Title: PSI 2015/2016 Lie Groups and Algebras - Gang Xu -4

Date: Sep 02, 2015 09:00 AM

URL: http://pirsa.org/15090012

Abstract:

Everything about SU(2)

Recap

$$[t_a, t_b] = i f_{abc} \underbrace{t_c}_{}$$

$$[T_a]_{bc} = -i f_{abc}$$

about SU(2)

$$[t_a, t_b] = i f_{abc} \underbrace{t_c} \rightsquigarrow [t_a, t_b] = i f_{ab}{}^{c} t_c$$

$$[T_a]_{bc} = -i f_{abc}$$

$[t_a, t_b] = i f_{ab}{}^{c} t_c$

mpact n-Vectorspace $\mathrm{Span}\{t_a\}$

$\langle t_a | t_b \rangle = \mathrm{Tr}[t_a t_b]$

$= (t_a)_{cd} (t_b)_{dc}$

$\langle t_a | t_b \rangle$

$= \lambda \delta_{ab}$

$i f_{ab}{}^{c}\, t_c$

n-Vector space $\quad \mathrm{Span}\{t_a\}$

$$\langle t_a | t_b \rangle = \mathrm{Tr}[t_a t_b]$$

$$= (t_a)_{cd} (t_b)_{dc}$$

$$\langle t_a | t_b \rangle = \lambda\, \delta_{ab}$$

Compact $\rightarrow$ Euclidean metric

f_{abc} : real

anti-symmetric ~~first two~~ all indices

Jacobi identity

Compact Lie algebra → finite

→ unitary

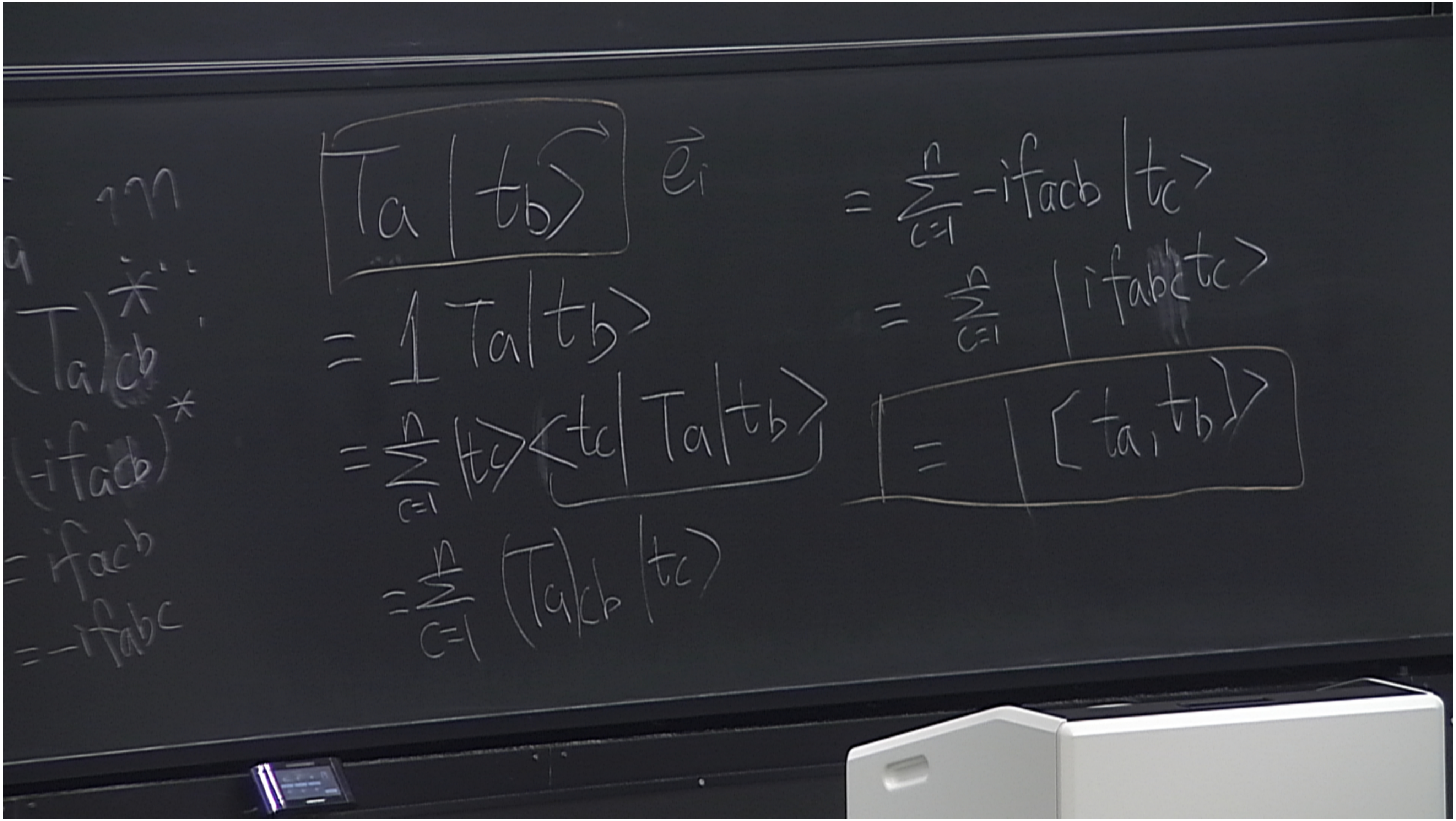
$$T_a^\dagger = T_a$$

$$(T_a^\dagger)_{bc} = (T_a)^*_{cb}$$

$$= (-i f_{acb})^*$$

$$= i f_{acb}$$

$$= -i f_{abc}$$



$\boxed{T_a \, | t_b \rangle}$ $\vec{e}_i$
$= 1 \, T_a | t_b \rangle$
$= \sum_{c=1}^{n} | t_c \rangle \langle t_c | T_a | t_b \rangle$
$= \sum_{c=1}^{n} (T_a)_{cb} | t_c \rangle$
$= \sum_{c=1}^{n} -i f_{acb} | t_c \rangle$
$= \sum_{c=1}^{n} | i f_{abc} t_c \rangle$
$= | [t_a, t_b] \rangle$
$(T_a)^*_{cb}$
$(-i f_{acb})^*$
$= i f_{acb}$
$= -i f_{abc}$

OB(1) is our only help — Lie

$R^2 \times D_2$ . . . .

$SU(2)$ . . . . ✓

— Lie

$$[t_a, t_b] = i f_{abc} t_c$$

$$= i \varepsilon_{abc} t_c \rightarrow$$

$$[J_i, J_j] = i \varepsilon_{ijk} J_k$$

General Recipe. A_1

1) Pick a maximal set of generators that are mutually commuting.

