Title: PSI 2015/2016 Special Functions - Dan Wohns -2

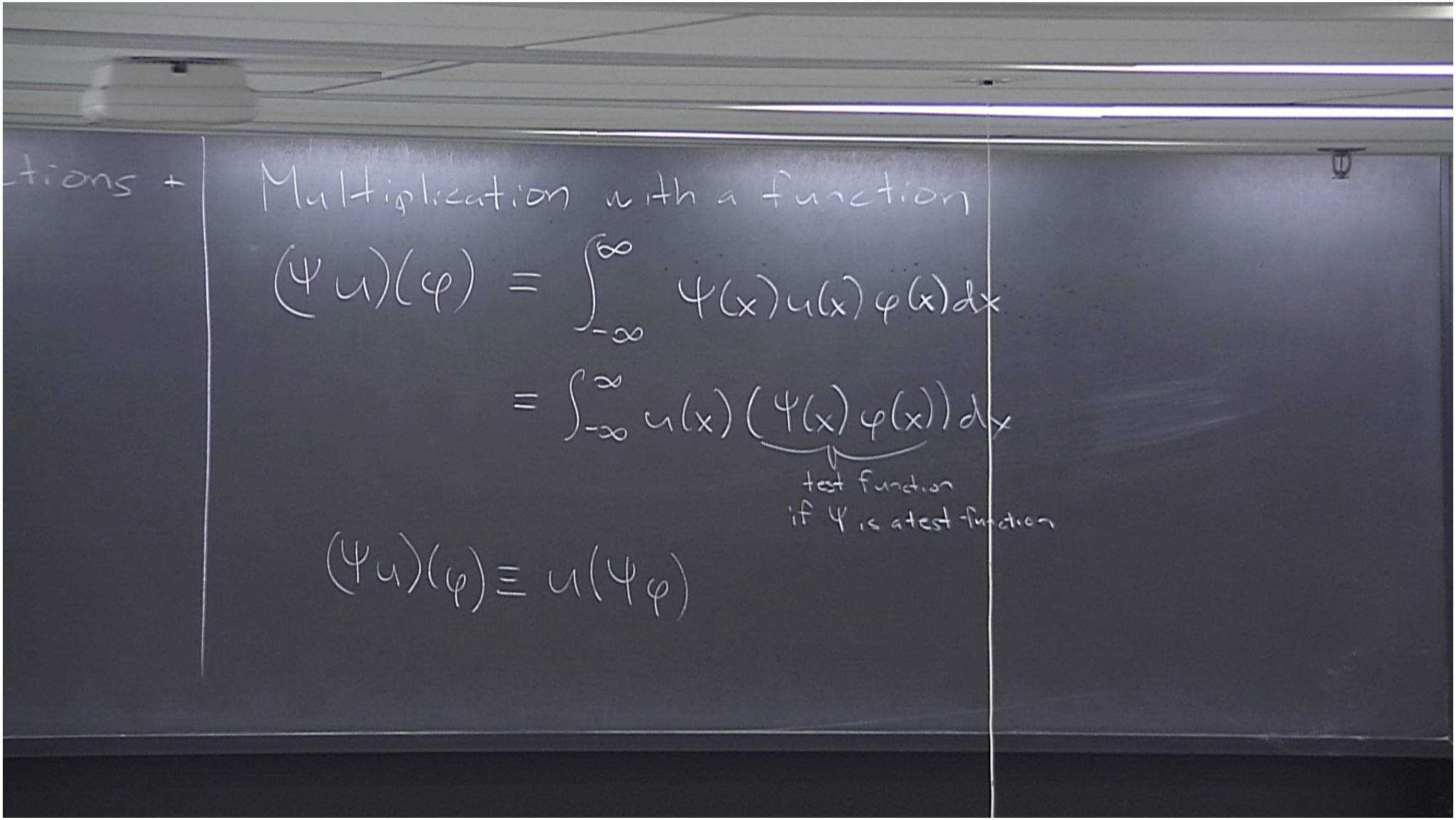
Date: Sep 01, 2015 10:30 AM

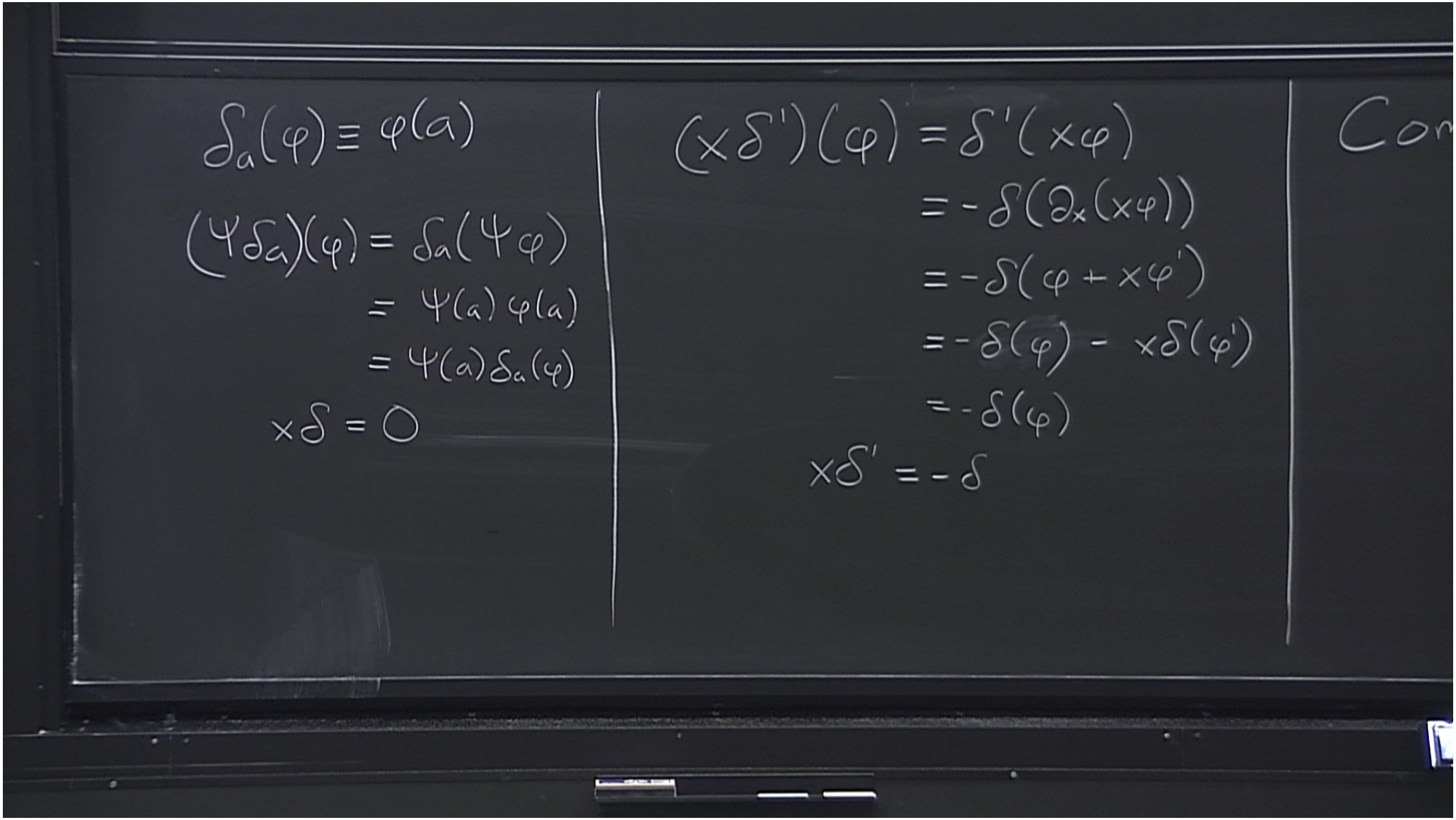
URL: http://pirsa.org/15090014

Abstract:

