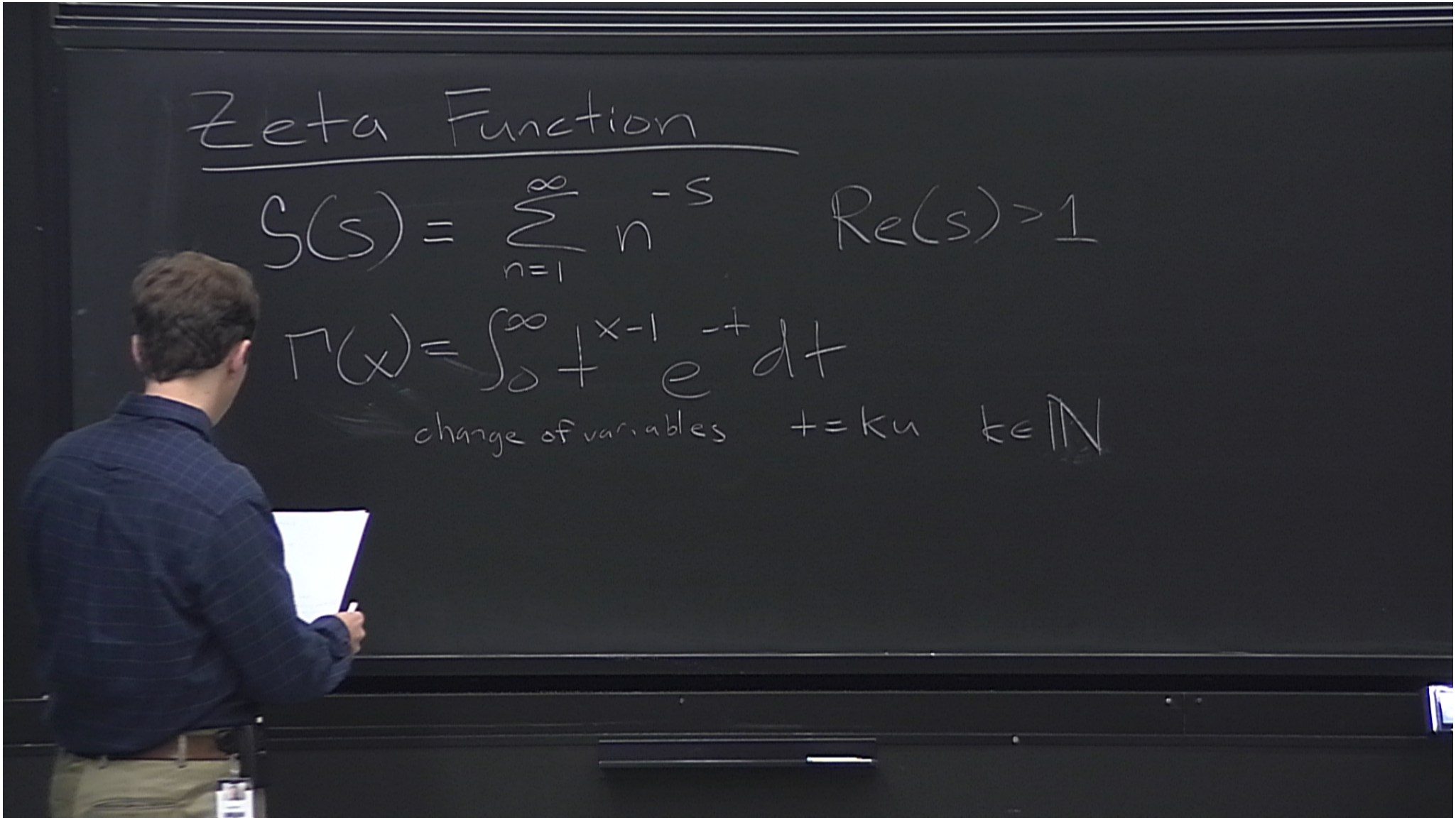
Date: Sep 02, 2015 10:30 AM

URL: http://pirsa.org/15090015

Abstract:


Zeta Function


Zeta Function
$\zeta(s) = \sum_{n}^{\infty} n^{-s}$


Zeta Function
$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $\mathrm{Re}(s) > 1$
$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$
change of variables $t = ku$ $k \in \mathbb{N}$

Zeta Function

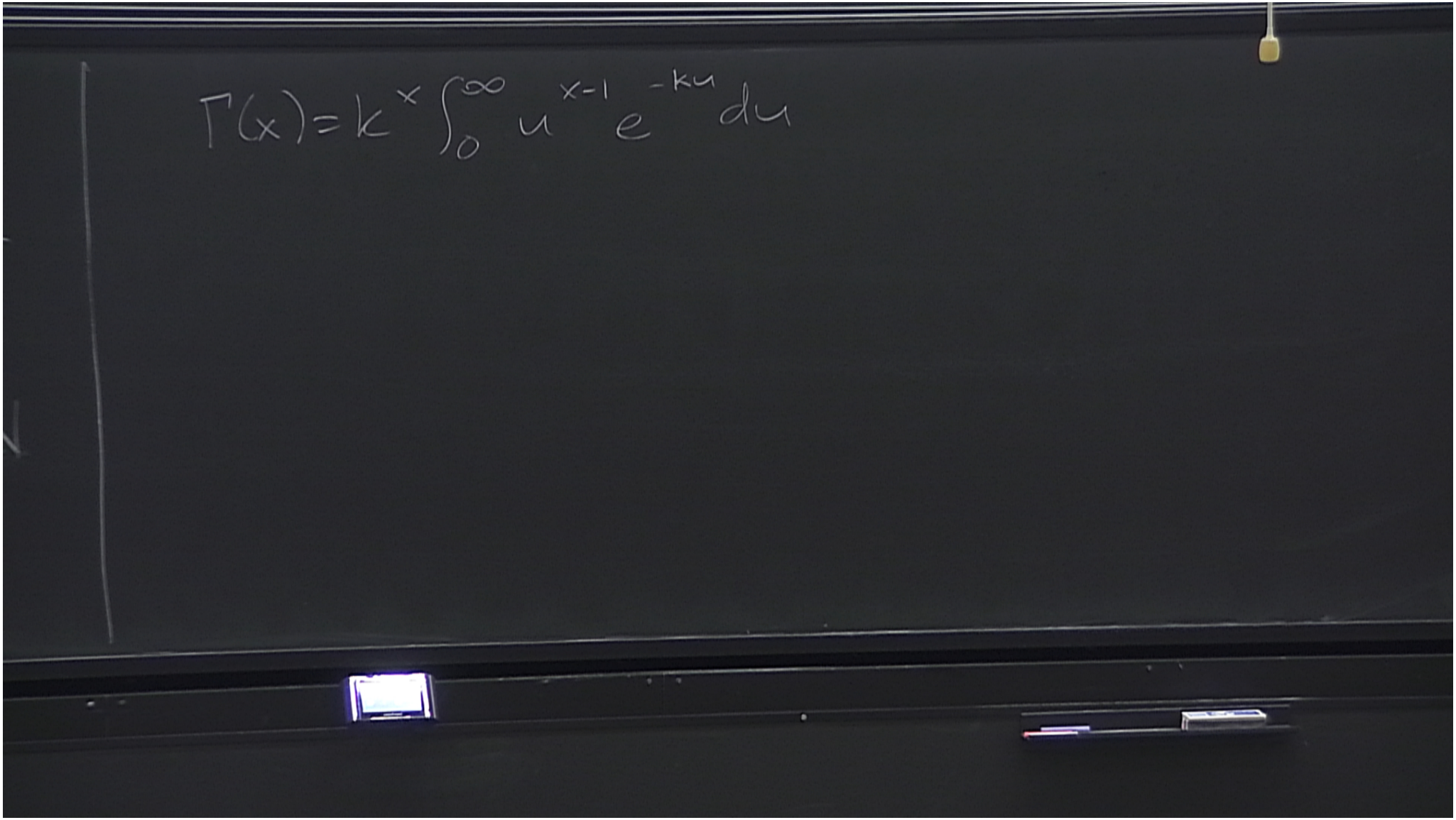
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \mathrm{Re}(s) > 1$$

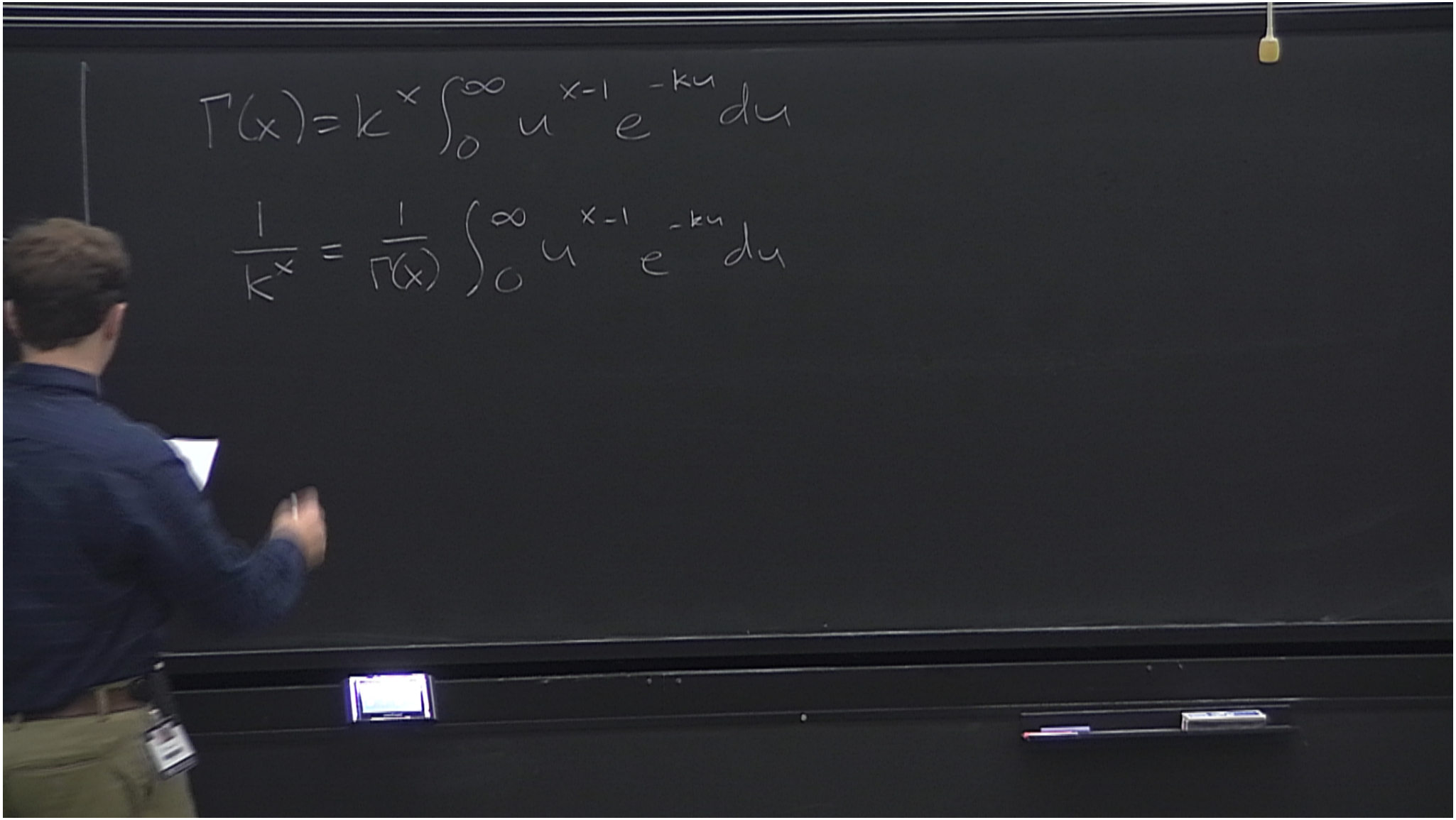
$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$$

change of variables $t = ku$ $k \in \mathbb{N}$

$$\Gamma(x) = \int_0^{\infty} (ku)^{x-1} e^{-ku}\, k\, du$$

$\Gamma(x) =$


$\Gamma(x) = k^x \int_0^\infty u^{x-1} e^{-ku} du$

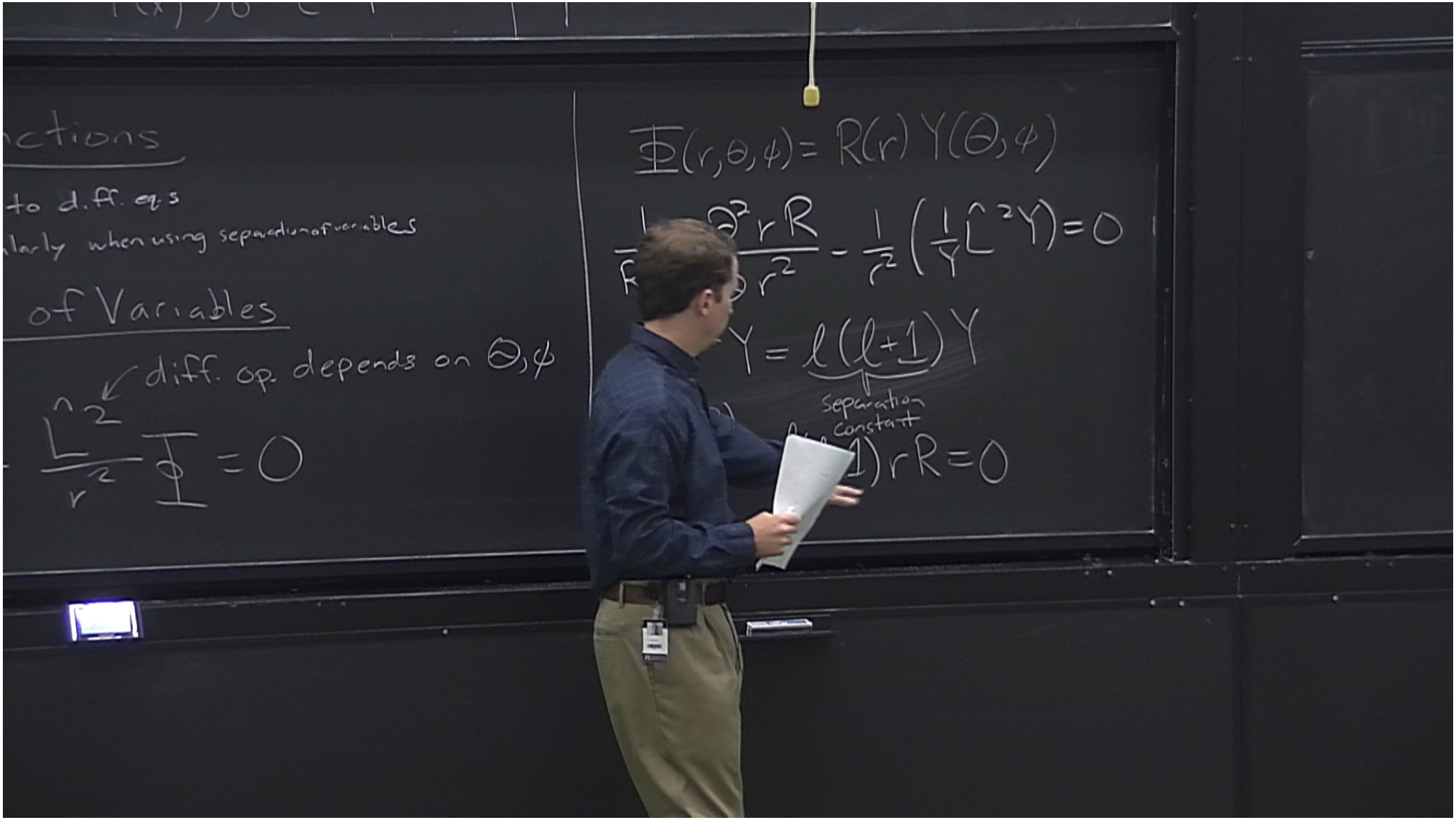

$\Gamma(x) = k^x \int_0^\infty u^{x-1} e^{-ku} du$
$\frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} e^{-ku} du$