$r = \text{rank}$ $\{H_1, H_2, \dots, H_r\}$

special case $\{J_3\}$

General Recipe · A_1

1) Pick a maximal set of generators that are mutually commuting. $[H_i, H_j]=0$

r = rank $\{H_1, H_2, \ldots, H_r\}$ → Cartan

special case $\{J_3\}$

2) Find combination o

other generators su

that they are eig

of

2) Find combination of other generators such that they are eigen-generators of Cartan generators.

$$[H_i, E_\alpha] = \alpha E_\alpha$$

3)

...ion of

...s such.

...e eigengenerators

...generators.

$$H_i|E_\alpha\rangle = \alpha|E_\alpha\rangle$$

$$...] = \alpha E_\alpha$$

3) Build a vector space

highest weight method.

Representation is the matrix acting on the vector space.

step 1: J_3

step 2: find $J = J_1 + a J_2$

such $[J_3, J] = bJ$

$$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm i J_2)$$

eigenvalues

$$[J_3, J_\pm] = \pm J_\pm$$

$$[J_+, J_-] = J_3$$

$J_2)$

$J_\pm$

step 3

$$J_3|m\rangle = m|m\rangle$$

$$J_3\left(J_\pm|m\rangle\right) = \left([J_3, J_\pm] + J_\pm J_3\right)|m\rangle$$

$$= \pm J_\pm|m\rangle + J_\pm m|m\rangle$$

$$= (m\pm 1)\, J_\pm|m\rangle$$

$$J_\pm |m\rangle \propto |m \pm 1\rangle \qquad \langle m|m'\rangle = \delta_{mm'}$$

$$J_- |m\rangle = N(m)\, |m-1\rangle$$

$$\langle m| J_+ J_- |m\rangle = \| J_- |m\rangle \|^2$$

$$= N^2(m)$$

$$\langle m| [J_+, J_-] + J_- J_+ |m\rangle$$

$$= m + \| J_+ |m\rangle \|^2$$

$$\langle m-1 | \underbrace{J_- | m\rangle} = N(m)$$

$$-J_+ | m\rangle$$

$$2$$

$$(J_+ | m-1\rangle, | m\rangle = N(m)$$

$$J_+ | m-1\rangle = N(m) | m\rangle$$

$$J_+ | m\rangle = N(m+1) | m+1\rangle$$

$r = \text{rank}\{H_1, H_2, \ldots, H_r\}$ special case $\{J_3\}$ $[H_i, L_g$

$J_\pm |m\rangle \propto |m \pm 1\rangle$ $\quad \langle m|m'\rangle = \delta_{mm'}$.