Recap: Distributions are functionals that act on test functions + are
- finite
- linear
- continuous

weak derivative: $u'(\varphi) = -u(\varphi')$


tions +
Multiplication with a function
$(\Psi u)(\varphi) = \int_{-\infty}^{\infty} \Psi(x)\, u(x)\, \varphi(x)\, dx$
$= \int_{-\infty}^{\infty} u(x)\, (\Psi(x)\, \varphi(x))\, dx$
test function
if Ψ is a test function
$(\Psi u)(\varphi) \equiv u(\Psi\varphi)$


$\delta_a(\varphi) \equiv \varphi(a)$
$(\psi\delta_a)(\varphi) = \delta_a(\psi\varphi)$
$= \psi(a)\varphi(a)$
$= \psi(a)\delta_a(\varphi)$
$x\delta = 0$
$(x\delta')(\varphi) = \delta'(x\varphi)$
$= -\delta(\partial_x(x\varphi))$
$= -\delta(\varphi + x\varphi')$
$= -\delta(\varphi) - x\delta(\varphi')$
$= -\delta(\varphi)$
$x\delta' = -\delta$
Con

Composition with a Function

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\,\varphi(x)\,dx$$

$(u \circ f)(\varphi) \equiv u(\varphi \circ g \cdot |g'|)$

change of variables $y = f(x)$

exists if
- f is C^∞
- $y = f(x)$ has a unique solution $x = g(y)$
- $g'(y)$ does not change sign

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(y)\,\underbrace{\varphi(g(y))\,|g'(y)|}_{\text{test function}}\,dy$$

Composition with a Function

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\,\varphi(x)\,dx$$

$$(u \circ f)(\varphi) \equiv u(\varphi \circ g \cdot |g'|)$$

change of variables $y = f(x)$

exists if
- f is C^∞
- $y = f(x)$ has a unique solution $x = g(y)$ } sufficient but not necessary
- $g'(y)$ does not change sign

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(y)\,\underbrace{\varphi(g(y))\,|g'(y)|}_{\text{test function}}\,dy$$

For δ:

$$(\delta \circ f)(\varphi) = \sum_{x_i \text{ roots of } f} \frac{\delta_{x_i}}{|f'(x_i)|}(\varphi)$$

- works if f has only simple roots

test function

Functional Derivatives

- Example: point particle

$$S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}[q, \dot{q}] = \int dt \left[\frac{1}{2}\dot{q}^2 - V(q)\right]$$

$$\delta S = S[q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t)] - S[q(t), \dot{q}(t)]$$

$$= \int dt\, \delta q(t) \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] + \mathcal{O}(\delta q^2)$$

test function

Functional Derivatives

- Example: point particle

$$S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}[q, \dot{q}] = \int dt \left[\frac{1}{2}\dot{q}^2 - V(q)\right]$$

$$\delta S = S[q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t)] - S[q(t), \dot{q}(t)]$$

$$= \int dt\, \delta q(t) \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] + \mathcal{O}(\delta q^2)$$

$$\frac{\delta S}{\delta q(t)}$$

test function

Functional Derivatives

- Example: point particle

$$S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}[q, \dot{q}] = \int dt \left[\frac{1}{2}\dot{q}^2 - V(q)\right]$$

$$\delta S = S[q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t)] - S[q(t), \dot{q}(t)]$$

$$= \int dt\, \delta q(t) \underbrace{\left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]}_{\int \frac{\delta S}{\delta q(t)} dt} + \mathcal{O}(\delta q^2)$$

test function

Functional Derivatives

- Example: point particle

$$S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}[q, \dot{q}] = \int dt \left[\frac{1}{2}\dot{q}^2 - V(q)\right]$$

$$\delta S = S[q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t)] - S[q(t), \dot{q}(t)]$$

$$= \int dt\, \delta q(t) \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] + O(\delta q^2)$$

$$= \int dt\, \frac{\delta S}{\delta q(t)}\, \delta q(t)$$

test function

Functional Derivatives

- Example: point particle

$$S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}[q, \dot{q}] = \int dt \left[\frac{1}{2}\dot{q}^2 - V(q)\right]$$

$$\delta S = S[q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t)] - S[q(t), \dot{q}(t)]$$

$$= \int dt\, \delta q(t) \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] + O(\delta q^2)$$