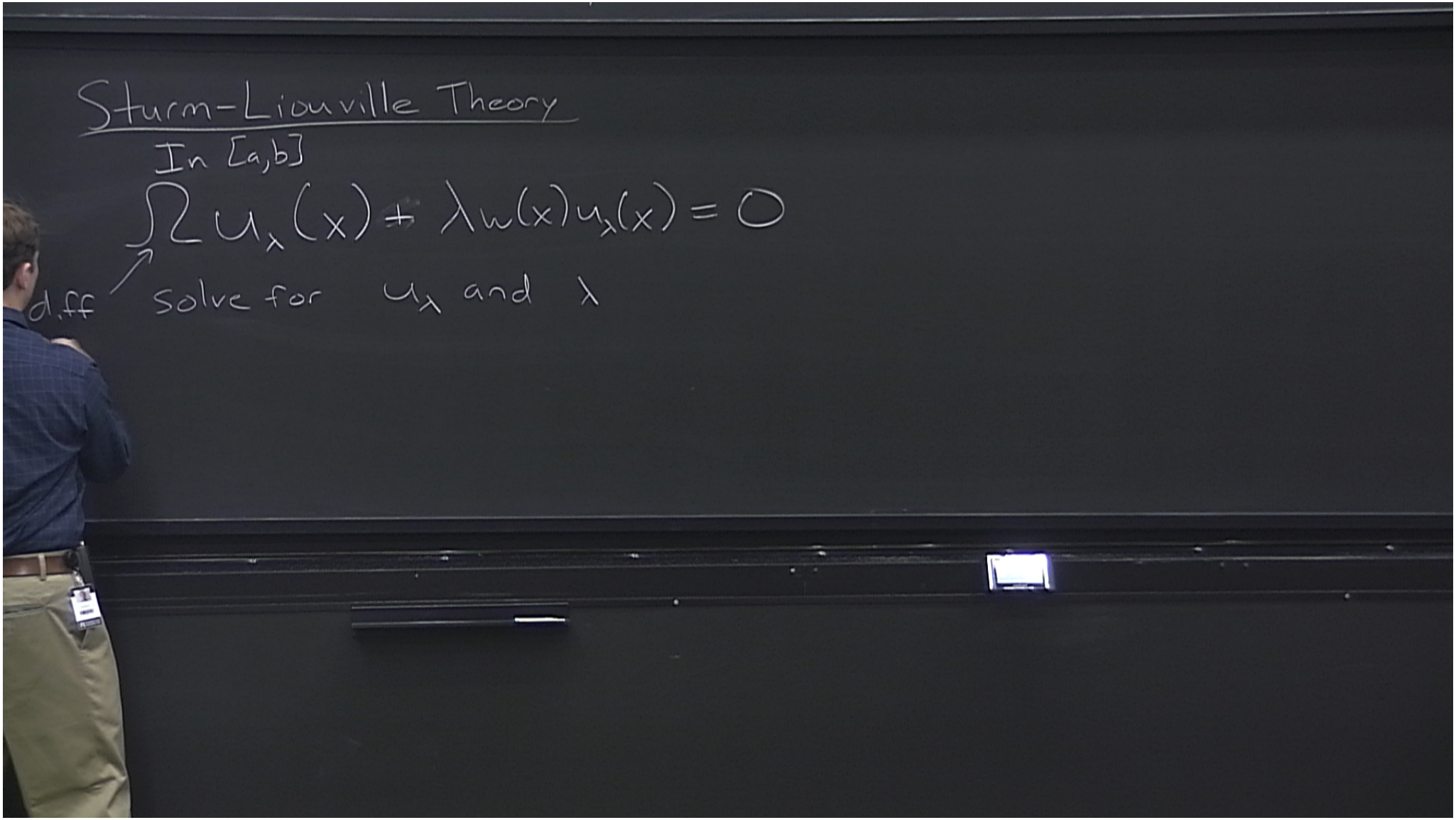
$$\Gamma(x) = k^x \int_0^\infty u^{x-1} e^{-ku} du$$

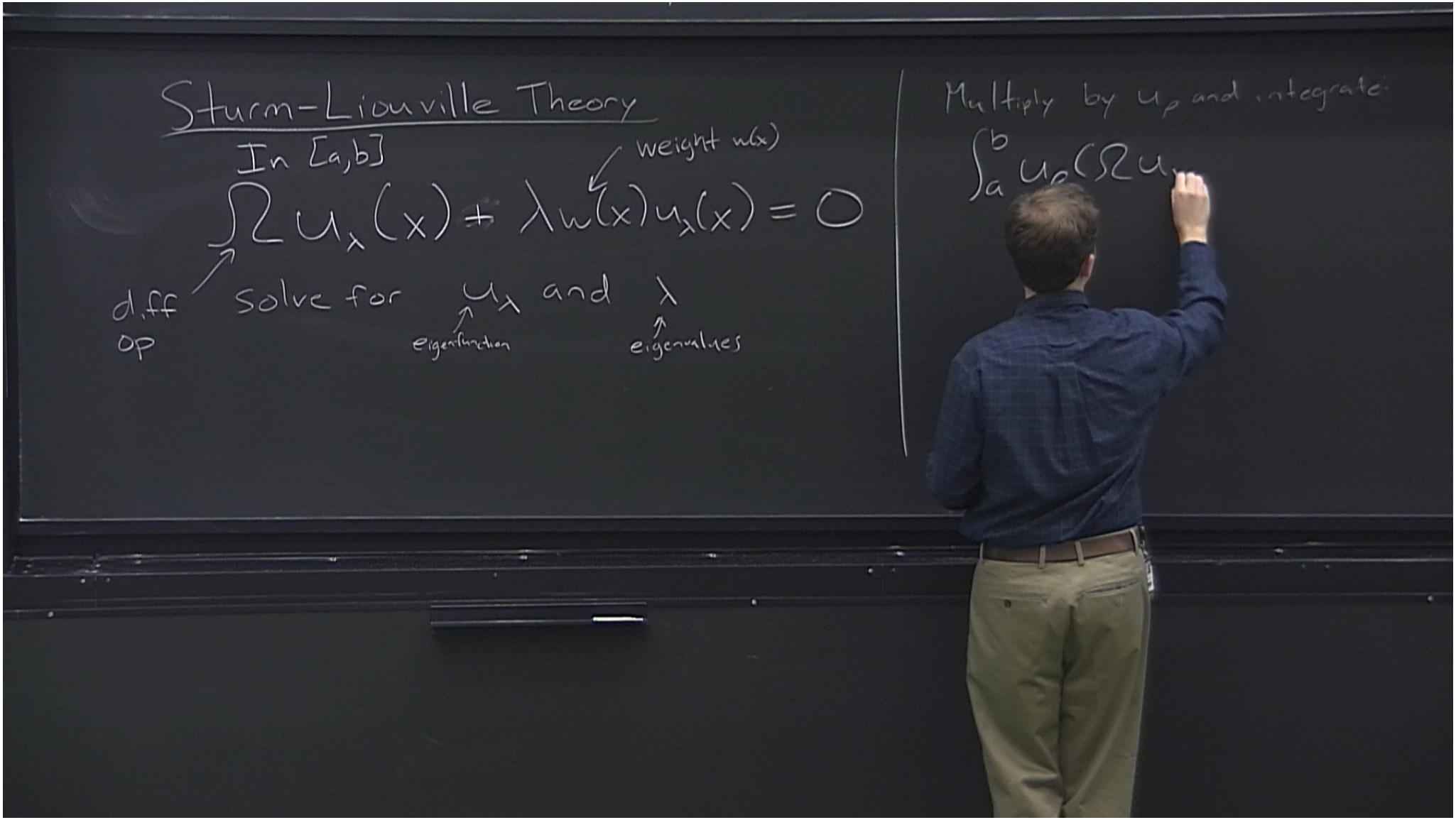
$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \sum_{k=1}^{\infty} \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} e^{-ku} du$$

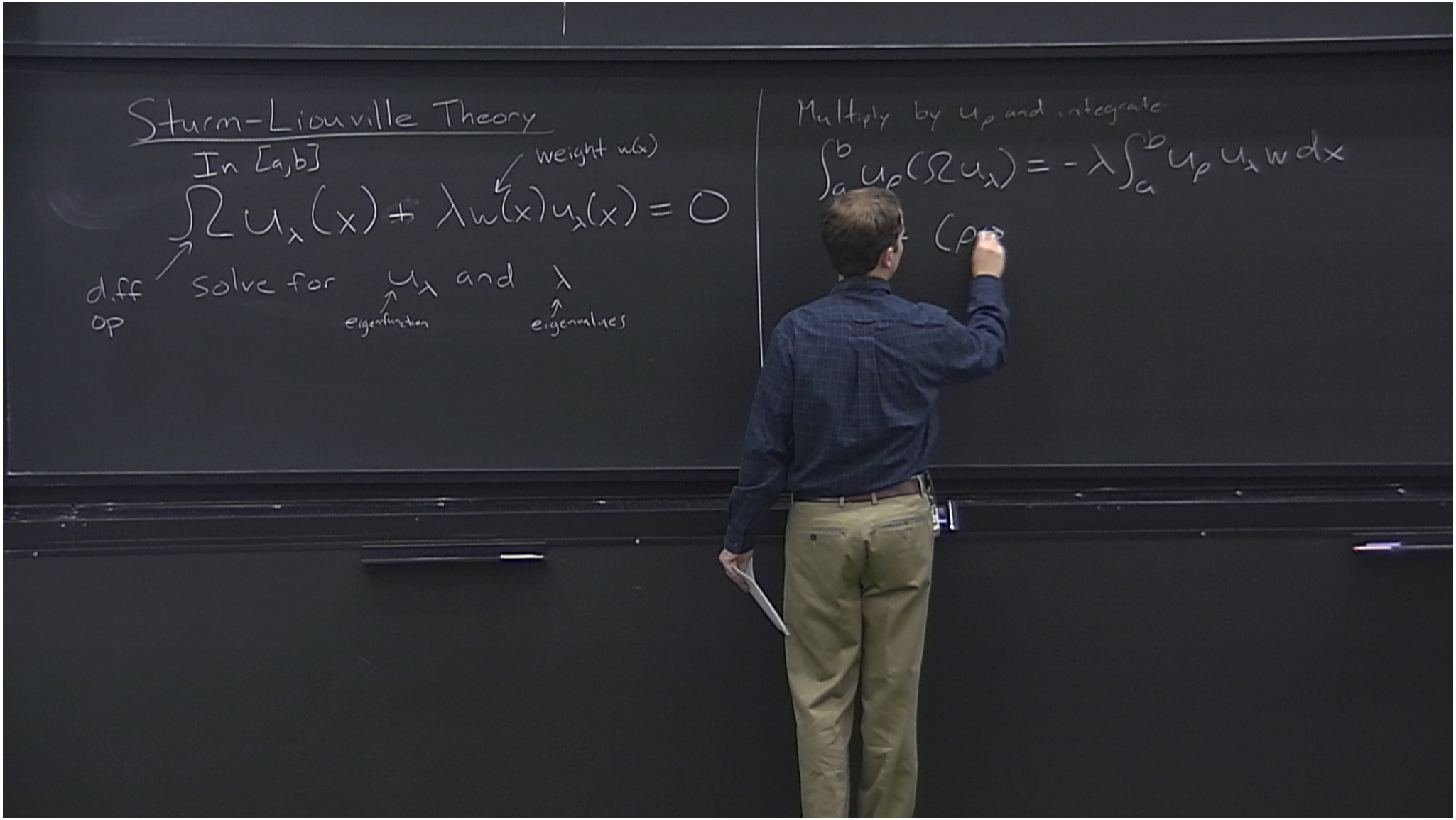
$$S(x) = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} \frac{e^{-u}}{1-e^{-u}} du$$

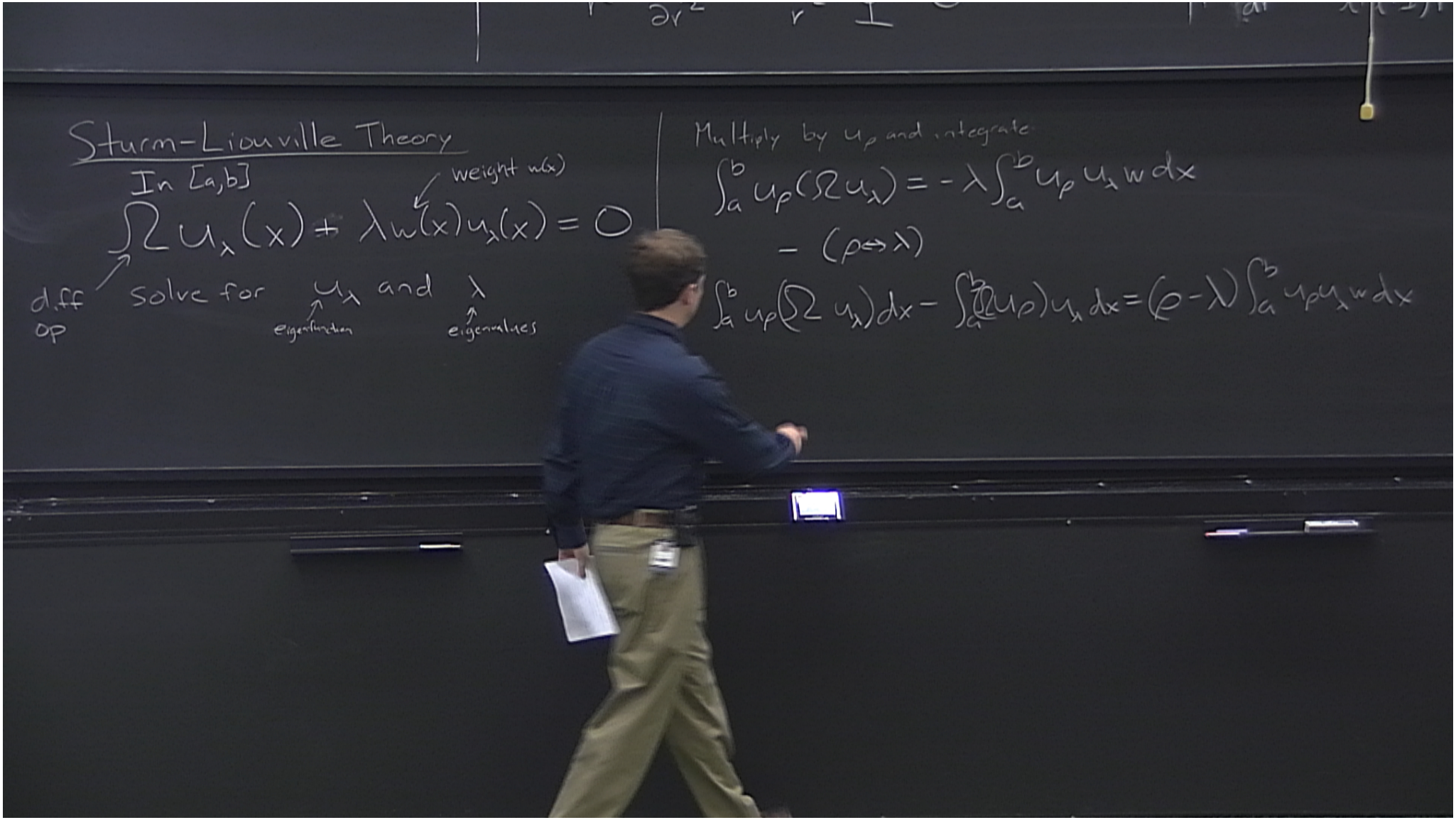
$$\boxed{S(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du}$$