$\underline{J_- |m\rangle = N(m) |m-1\rangle}$

$\langle m| [J_+, J_-] + J_- J_+ |m\rangle$

$= m + \| J_+ |m\rangle \|^2$

$\langle m| J_+ J_- |m\rangle = \| J_- |m\rangle \|^2$

$= N^2(m) = m + N^2(m+1)$

Resolve

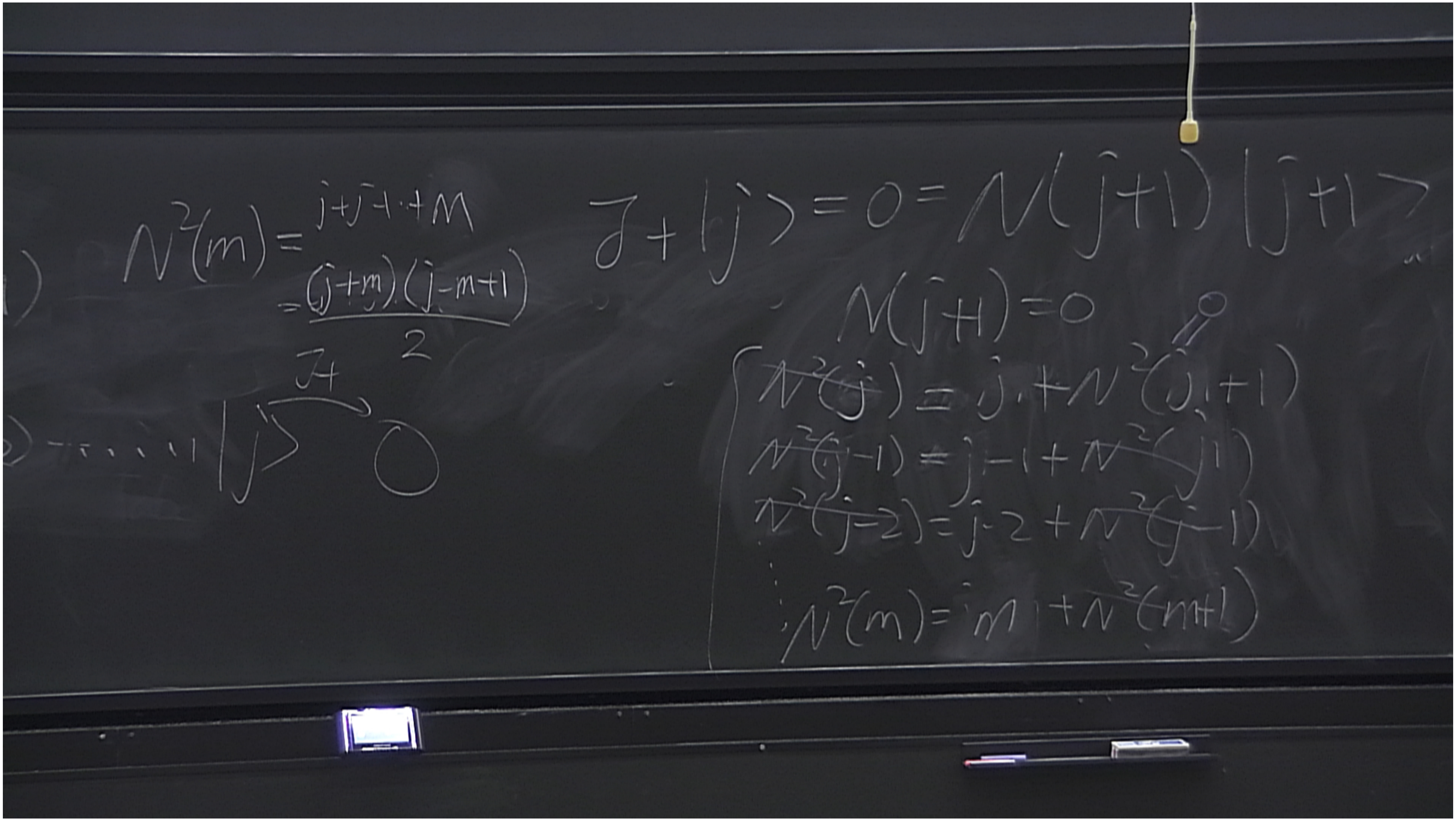
$$N^2(m) = m + N^2(m+1)$$

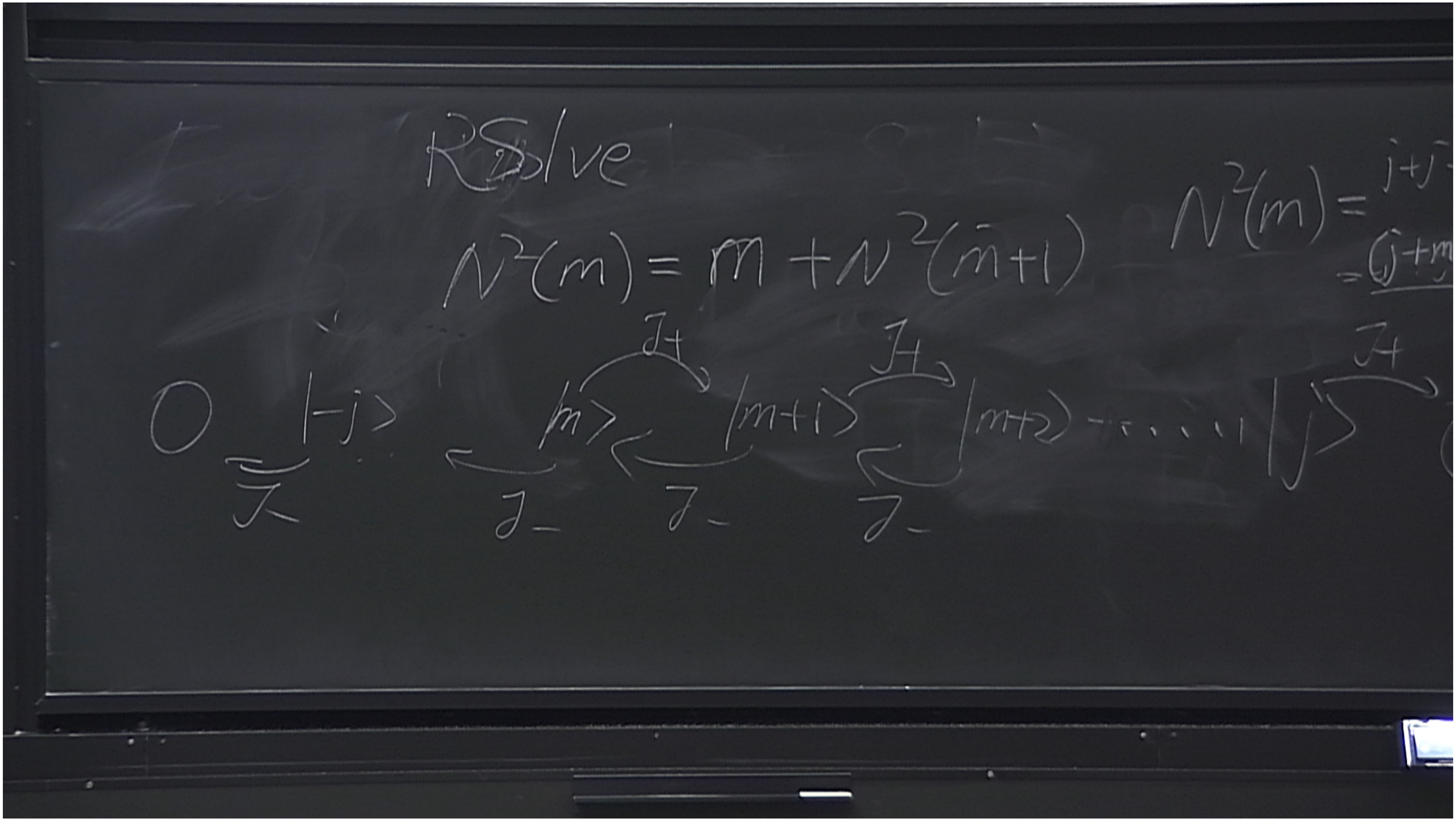
$$|m\rangle \xrightarrow{J_+} |m+1\rangle \xrightarrow{J_+} |m+2\rangle \cdots\cdots |j\rangle \xrightarrow{J_+} 0$$