$$= \int dt\, \frac{\delta S}{\delta q(t)}\, \delta q(t)$$

test function

Functional Derivatives

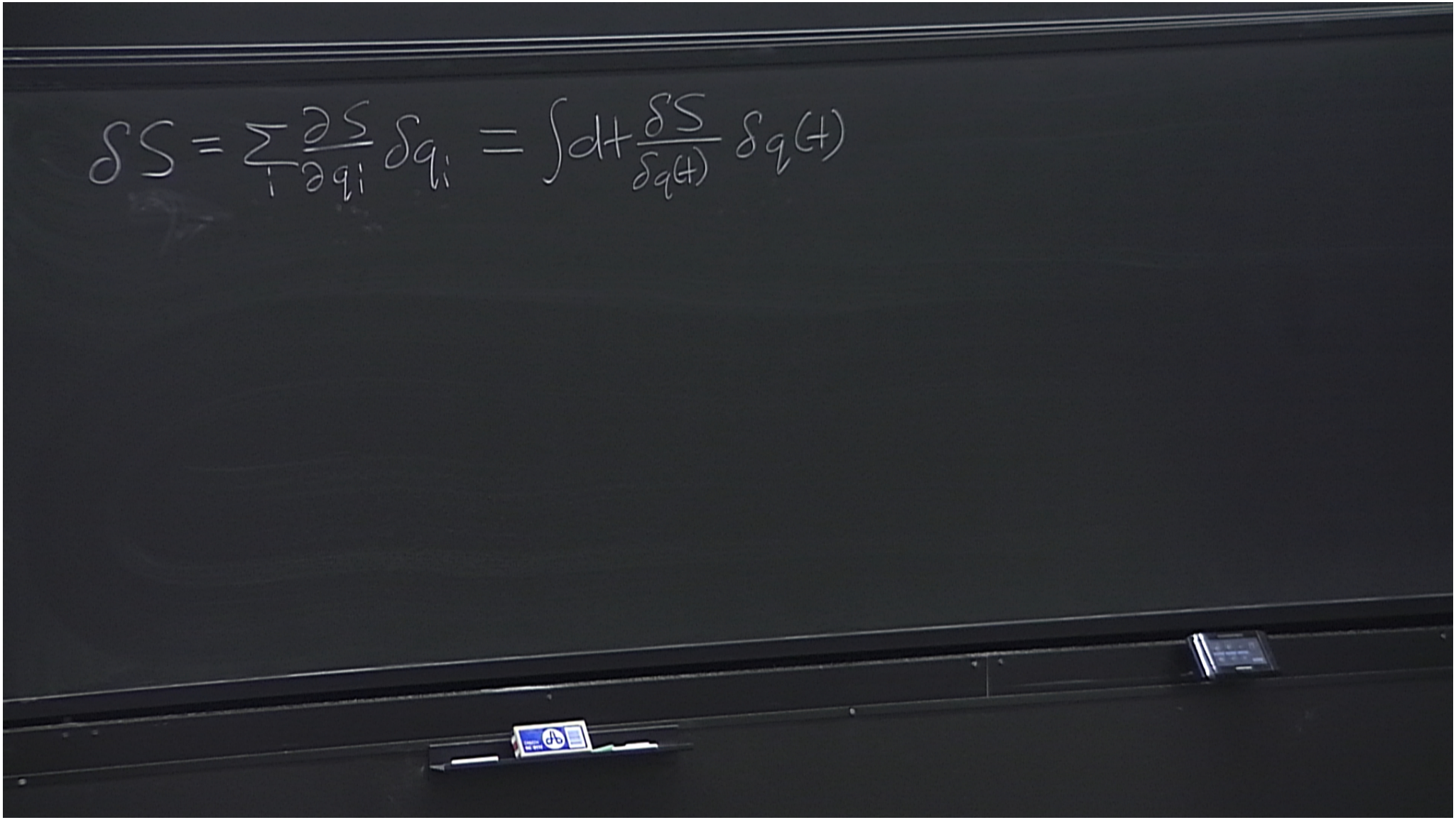
- Example: point particle

$$S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}[q, \dot{q}] = \int dt \left[\frac{1}{2}\dot{q}^2 - V(q)\right]$$

$$\delta S = S[q(t)+\delta q(t), \dot{q}(t)+\delta\dot{q}(t)] - S[q(t), \dot{q}(t)]$$

$$= \int dt\, \delta q(t) \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] + O(\delta q^2)$$

$$= \int dt\, \frac{\delta S}{\delta q(t)}\, \delta q(t)$$


$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i = \int dt \frac{\delta S}{\delta q(t)} \delta q(t)$

$$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i = \int dt \frac{\delta S}{\delta q(t)} \delta q(t)$$

$$\delta q(t) = \epsilon \phi(t)$$

$\phi(t)$: test function

$$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i = \int dt \frac{\delta S}{\delta q(t)} \delta q(t)$$

$$\delta q(t) = \epsilon \phi(t)$$

$\phi(t)$: test function

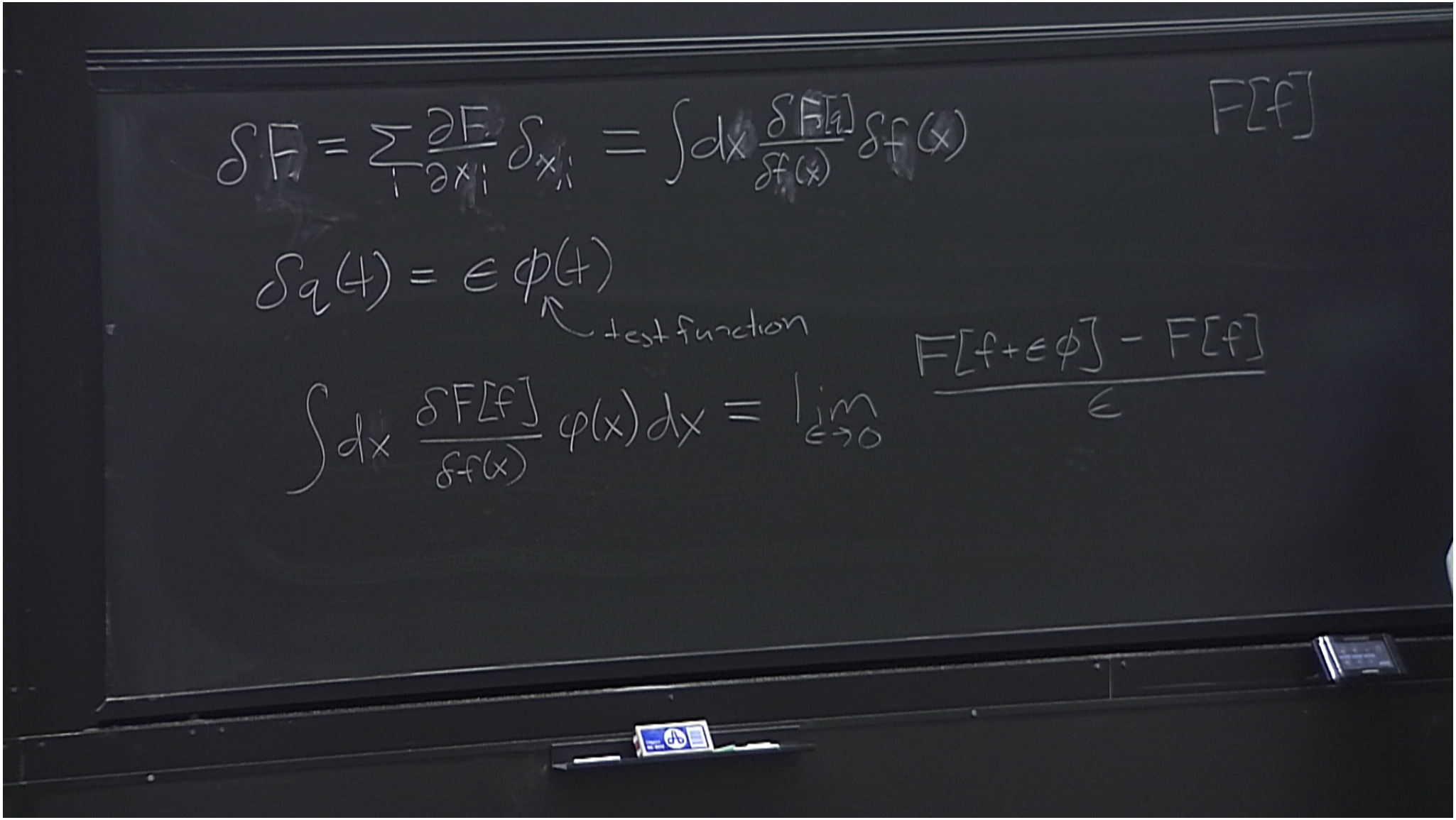
$$\int dx \frac{\delta F[f]}{\delta f(x)} \varphi(x)\, dx = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\phi] - F[f]}{\epsilon}$$