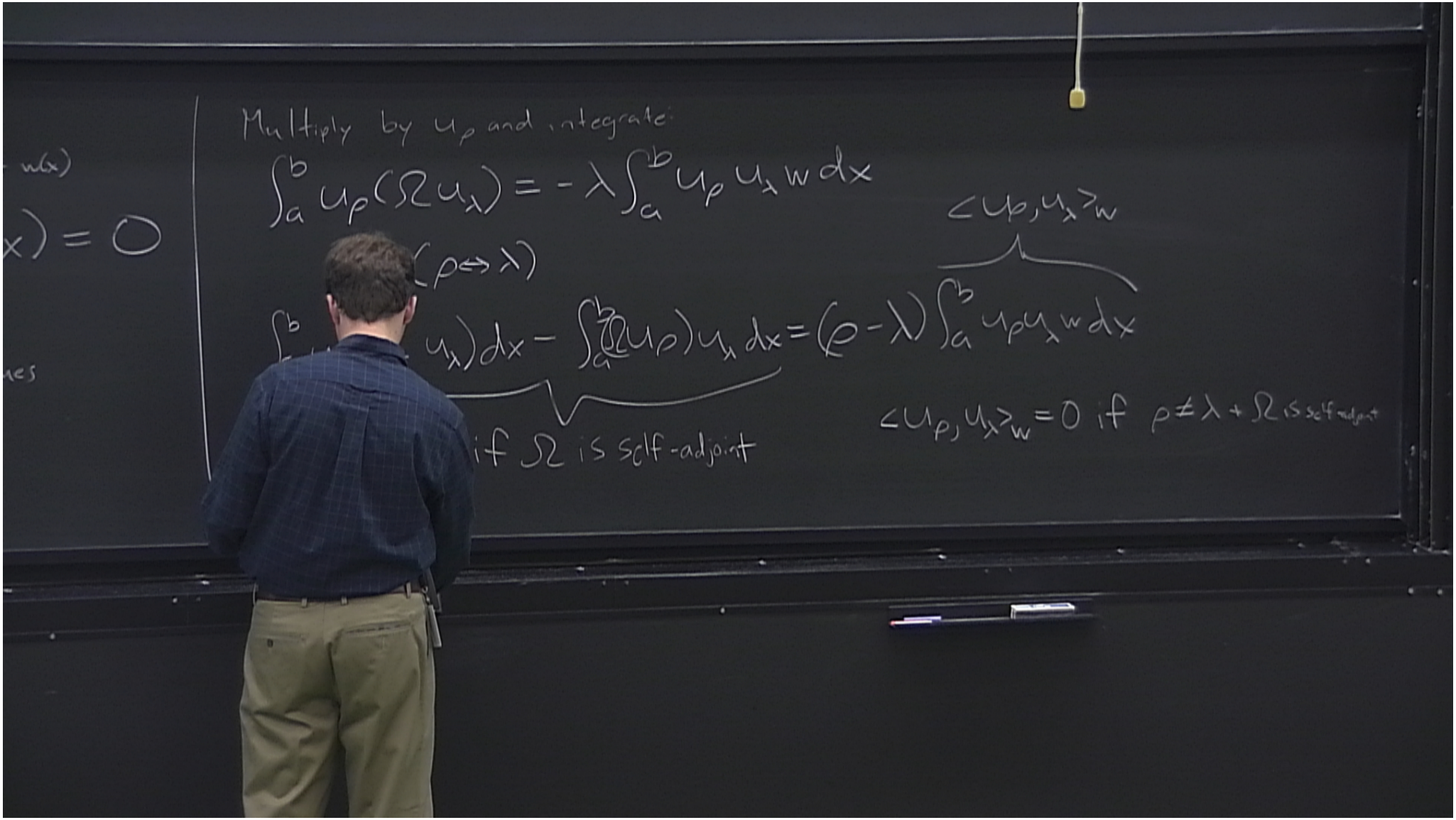

ctions
to d.ff. eq.s
larly when using separation of variables
of Variables
diff. op. depends on Θ,φ
$\frac{\hat{L}^2}{r^2}\Phi = 0$
$\Phi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$
$-\frac{1}{r^2}\left(\frac{1}{Y}\hat{L}^2 Y\right) = 0$
$\hat{L}^2 Y = \ell(\ell+1)\,Y$
separation constant
$\ell(\ell+1)\,rR = 0$

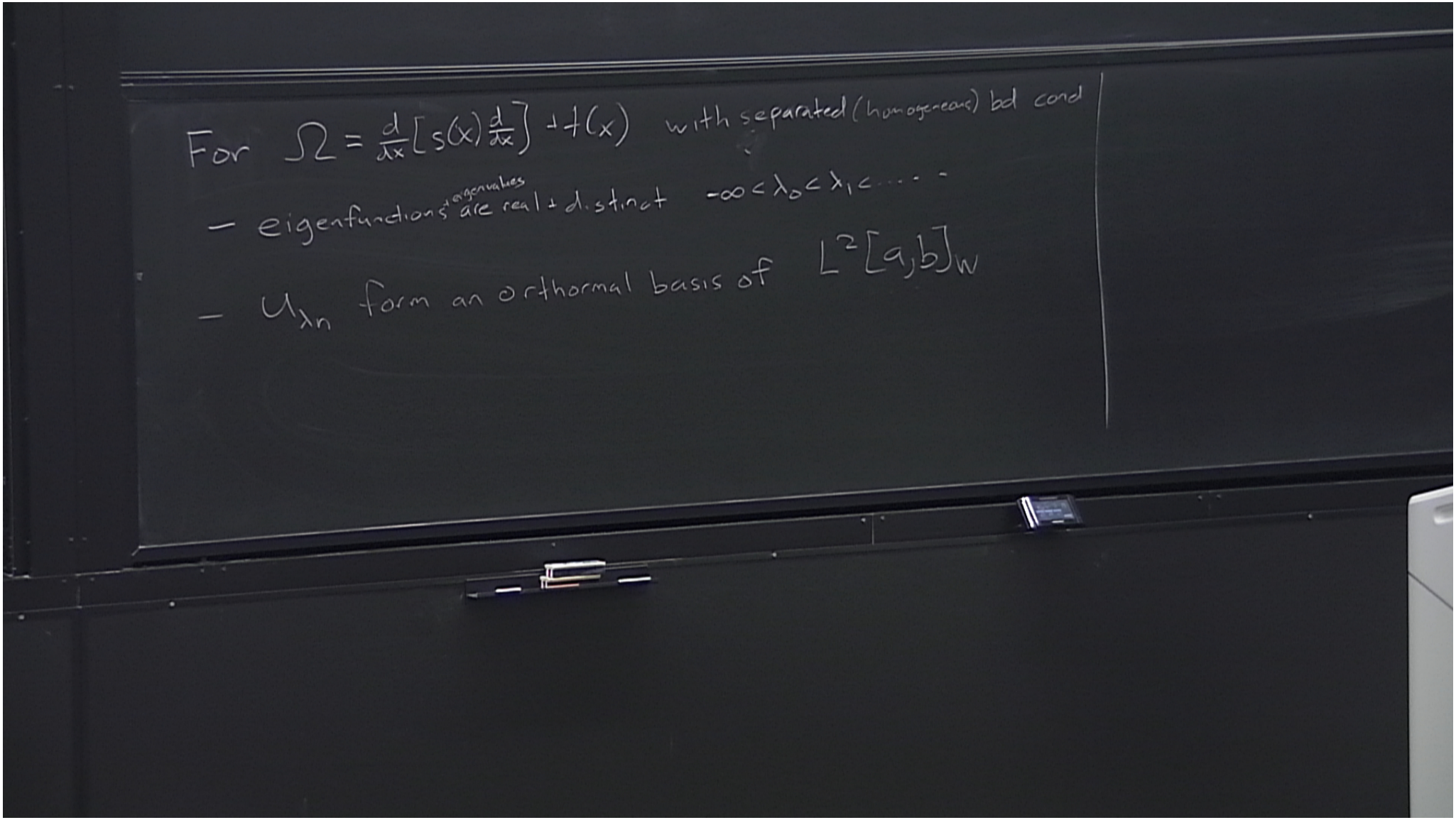

Sturm-Liouville Theory
In [a,b]
$\mathcal{L} u_\lambda(x) + \lambda w(x) u_\lambda(x) = 0$
diff
solve for u_λ and λ

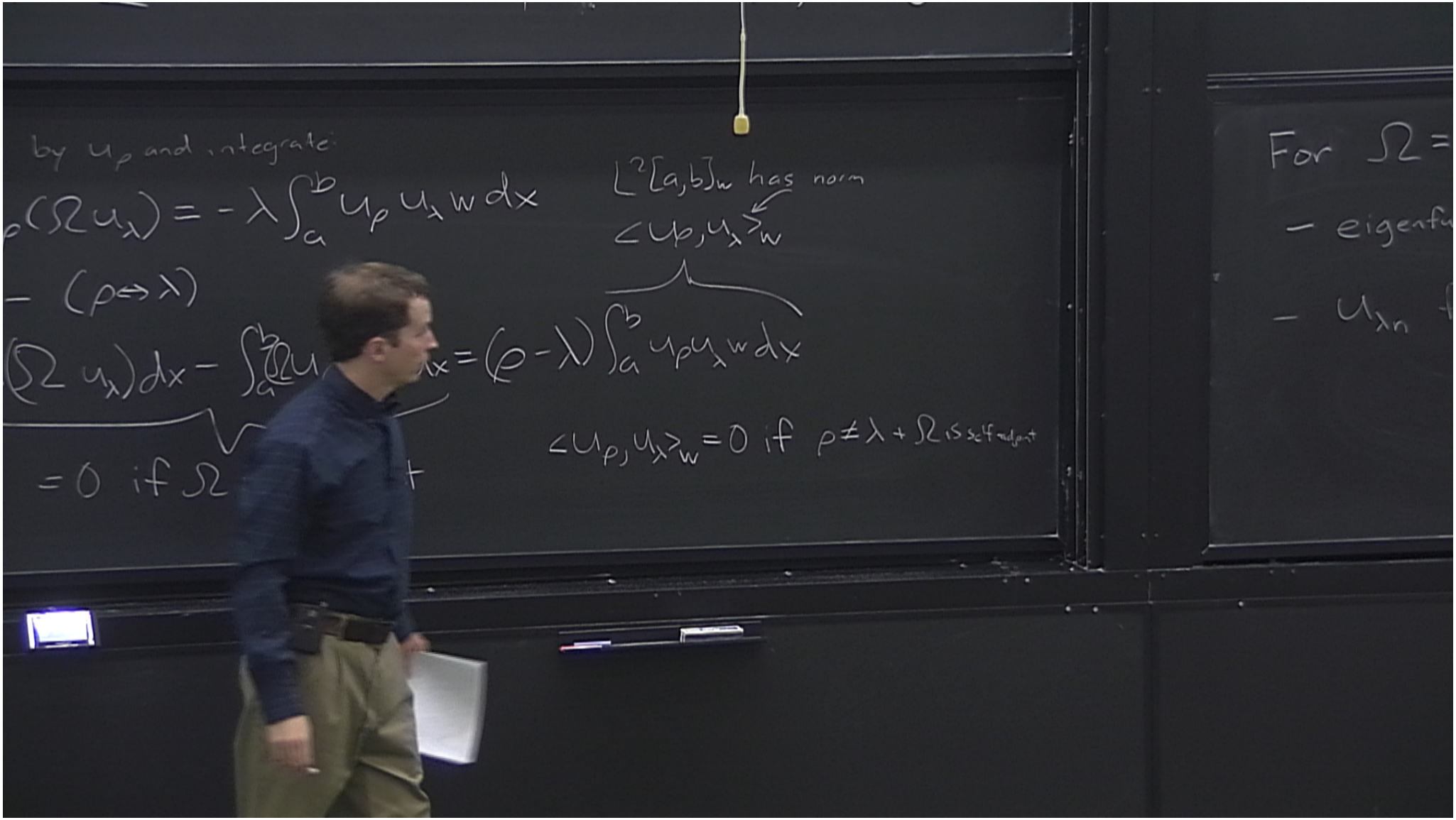

Sturm-Liouville Theory
In [a,b]
weight w(x)
$\hat{\Omega} u_\lambda(x) + \lambda w(x) u_\lambda(x) = 0$
diff op
solve for u_λ and λ
eigenfunction
eigenvalues
Multiply by u_ρ and integrate:
$\int_a^b u_\rho (\Omega u_\lambda$

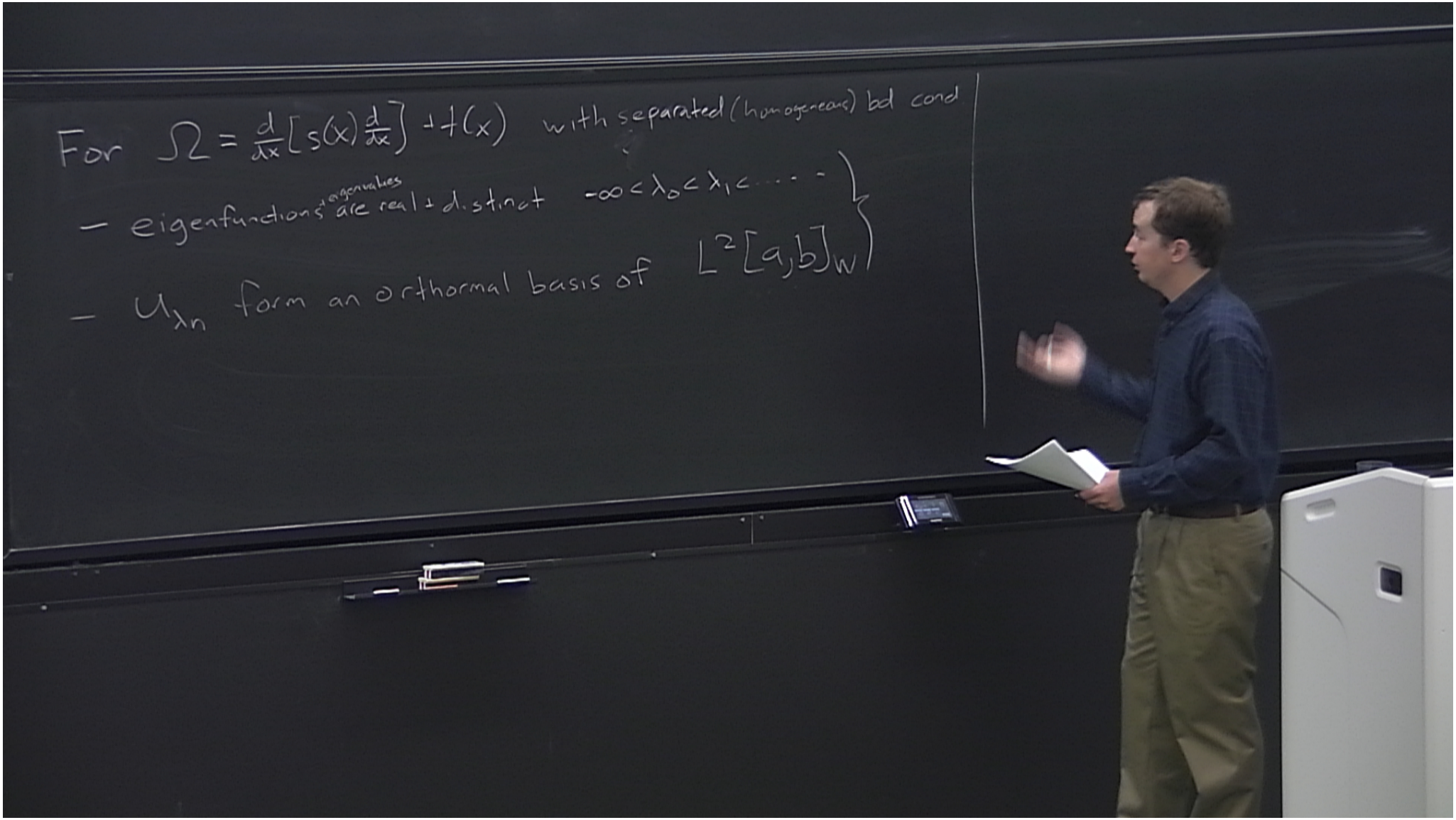

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$\int_a^b u_\rho(\Omega u_\lambda) = -\lambda \int_a^b u_\rho u_\lambda w\, dx$

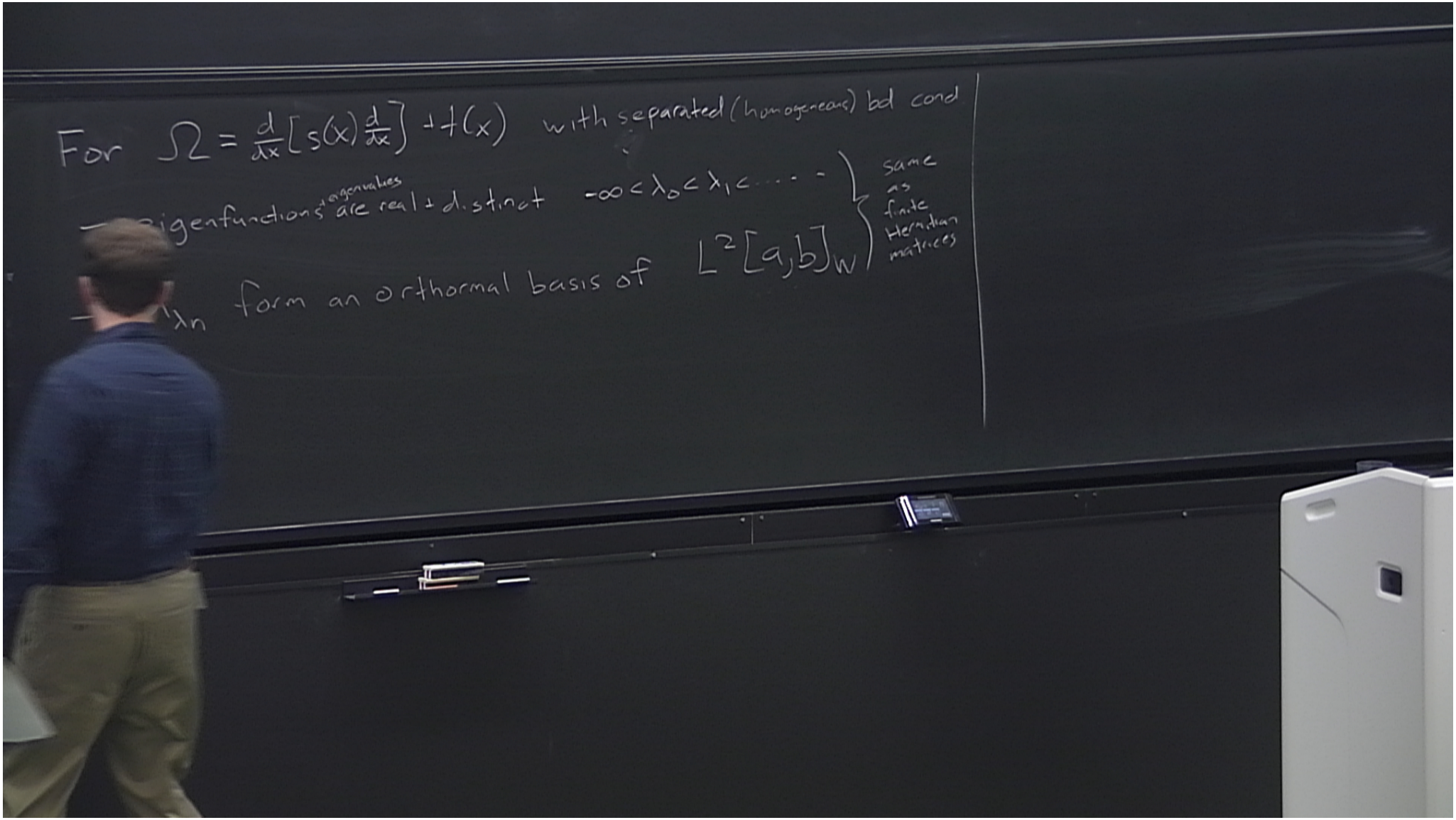

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Multiply by u_ρ and integrate
$\int_a^b u_\rho(\hat{\Omega} u_\lambda) = -\lambda \int_a^b u_\rho u_\lambda w\, dx$
$-(\rho \leftrightarrow \lambda)$
$\int_a^b u_\rho(\hat{\Omega} u_\lambda)dx - \int_a^b (\hat{\Omega} u_\rho) u_\lambda dx = (\rho - \lambda)\int_a^b u_\rho u_\lambda w\, dx$

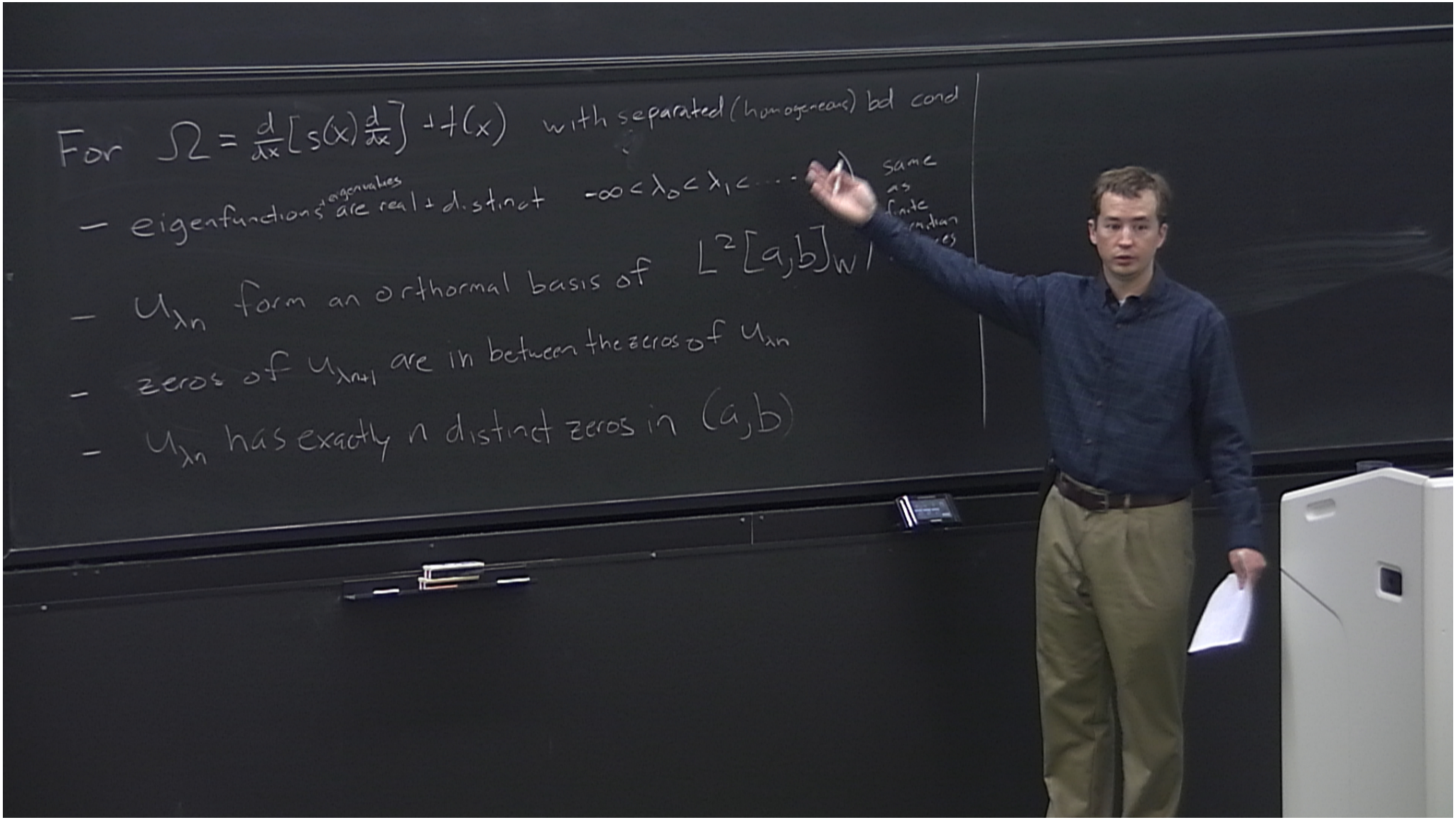

Multiply by u_ρ and integrate:
$\int_a^b u_\rho(\Omega u_\lambda) = -\lambda \int_a^b u_\rho u_\lambda w\,dx$
$(\rho \leftrightarrow \lambda)$
$\int_a^b (\Omega u_\rho) u_\lambda\,dx - \int_a^b (\Omega u_\rho) u_\lambda\,dx = (\rho - \lambda) \int_a^b u_\rho u_\lambda w\,dx$
$\langle u_\rho, u_\lambda \rangle_w$
$\langle u_\rho, u_\lambda \rangle_w = 0$ if $\rho \neq \lambda$ + Ω is self-adjoint
if Ω is self-adjoint

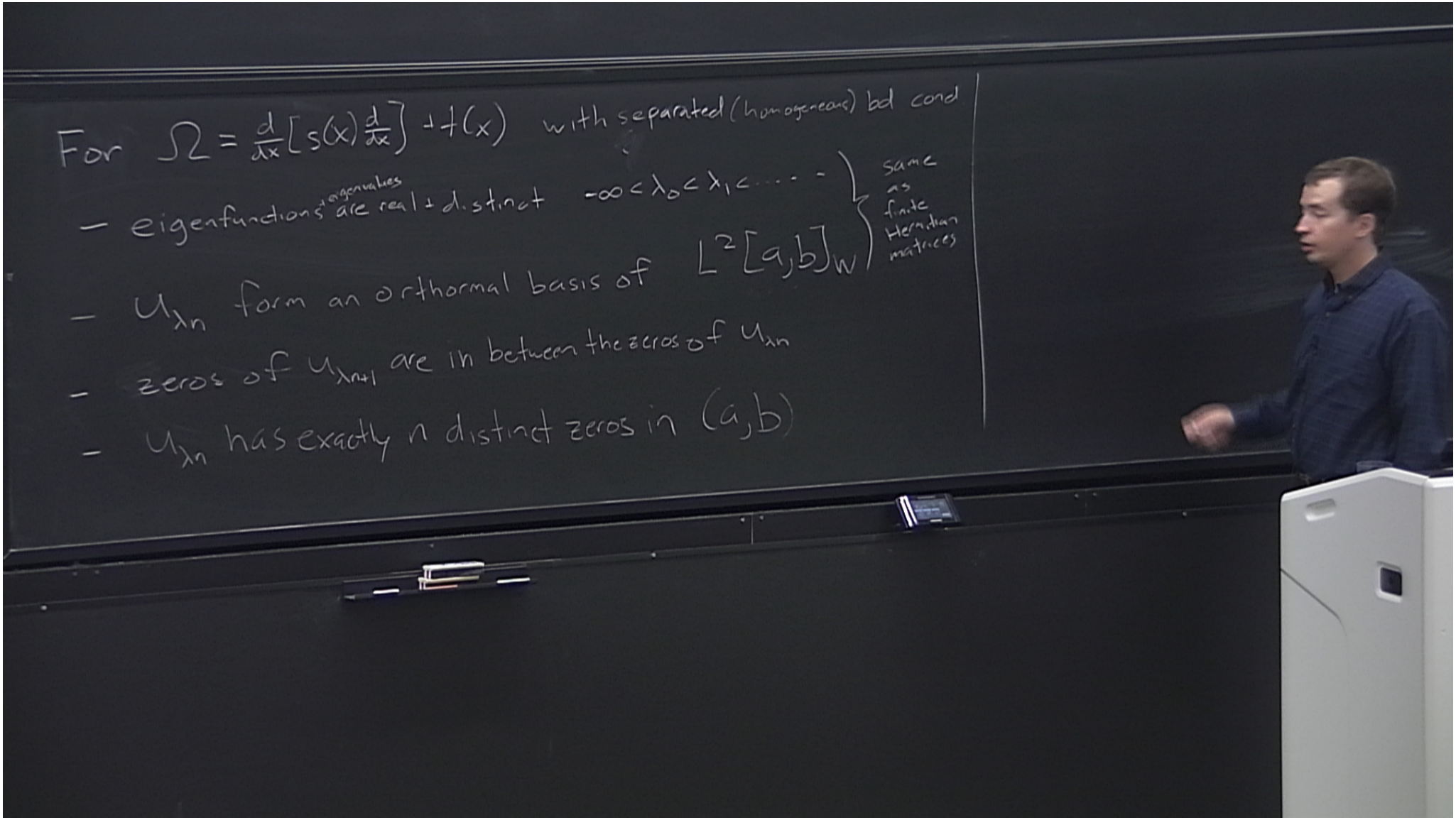

For $\Omega = \frac{d}{dx}\left[s(x)\frac{d}{dx}\right] + t(x)$ with separated (homogeneous) bd cond
- eigenfunctions & eigenvalues are real & distinct $-\infty < \lambda_0 < \lambda_1 < \cdots$
- u_{λ_n} form an orthormal basis of $L^2[a,b]_w$

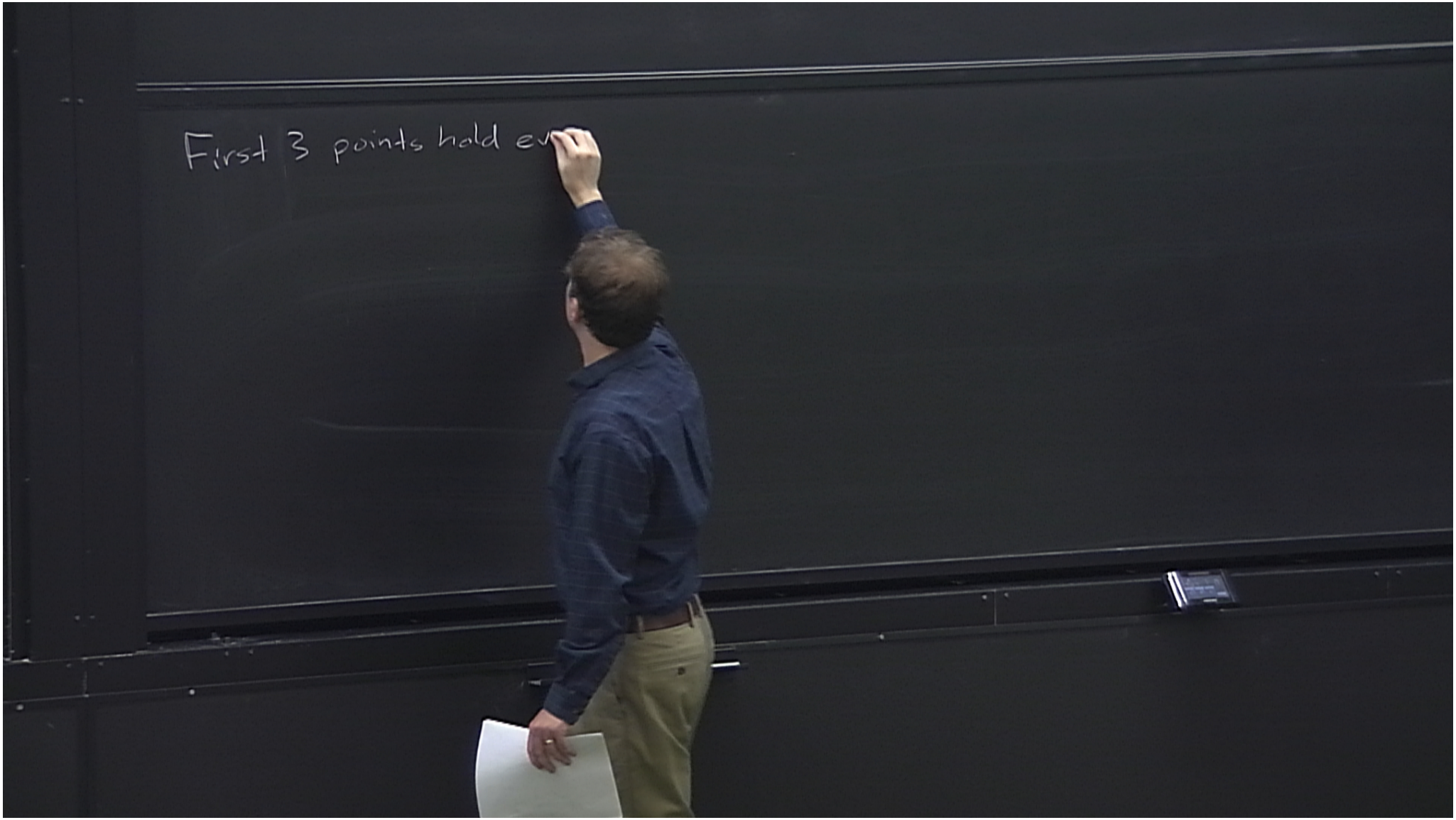

by u_ρ and integrate
$L^2[a,b]_w$ has norm
$\langle u_\rho, u_\lambda \rangle_w$
$\langle u_\rho, u_\lambda \rangle_w = 0$ if $\rho \neq \lambda$ + Ω is self-adjoint
For Ω =
– eigenfu
– $u_{\lambda n}$