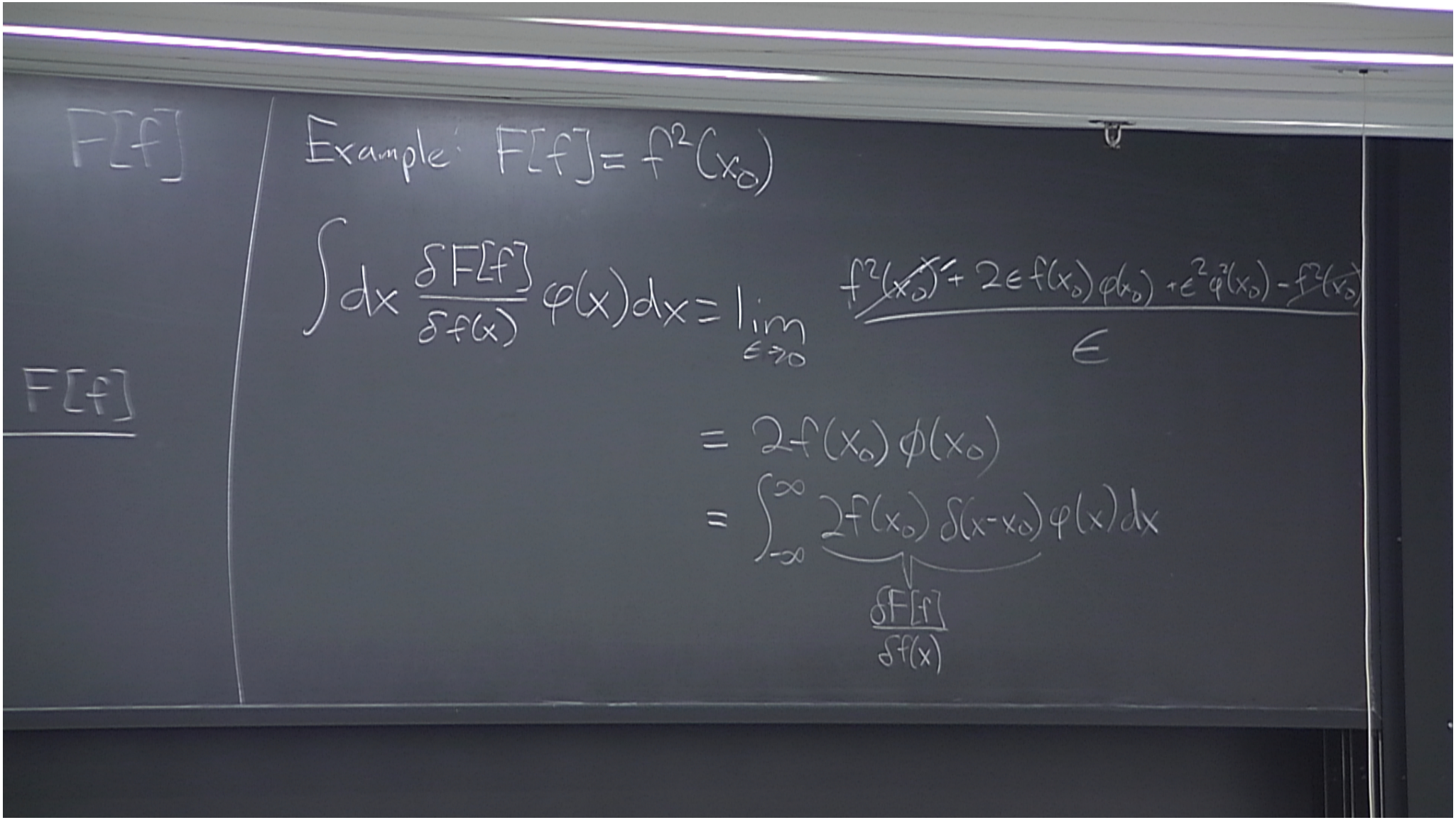
$$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i = \int dt \frac{\delta S}{\delta q(t)} \delta q(t)$$