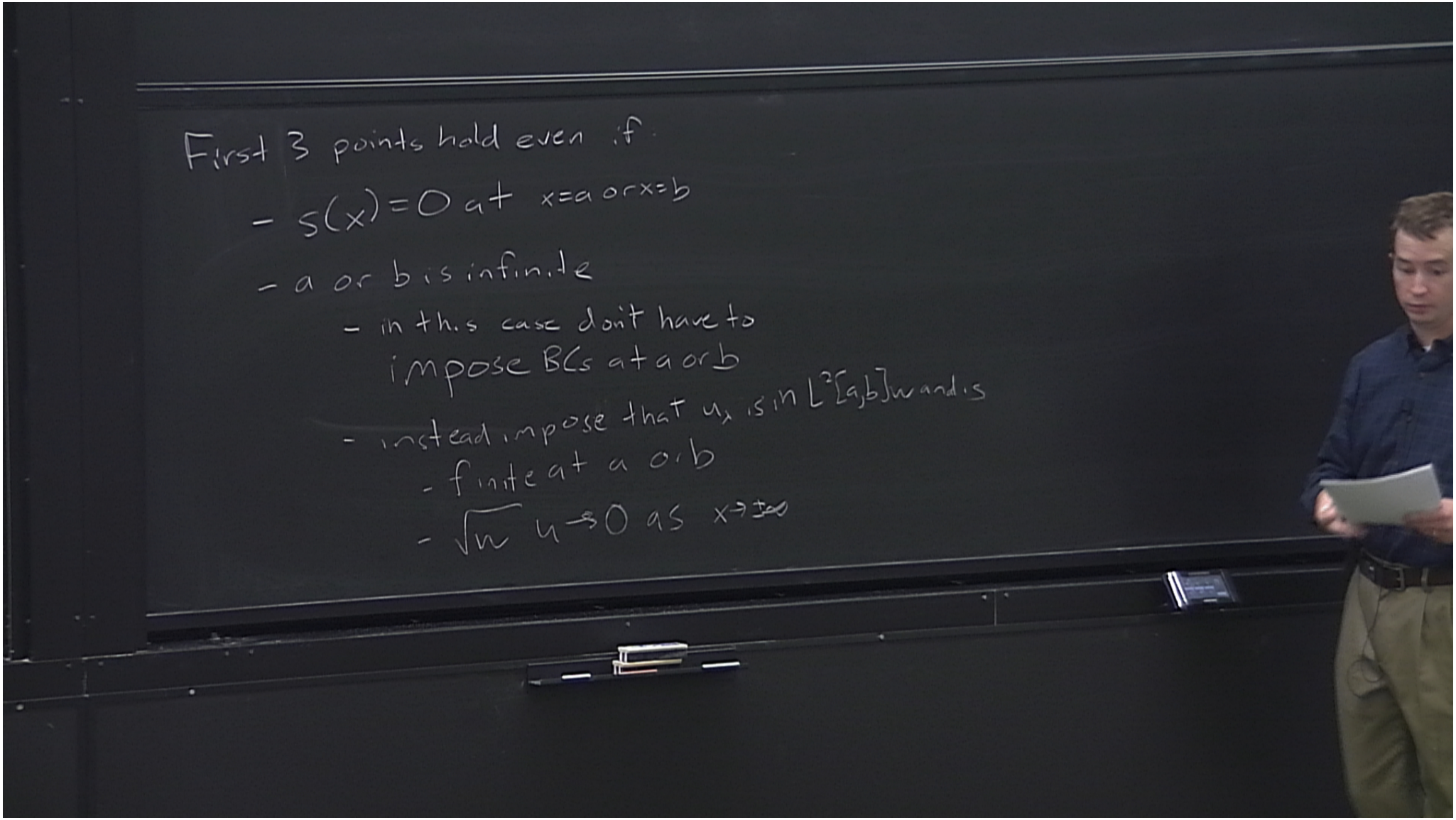

For $\Omega = \frac{d}{dx}\left[s(x)\frac{d}{dx}\right] + t(x)$ with separated (homogeneous) bd cond
- eigenfunctions' eigenvalues are real + distinct $-\infty < \lambda_0 < \lambda_1 < \cdots$
- $u_{\lambda n}$ form an orthonormal basis of $L^2[a,b]_w$


For $\Omega = \frac{d}{dx}\left[s(x)\frac{d}{dx}\right] + t(x)$ with separated (homogeneous) bd cond
eigenfunctions / eigenvalues are real + distinct $-\infty < \lambda_0 < \lambda_1 < \cdots$
λ_n form an orthonormal basis of $L^2[a,b]_w$
same as finite Hermitian matrices

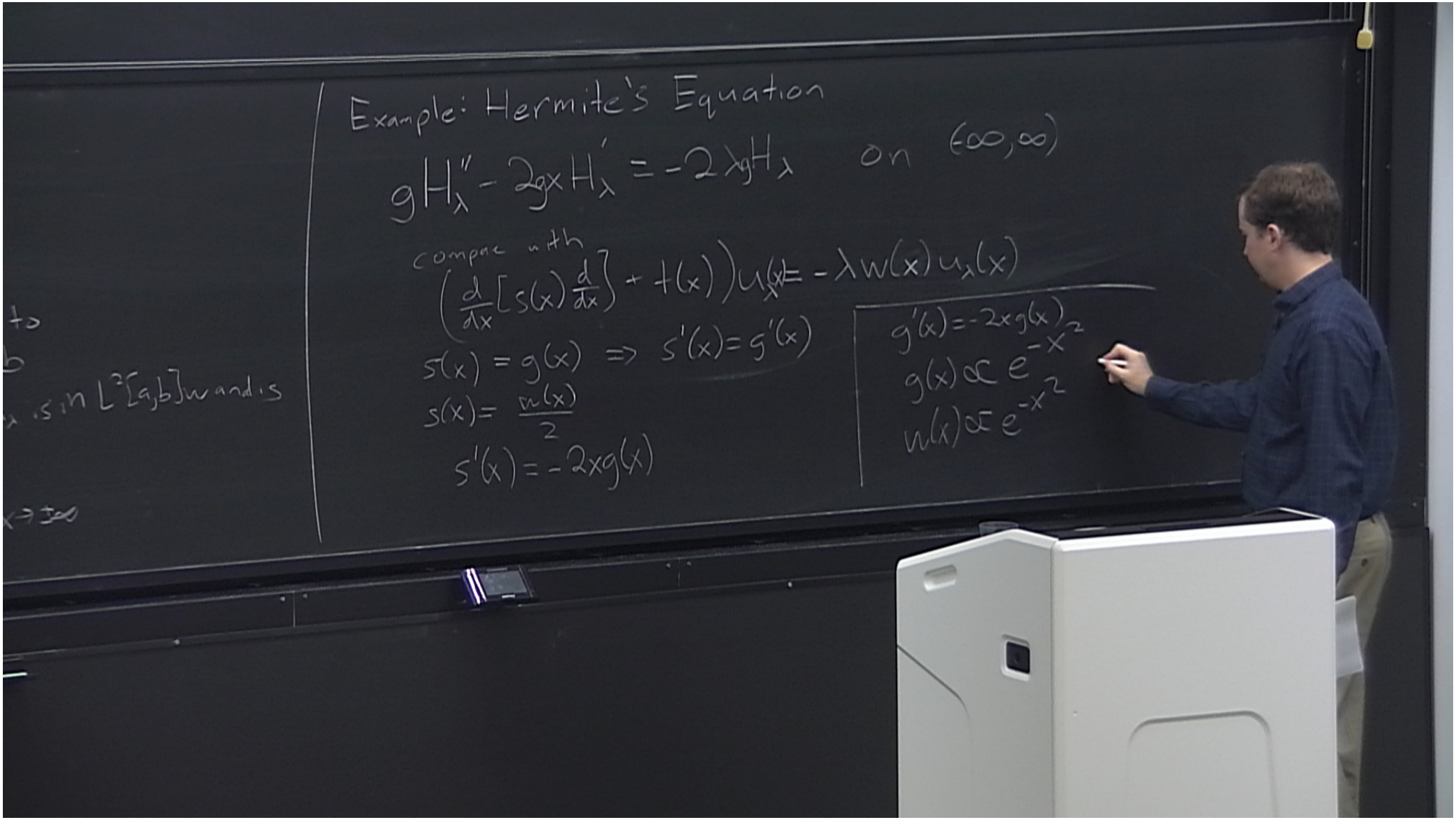

For $\Omega = \frac{d}{dx}\left[s(x)\frac{d}{dx}\right] + t(x)$ with separated (homogeneous) bd cond
- eigenfunctions (u_{λ_n}) (eigenvalues) are real + distinct $-\infty < \lambda_0 < \lambda_1 < \cdots$
same as finite
- u_{λ_n} form an orthonormal basis of $L^2[a,b]_{w/}$
- zeros of $u_{\lambda_{n+1}}$ are in between the zeros of u_{λ_n}
- u_{λ_n} has exactly n distinct zeros in (a,b)

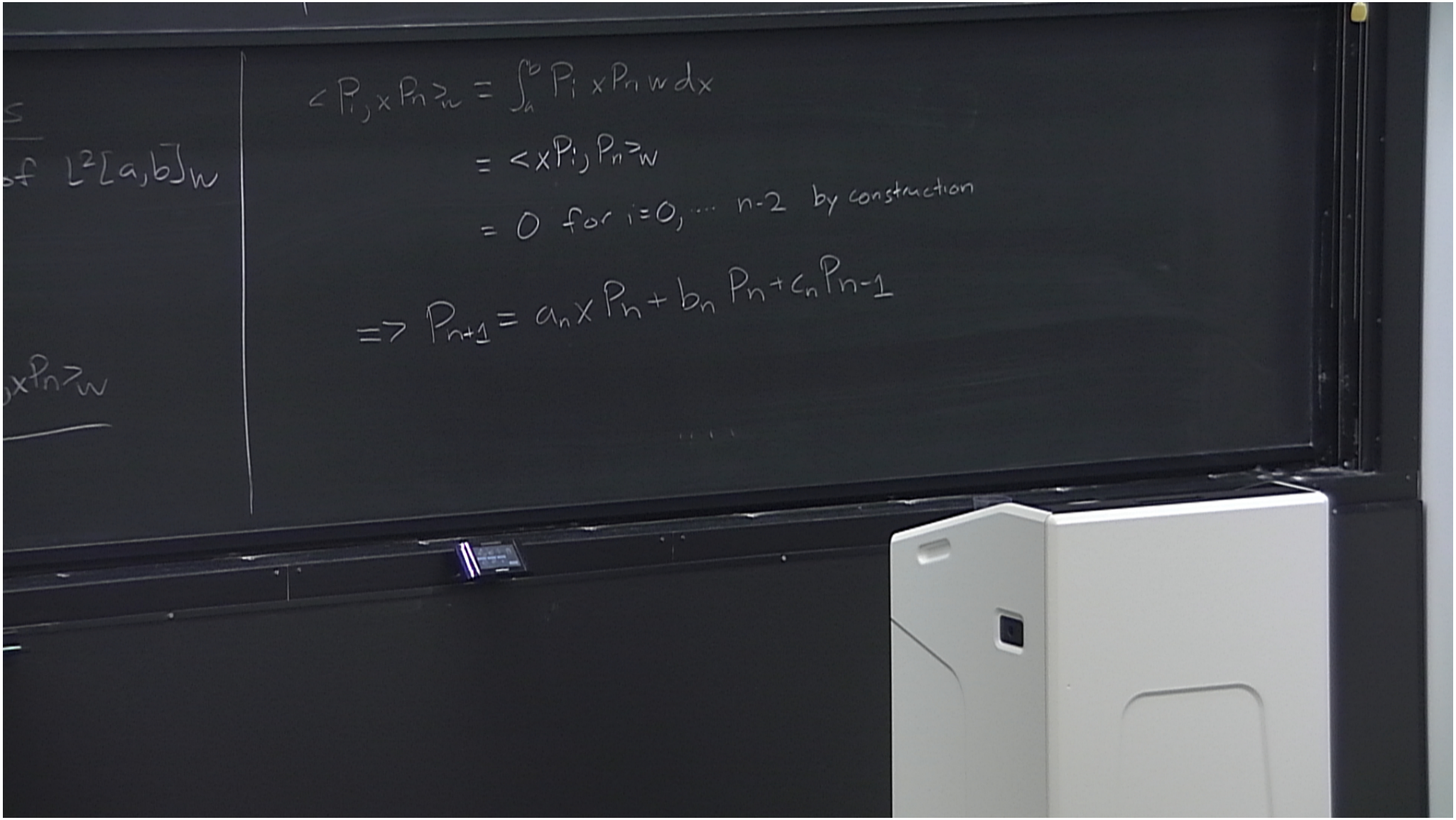

For $\Omega = \frac{d}{dx}\left[s(x)\frac{d}{dx}\right] + t(x)$ with separated (homogeneous) bd cond
- eigenfunctions (eigenvalues) are real + distinct $-\infty < \lambda_0 < \lambda_1 < \cdots$
- $u_{\lambda n}$ form an orthonormal basis of $L^2[a,b]_w$
- zeros of $u_{\lambda n+1}$ are in between the zeros of $u_{\lambda n}$
- $u_{\lambda n}$ has exactly n distinct zeros in (a,b)
same as finite Hermitian matrices

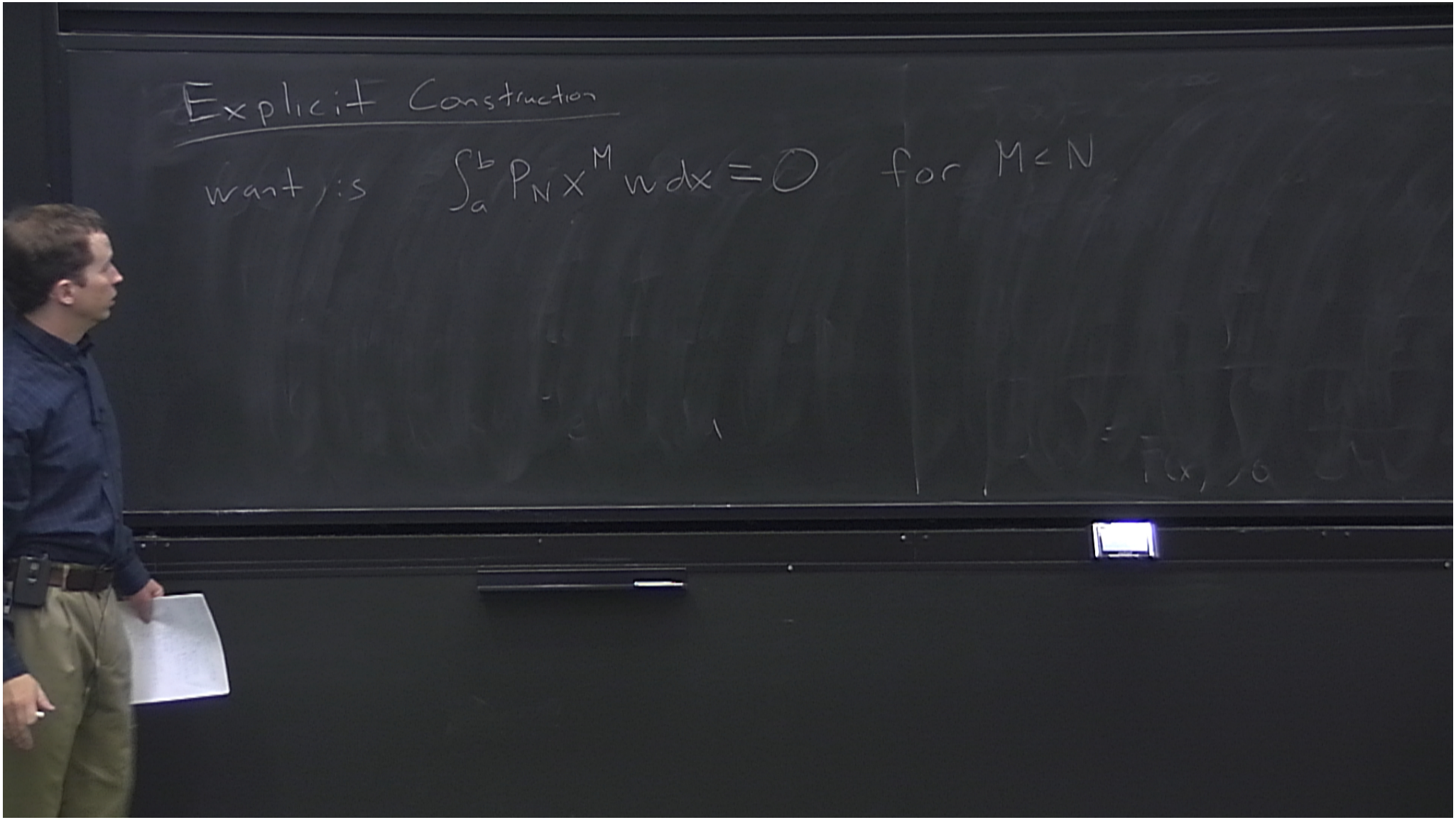

First 3 points hold ev

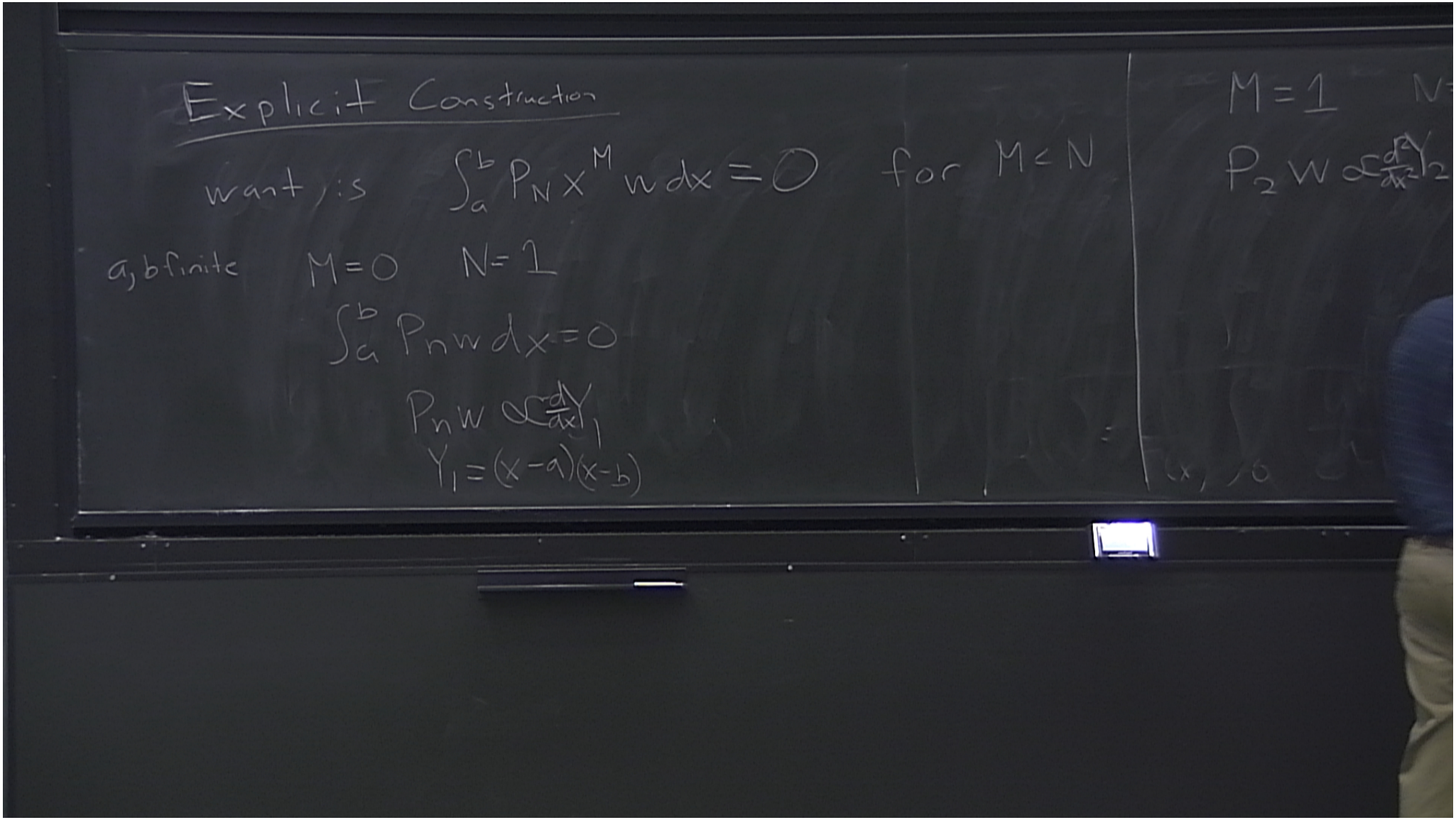

First 3 points hold even if
- $s(x) = 0$ at $x = a$ or $x = b$
- a or b is infinite
- in this case don't have to impose BCs at a or b
- instead impose that u_x is in $L^2[a,b]_w$ and is
- finite at a or b
- $\sqrt{w}\,u \to 0$ as $x \to \pm\infty$

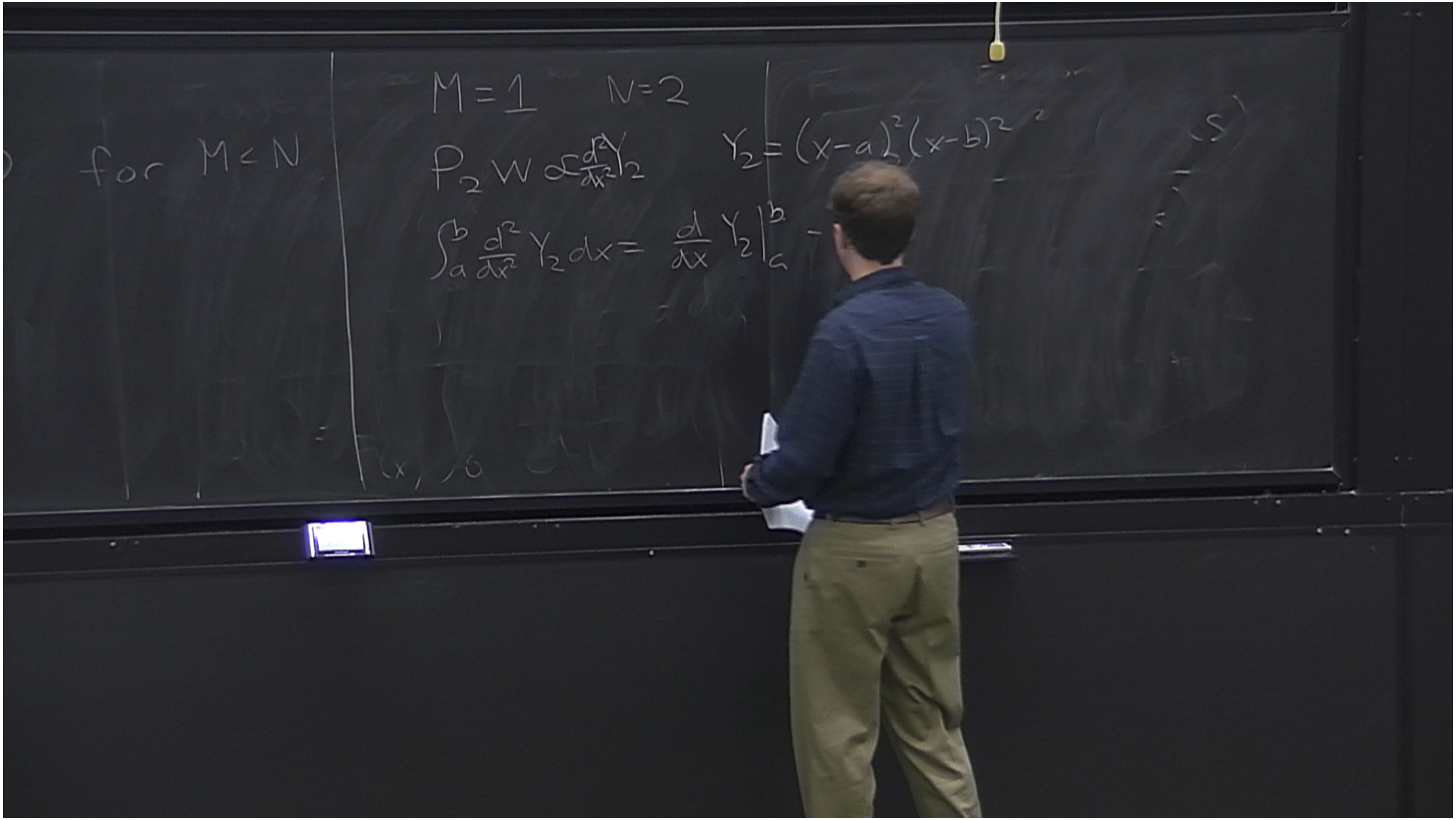

Example: Hermite's Equation
$H_\lambda'' - 2xH_\lambda' = -2\lambda H_\lambda$ on $(-\infty,\infty)$
$L^2[a,b]_w$

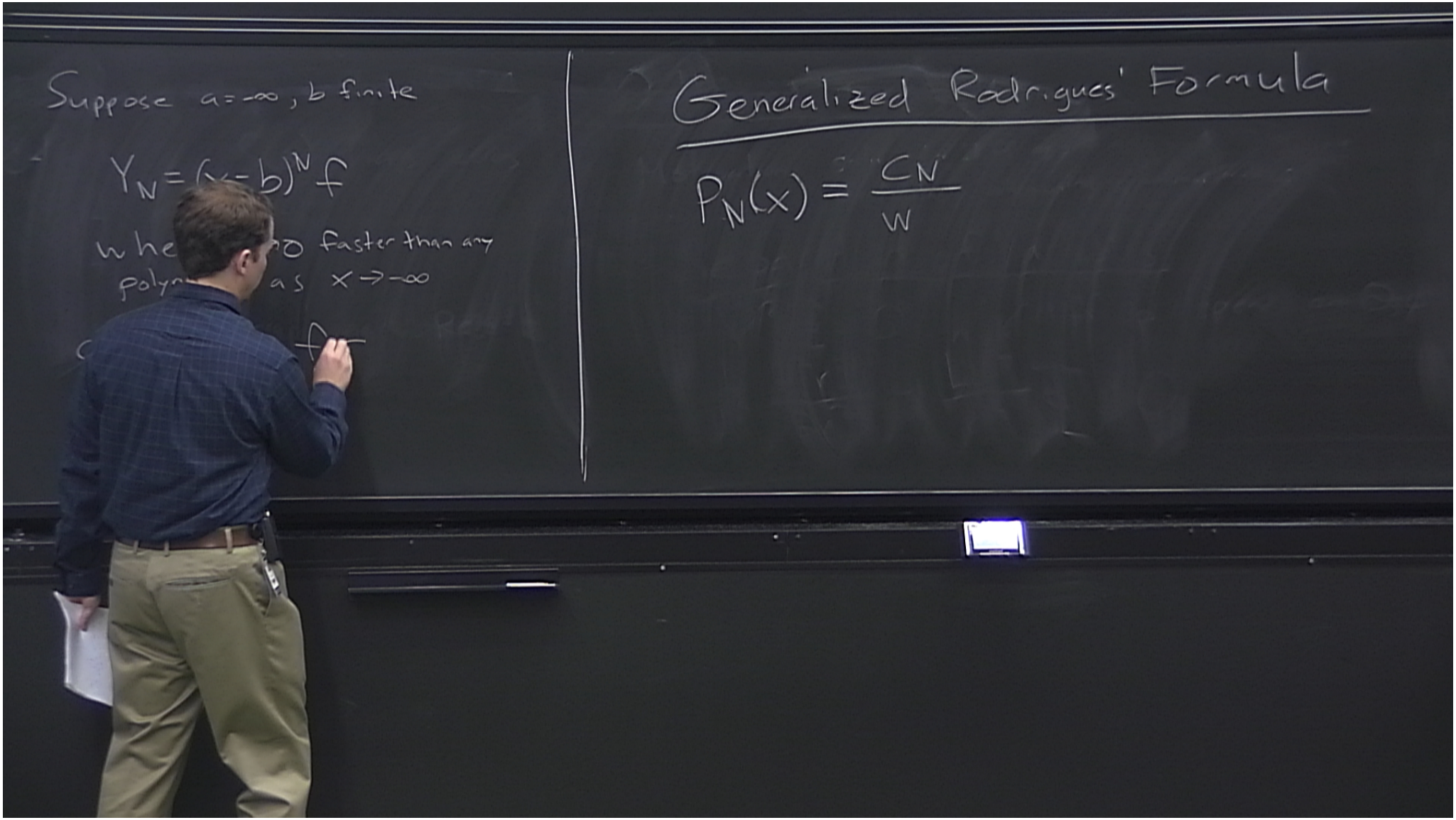

Example: Hermite's Equation
$gH_\lambda'' - 2gxH_\lambda' = -2\lambda gH_\lambda$ on $(-\infty,\infty)$
compare with
$\left(\frac{d}{dx}[s(x)\frac{d}{dx}] + t(x)\right)u_\lambda(x) = -\lambda w(x)u_\lambda(x)$
$s(x) = g(x) \Rightarrow s'(x) = g'(x)$
$s(x) = \frac{w(x)}{2}$
$s'(x) = -2xg(x)$
$g'(x) = -2xg(x)$
$g(x) \propto e^{-x^2}$
$w(x) \propto e^{-x^2}$