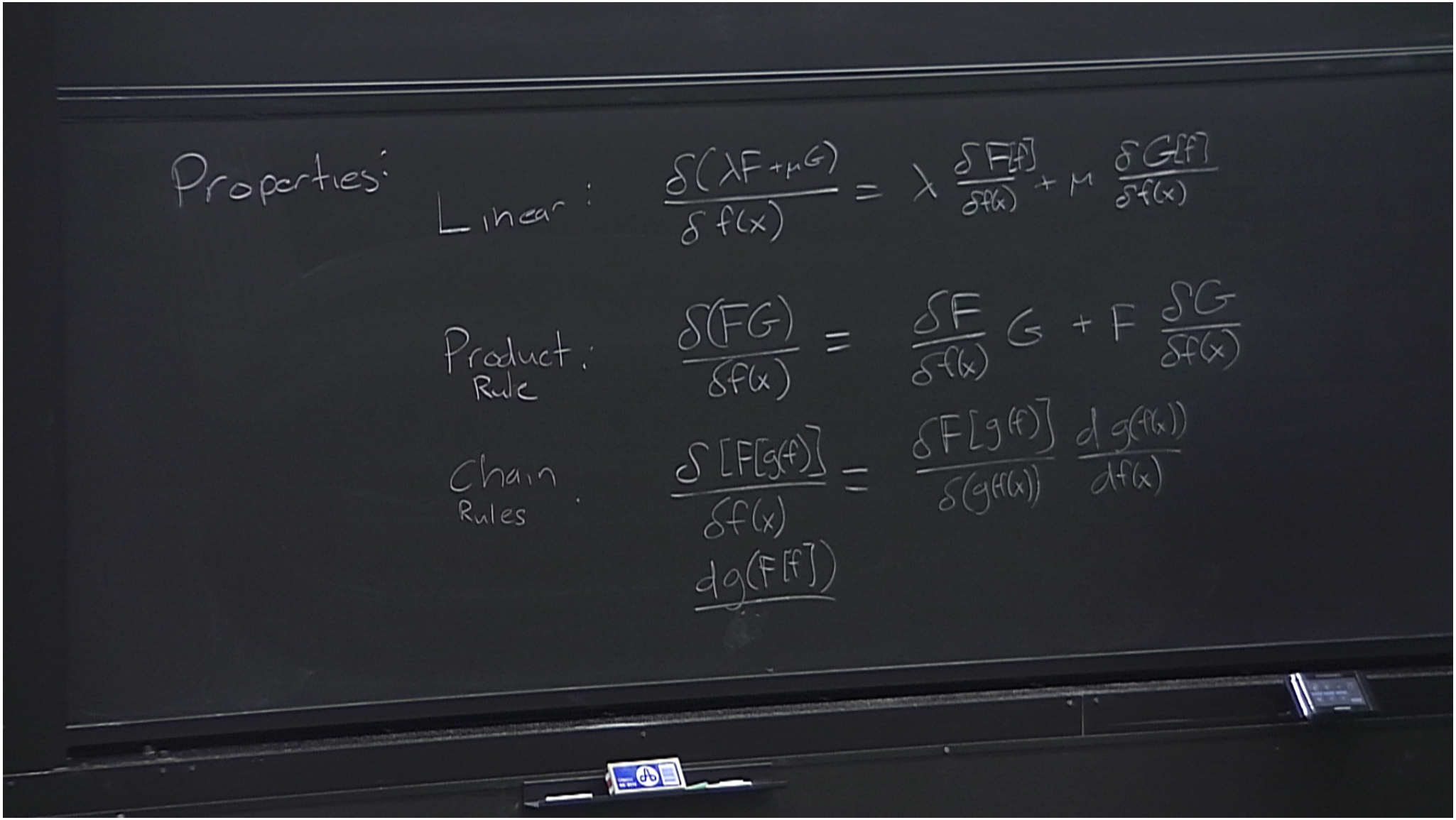
$$\delta q(t) = \epsilon \phi(t)$$

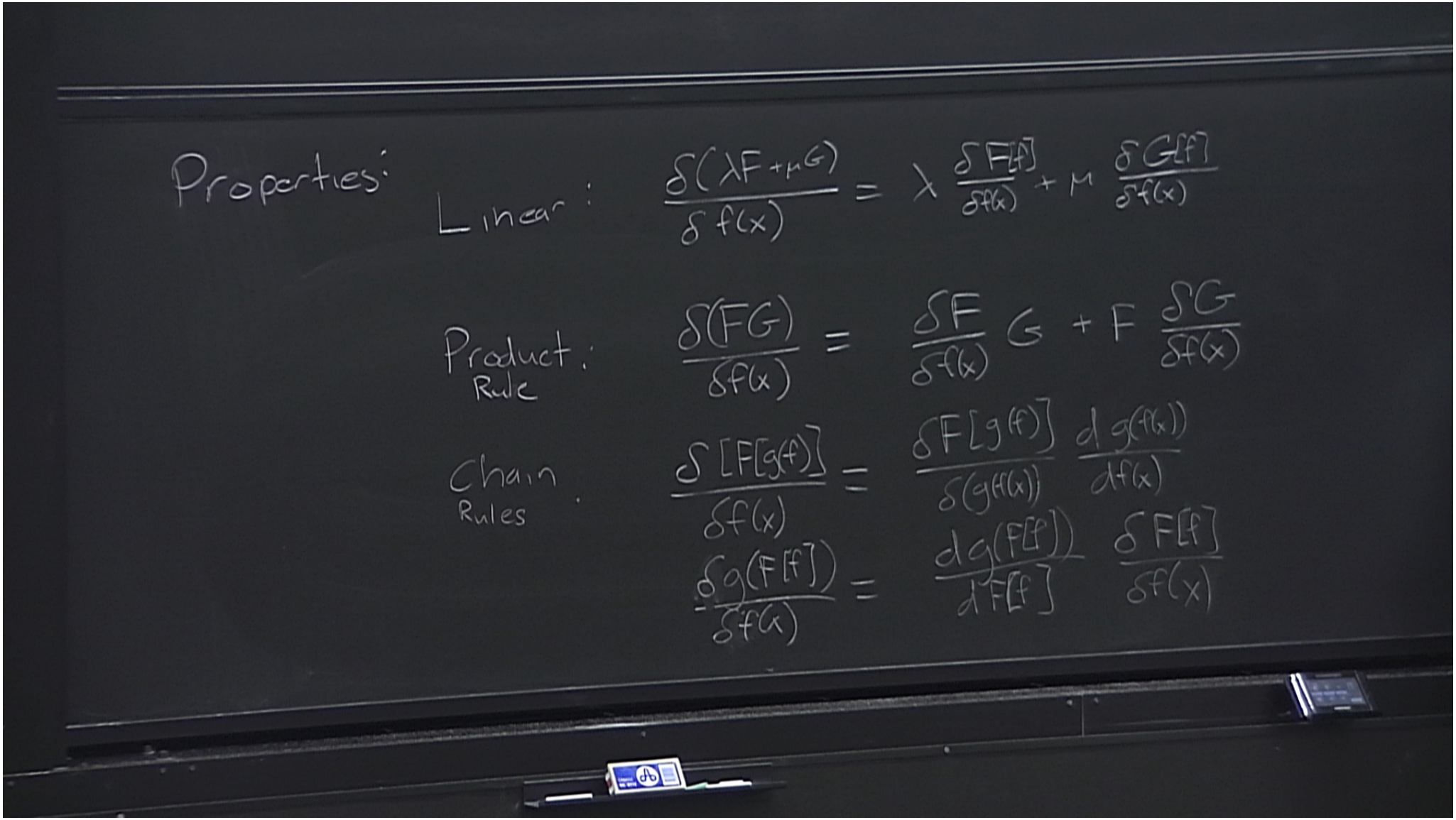
test function

$$\int dx \frac{\delta F[f]}{\delta f(x)} \varphi(x) dx = \lim_{\epsilon \to 0} \frac{F[f + \epsilon \phi] - F[f]}{\epsilon}$$


$\delta F = \sum_i \frac{\partial F}{\partial x_i} \delta x_i = \int dx \frac{\delta F[q]}{\delta f(x)} \delta f(x)$
F[f]
$\delta q(t) = \epsilon \phi(t)$
test function
$\int dx \frac{\delta F[f]}{\delta f(x)} \varphi(x) dx = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\phi] - F[f]}{\epsilon}$


F[f]
Example: F[f] = f²(x₀)
∫ dx (δF[f]/δf(x)) φ(x) dx = lim_{ε→0} (f²(x₀) + 2ε f(x₀) φ(x₀) + ε² φ²(x₀) − f²(x₀)) / ε
F[f]
= 2f(x₀) φ(x₀)
= ∫_{−∞}^{∞} 2f(x₀) δ(x − x₀) φ(x) dx
δF[f]/δf(x)


Properties:
Linear: $\frac{\delta(\lambda F + \mu G)}{\delta f(x)} = \lambda \frac{\delta F[f]}{\delta f(x)} + \mu \frac{\delta G[f]}{\delta f(x)}$
Product Rule: $\frac{\delta(FG)}{\delta f(x)} = \frac{\delta F}{\delta f(x)} G + F \frac{\delta G}{\delta f(x)}$
Chain Rules: $\frac{\delta [F[g(f)]]}{\delta f(x)} = \frac{\delta F[g(f)]}{\delta (g(f(x)))} \frac{d\, g(f(x))}{d f(x)}$
$\frac{dg(F[f])}{}$