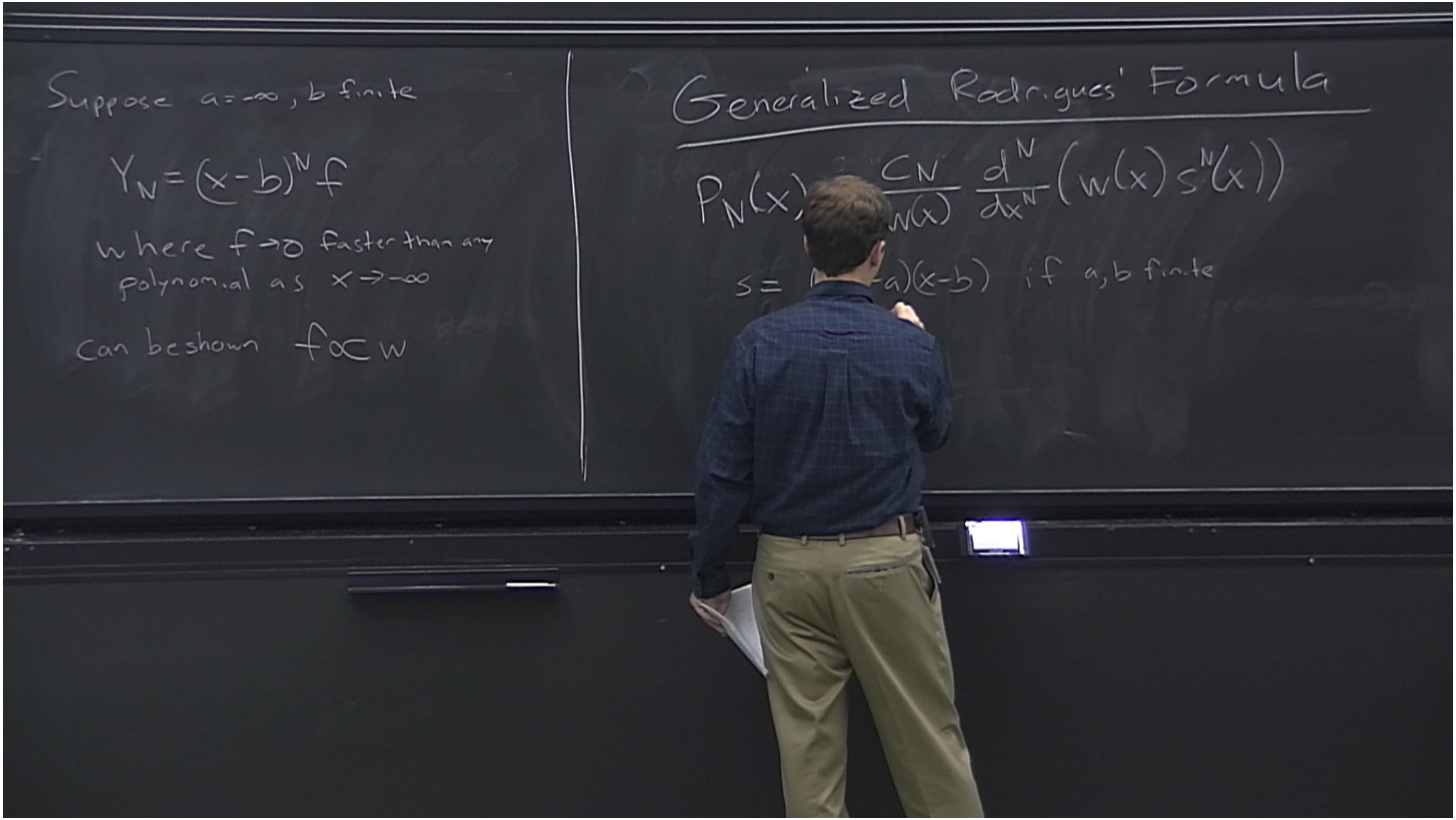

$\langle P_i, x P_n \rangle_w = \int_a^b P_i \, x P_n \, w \, dx$
$= \langle x P_i, P_n \rangle_w$
$= 0$ for $i = 0, \cdots n-2$ by construction
$\Rightarrow P_{n+1} = a_n x P_n + b_n P_n + c_n P_{n-1}$


Explicit Construction
want is $\int_a^b P_N x^M w\,dx = 0$ for $M < N$


Explicit Construction
want is $\int_a^b P_N x^M w\,dx = 0$ for $M < N$
a, b finite
$M = 0$ $N = 1$
$\int_a^b P_n w\,dx = 0$
$P_n w \propto \frac{d}{dx} Y_1$
$Y_1 = (x-a)(x-b)$
$M = 1$
$P_2 w \propto \frac{d}{dx} Y_2$


for M<N
M=1 N=2
$P_2 W \propto \frac{d^2}{dx^2} Y_2$
$Y_2 = (x-a)^2 (x-b)^2$
$\int_a^b \frac{d^2}{dx^2} Y_2 dx = \frac{d}{dx} Y_2 \Big|_a^b$


Suppose $a=-\infty$, b finite
$Y_N = (x-b)^N f$
faster than any
as $x \to -\infty$
Generalized Rodrigues' Formula
$P_N(x) = \frac{C_N}{w}$


Suppose $a=-\infty$, b finite
$Y_N = (x-b)^N f$
where $f \to 0$ faster than any polynomial as $x \to -\infty$
Can be shown $f \propto w$
Generalized Rodrigues' Formula
$P_N(x) \propto \frac{c_N}{w(x)} \frac{d^N}{dx^N}(w(x)s^N(x))$
$s = (x-a)(x-b)$ if a, b finite

Suppose $a=-\infty$, b finite

$$Y_N = (x-b)^N f$$

where $f \to 0$ faster than any polynomial as $x \to -\infty$

Can be shown $f \propto w$

Generalized Rodrigues' Formula

$$P_N(x) = \frac{c_N}{w(x)} \frac{d^N}{dx^N}\left(w(x)\, s^N(x)\right)$$

$$s = \begin{cases} (x-a)(x-b) & \text{if } a, b \text{ finite} \\ x-a & \text{if } a \text{ finite } b=\infty \\ x-b & \text{if } b \quad a=-\infty \\ 1 & \text{otherwise} \end{cases}$$

Generalized Rodrigues' Formula

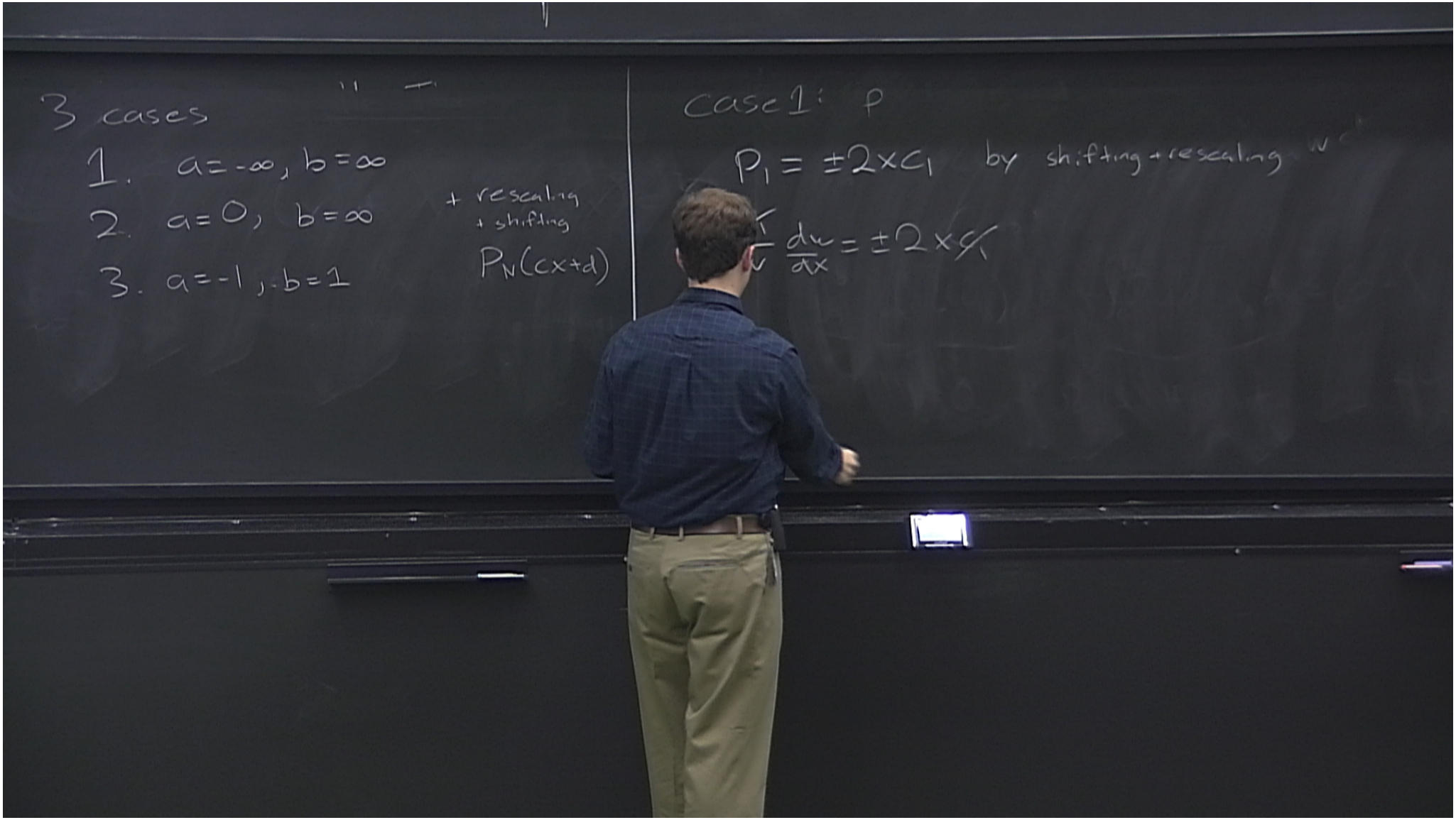
$$P_N(x) = \frac{C_N}{w(x)} \frac{d^N}{dx^N}\left(w(x)\, s^N(x)\right)$$

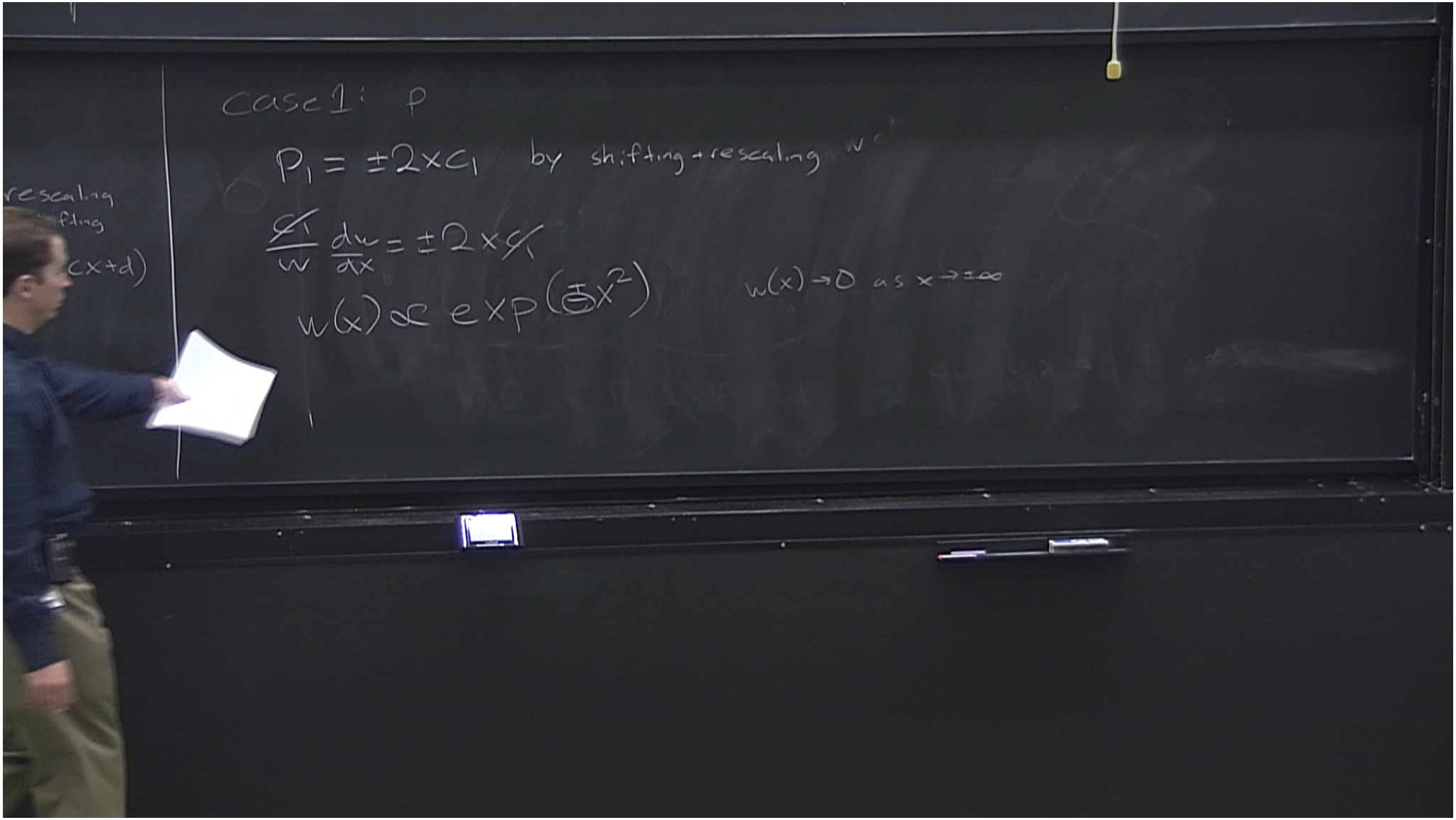
$$s = \begin{cases} (x-a)(x-b) & \text{if } a, b \text{ finite} \\ x-a & \text{if } a \text{ finite } b=\infty \\ x-b & \text{if } b \quad a=-\infty \\ 1 & \text{otherwise} \end{cases}$$