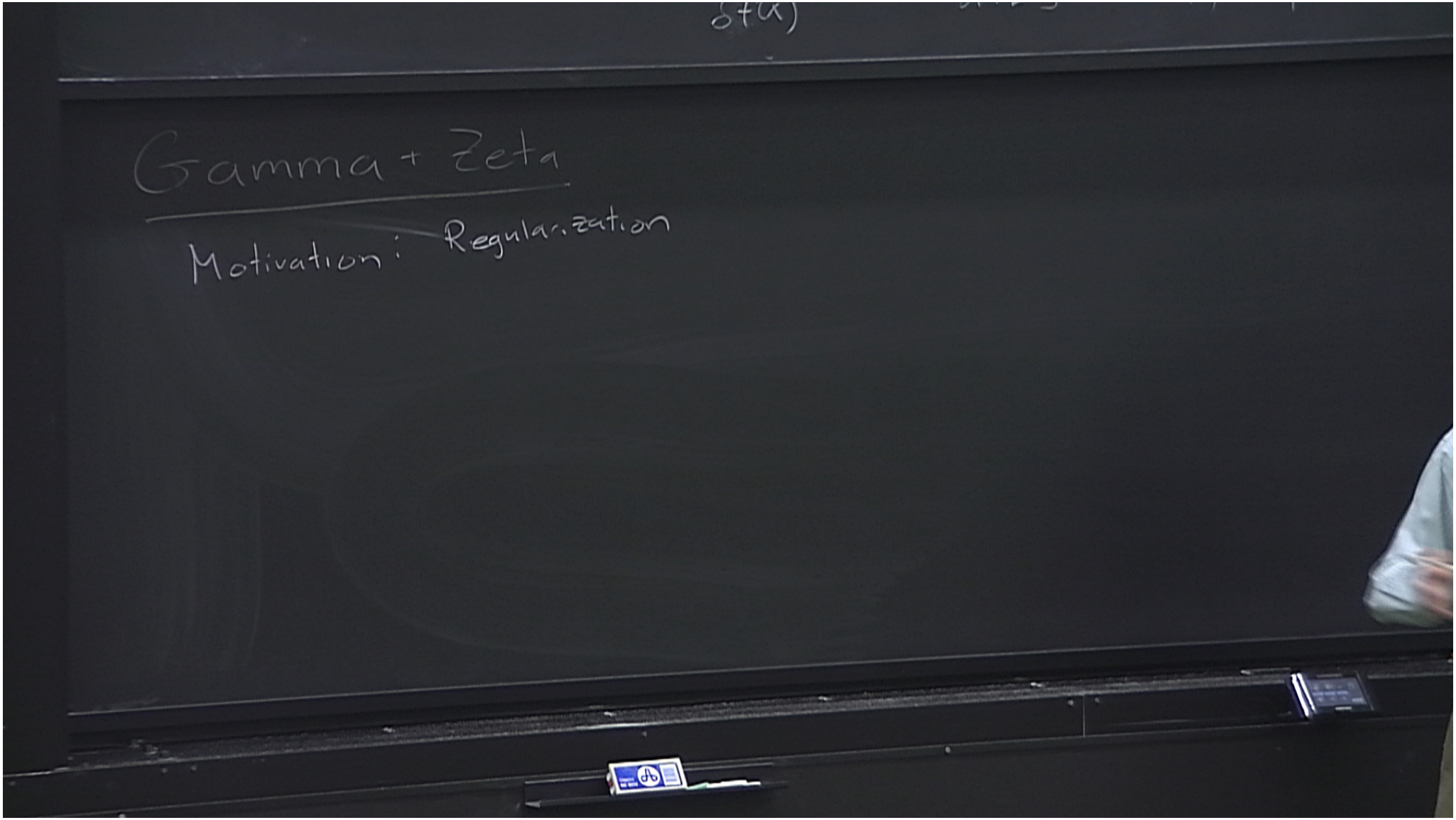

Properties:
Linear: $\frac{\delta(\lambda F + \mu G)}{\delta f(x)} = \lambda \frac{\delta F[f]}{\delta f(x)} + \mu \frac{\delta G[f]}{\delta f(x)}$
Product Rule: $\frac{\delta(FG)}{\delta f(x)} = \frac{\delta F}{\delta f(x)} G + F \frac{\delta G}{\delta f(x)}$
Chain Rules: $\frac{\delta [F[g(f)]]}{\delta f(x)} = \frac{\delta F[g(f)]}{\delta (g(f(x)))} \frac{d\, g(f(x))}{d f(x)}$
$\frac{\delta g(F[f])}{\delta f(x)} = \frac{d g(F[f])}{d F[f]} \frac{\delta F[f]}{\delta f(x)}$

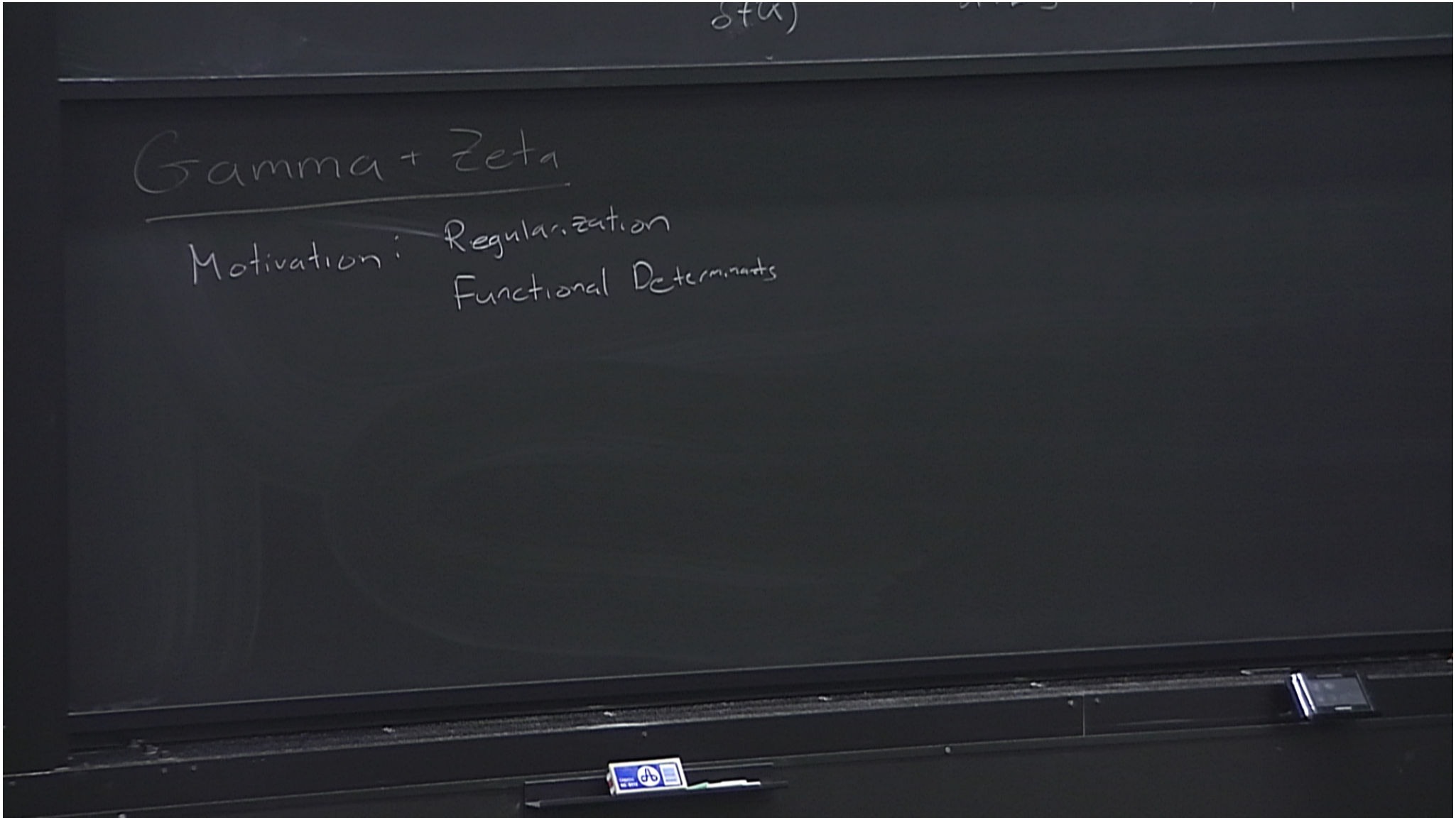
$\frac{\delta F[f]}{\delta f(x)}$

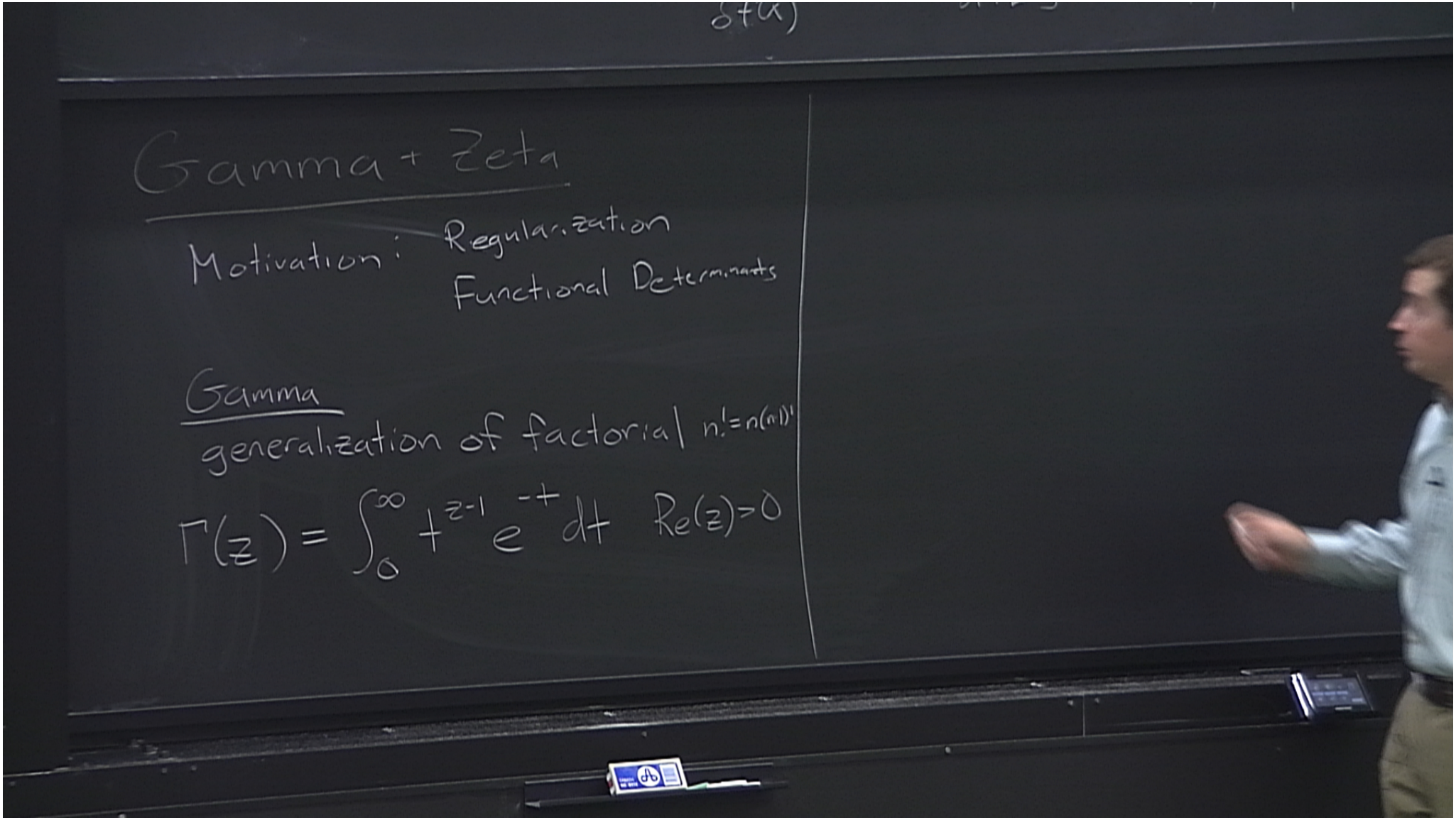
Warnings:

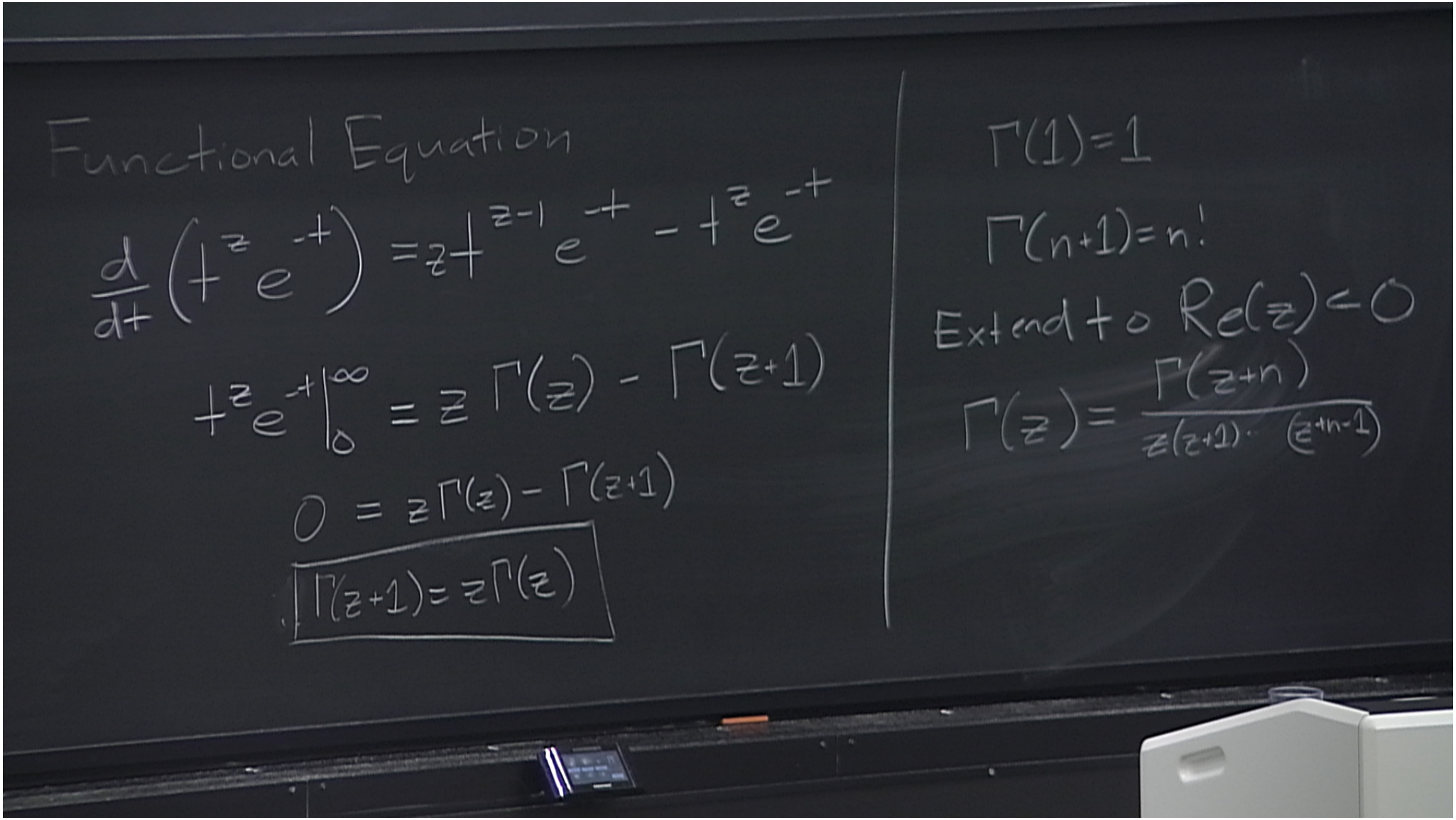
- $u(\varphi)$ + $u \cdot \varphi$ only make sense if φ is a test function
- $u \circ f$ requires strict conditions on f
- linear theory
- $f_n(x) \to f(x)$