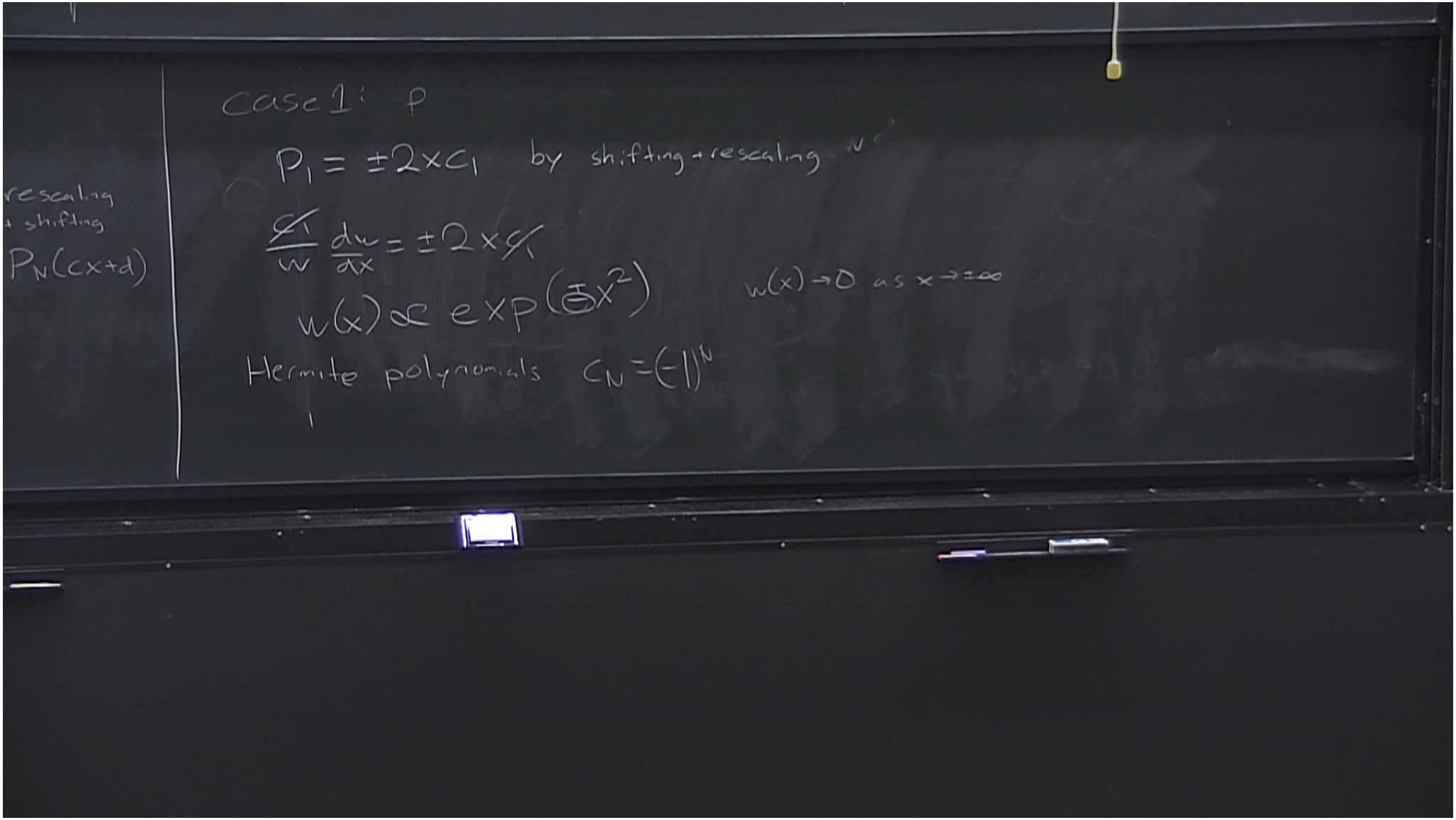
works if

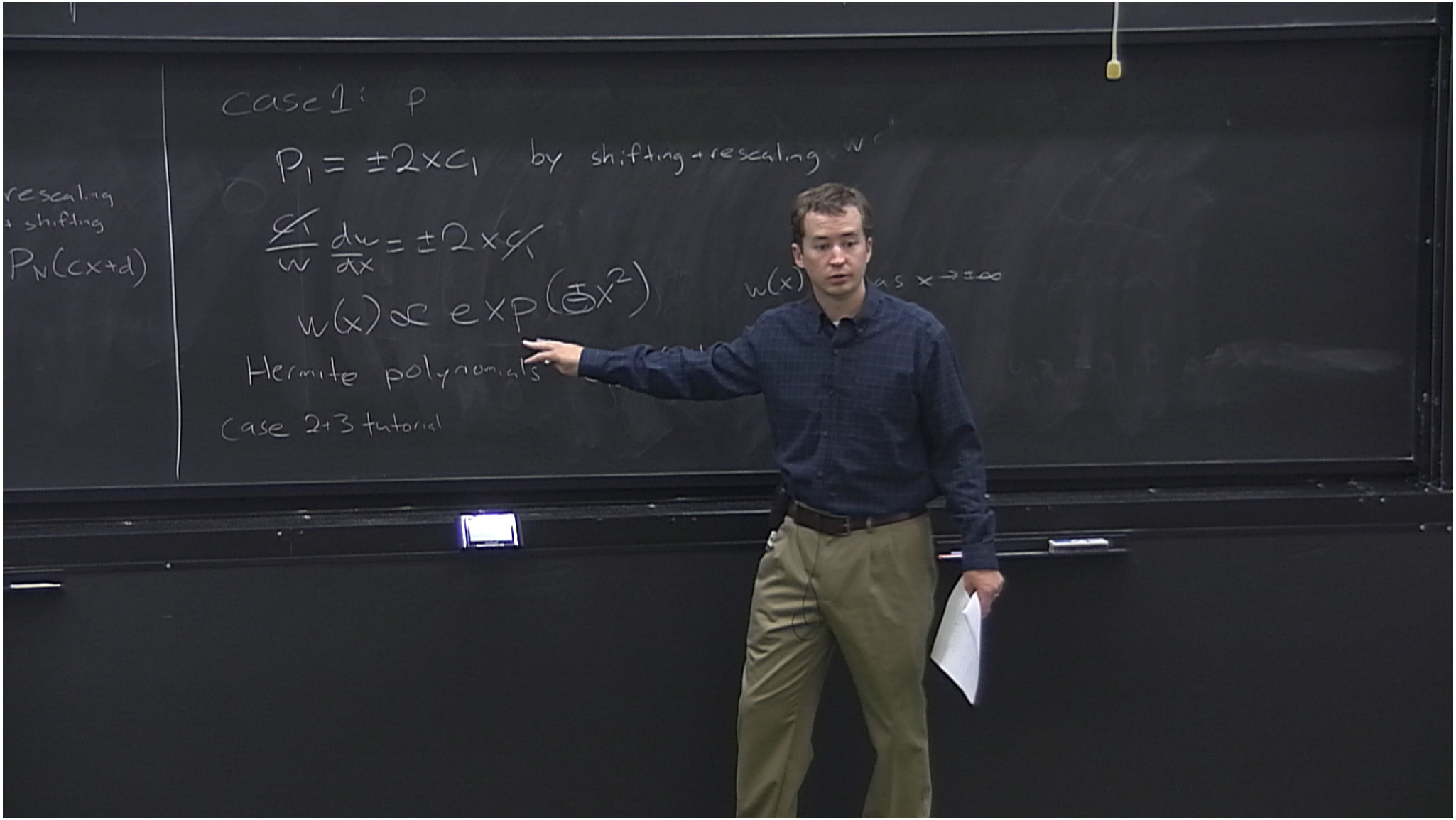
- $w(a)s(a) = 0 = w(b)s(b)$ if a, b finite
- $w(x)s(x) \to 0$ faster than any polynomial as $x \to \pm\infty$ if a, b infinite

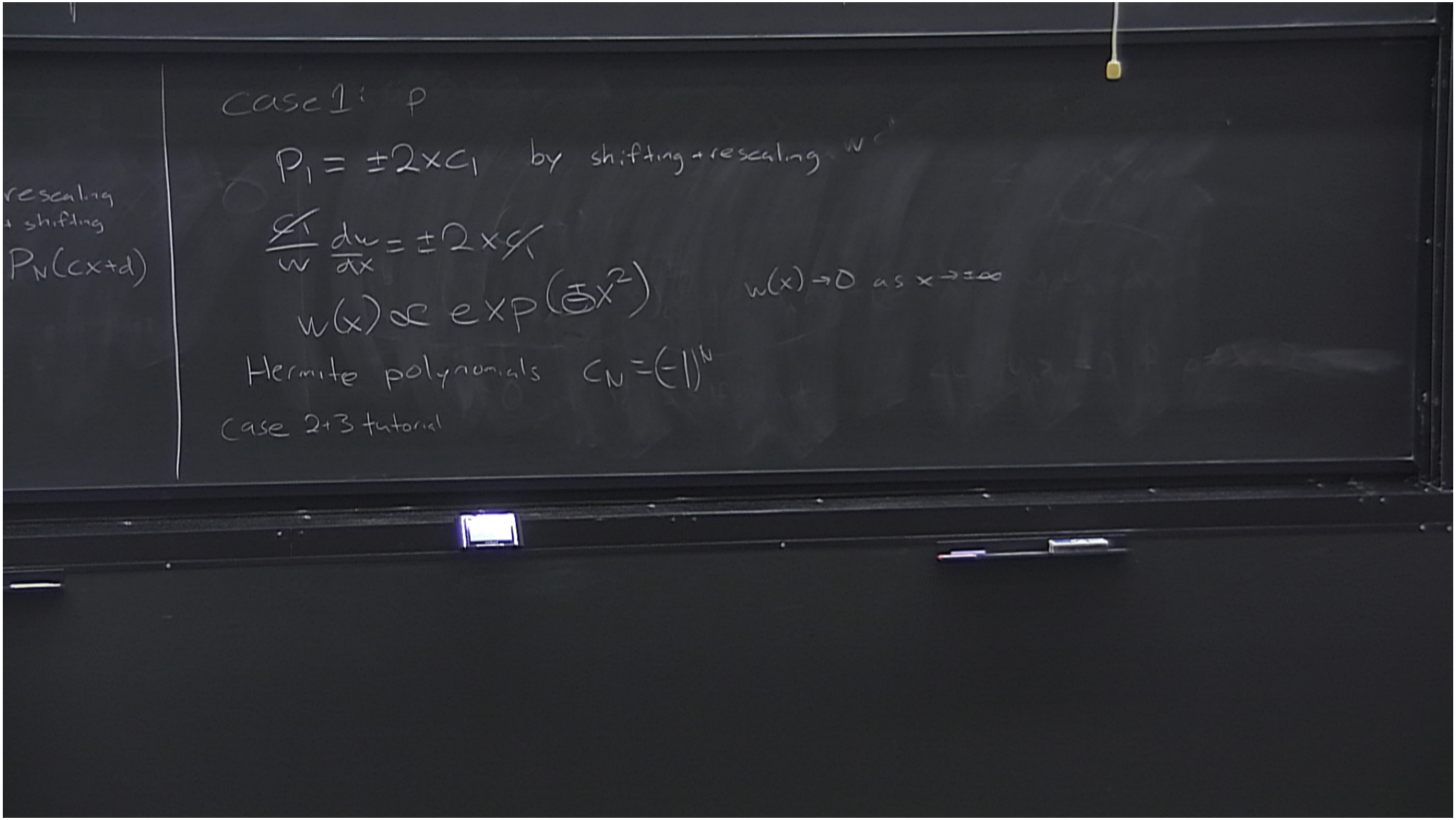
- P_i is a polynomial

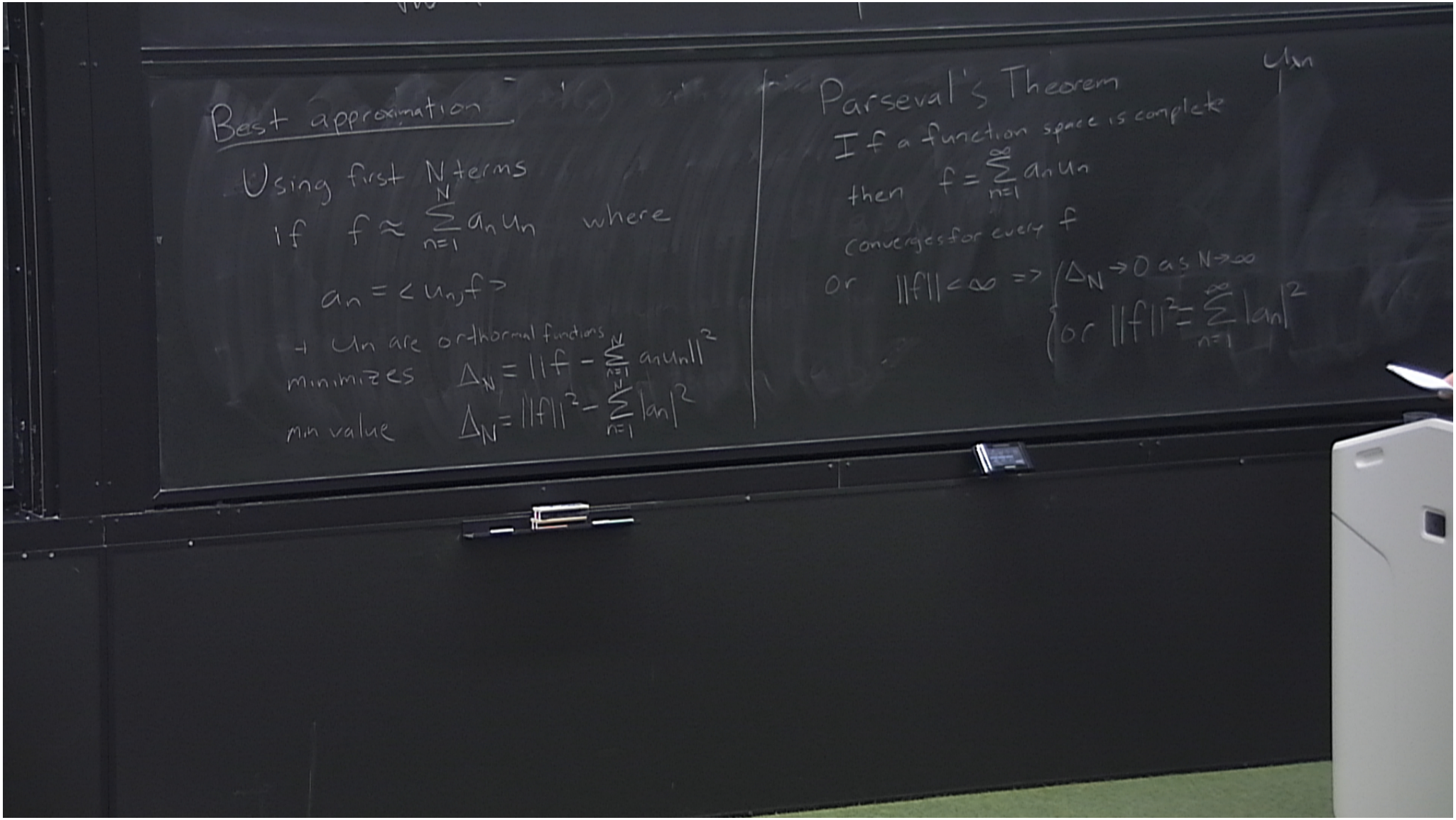

3 cases
1. a=-∞, b=∞
2. a=0, b=∞
3. a=-1, b=1
+ rescaling + shifting
$P_N(cx+d)$
case 1: P
$P_1 = \pm 2x c_1$ by shifting + rescaling
$\frac{dw}{dx} = \pm 2x c_1$


rescaling
fting
(cx+d)
case 1: p
$P_1 = \pm 2 x c_1$ by shifting + rescaling w
$\frac{c_1}{w}\frac{dw}{dx} = \pm 2 x c_1$
$w(x) \propto \exp(\mp x^2)$
$w(x) \to 0$ as $x \to \pm\infty$


rescaling
+ shifting
$P_N(cx+d)$
case 1: P
$P_1 = \pm 2xc_1$ by shifting + rescaling
$\frac{c_1}{w}\frac{dw}{dx} = \pm 2xc_1$
$w(x) \propto \exp(\mp x^2)$
$w(x) \to 0$ as $x \to \pm\infty$
Hermite polynomials $c_N = (-1)^N$


rescaling
+ shifting
$P_N(cx+d)$
case 1: P
$P_1 = \pm 2x c_1$ by shifting + rescaling
$\frac{c_1}{w}\frac{dw}{dx} = \pm 2x c_1$
$w(x) \propto \exp(\mp x^2)$
Hermite polynomials
case 2+3 tutorial


rescaling
+ shifting
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$w(x) \to 0$ as $x \to \pm\infty$
Hermite polynomials $c_N = (-1)^N$
case 2+3 tutorial


Best approximation
Using first N terms
if $f \approx \sum_{n=1}^{N} a_n u_n$ where
$a_n = \langle u_n, f \rangle$
+ u_n are orthonormal functions
minimizes $\Delta_N = \|f - \sum_{n=1}^{N} a_n u_n\|^2$
min value $\Delta_N = \|f\|^2 - \sum_{n=1}^{N} |a_n|^2$
Parseval's Theorem
If a function space is complete
then $f = \sum_{n=1}^{\infty} a_n u_n$
converges for every f
or $\|f\| < \infty \Rightarrow$ $\Delta_N \to 0$ as $N \to \infty$
or $\|f\|^2 = \sum_{n=1}^{\infty} |a_n|^2$
u_n