  distinct from $\int f_n \to \int f$


Gamma + Zeta
Motivation: Regularization


Gamma + Zeta
Motivation: Regularization
Functional Determinants

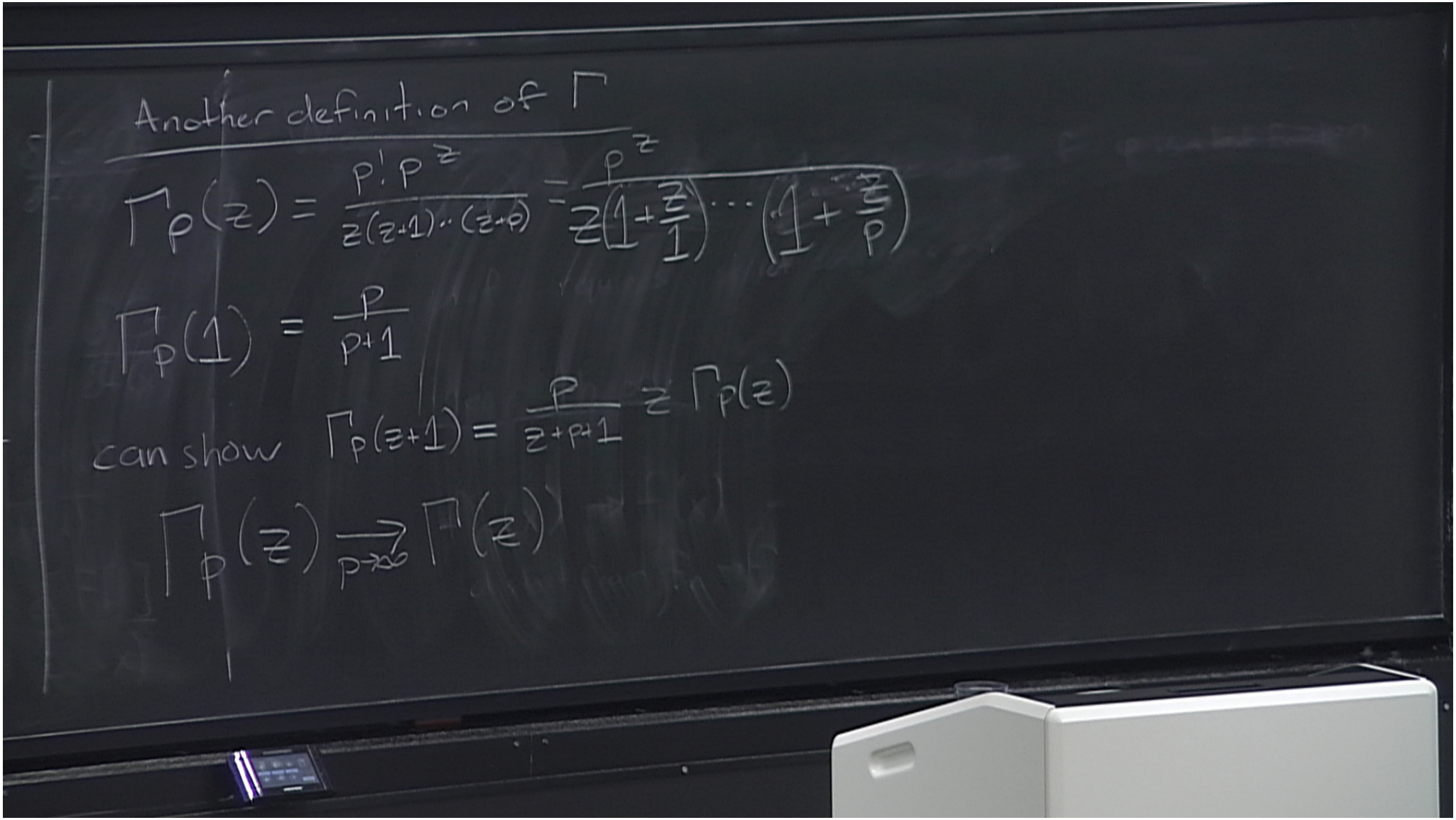

Gamma + Zeta
Motivation: Regularization
Functional Determinants
Gamma
generalization of factorial
$n! = n(n-1)!$
$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad Re(z) > 0$


Functional Equation
$\frac{d}{dt}\left(t^z e^{-t}\right) = z t^{z-1} e^{-t} - t^z e^{-t}$
$t^z e^{-t}\Big|_0^\infty = z\,\Gamma(z) - \Gamma(z+1)$
$0 = z\,\Gamma(z) - \Gamma(z+1)$
$\Gamma(z+1) = z\Gamma(z)$
$\Gamma(1) = 1$
$\Gamma(n+1) = n!$
Extend to $Re(z) < 0$
$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$

Let $z = -n + 1 + \epsilon$

$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{(\epsilon - n + 1) \cdots (\epsilon - 1)\epsilon} = \frac{(-1)^{n-1}}{(n-1)!\,\epsilon}$$

Γ has poles at $z = -n$ with residue $\frac{(-1)^n}{n!}$


Another definition of Γ
$\Gamma_p(z) = \frac{p!\,p^z}{z(z+1)\cdots(z+p)} = \frac{p^z}{z\left(1+\frac{z}{1}\right)\cdots\left(1+\frac{z}{p}\right)}$
$\Gamma_p(1) = \frac{p}{p+1}$
can show $\Gamma_p(z+1) = \frac{p}{z+p+1}\, z\,\Gamma_p(z)$
$\Gamma_p(z) \xrightarrow[p\to\infty]{} \Gamma(z)$

Weierstrass formula

$$p^z = e^{z \log p}$$

$$\Gamma_p(z) = \frac{p^z}{z\left(1+\frac{z}{1}\right)\cdots\left(1+\frac{z}{p}\right)}$$

$$= \frac{e^{z(\log p - 1 - 1/2 \cdots - 1/p)}\, e^{z + z/2 + \cdots z/p}}{z\left(1+\frac{z}{1}\right)\cdots\left(1+\frac{z}{p}\right)}$$

Euler-Mascheroni constant

$$\gamma = \lim_{p\to\infty} 1 + \frac{1}{2} + \cdots \frac{1}{p} - \log p \approx 0.577$$

test function

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}$$

test function

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}$$

found all of the poles already

Γ has no zeros

$$\frac{1}{\Gamma(z)}\frac{1}{\Gamma(-z)} = -z^2 \prod_{p=1}^{\infty} \left(1+\frac{z}{p}\right)e^{-\frac{z}{p}} \left(1-\frac{z}{p}\right)e^{\frac{z}{p}}$$

$$\frac{1}{\Gamma(z)}\frac{1}{\Gamma(1-z)} = -z^2 \prod_{p=1}^{\infty} \left(1-\frac{z^2}{p^2}\right) \qquad \frac{\sin(\pi z)}{\pi} = z \prod_{p=1}^{\infty} \left(1-\frac{z^2}{p^2}\right)$$

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}}$$