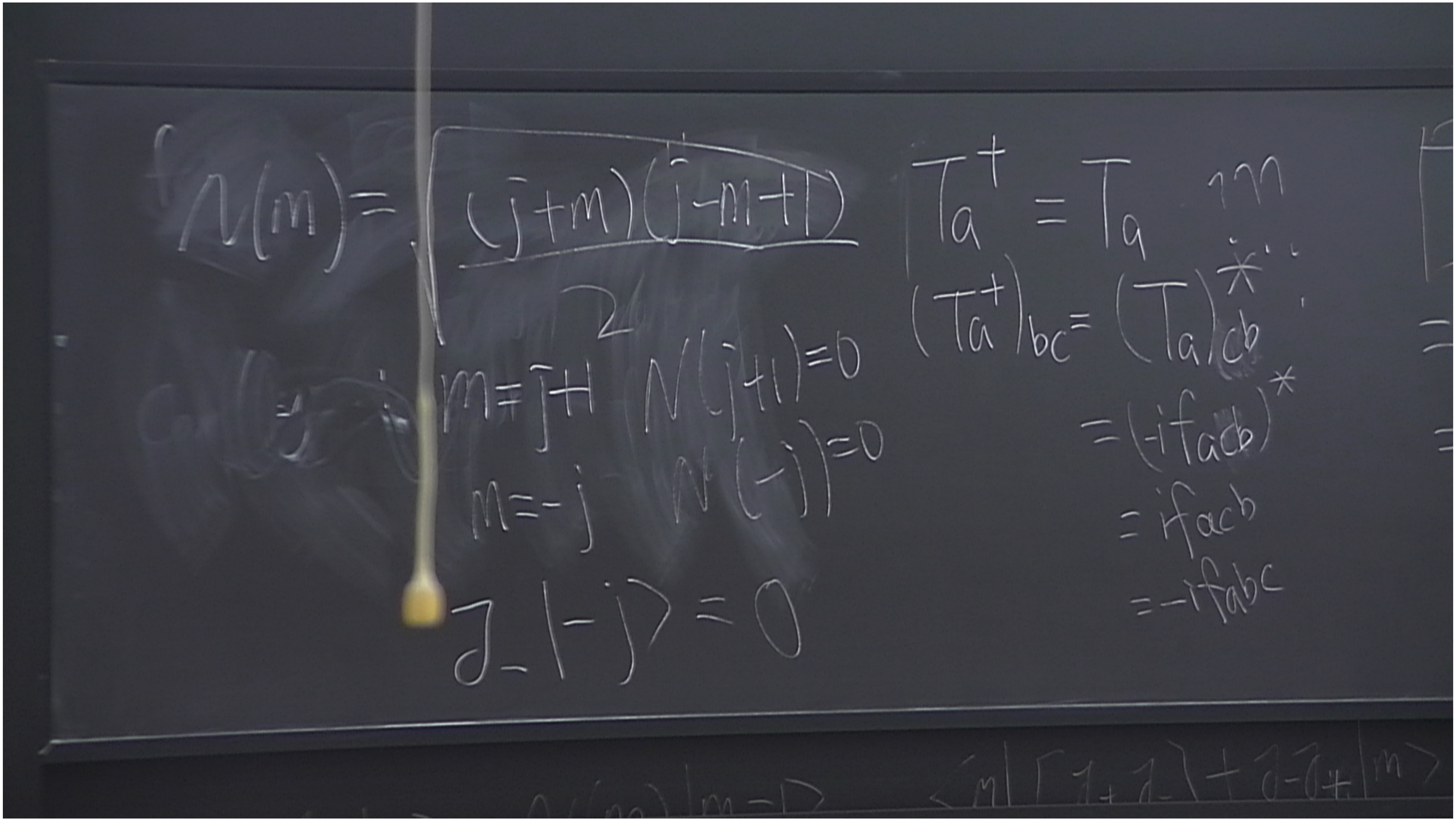
$$J_+|j\rangle = 0 = N(j+1)\,|j+1\rangle$$

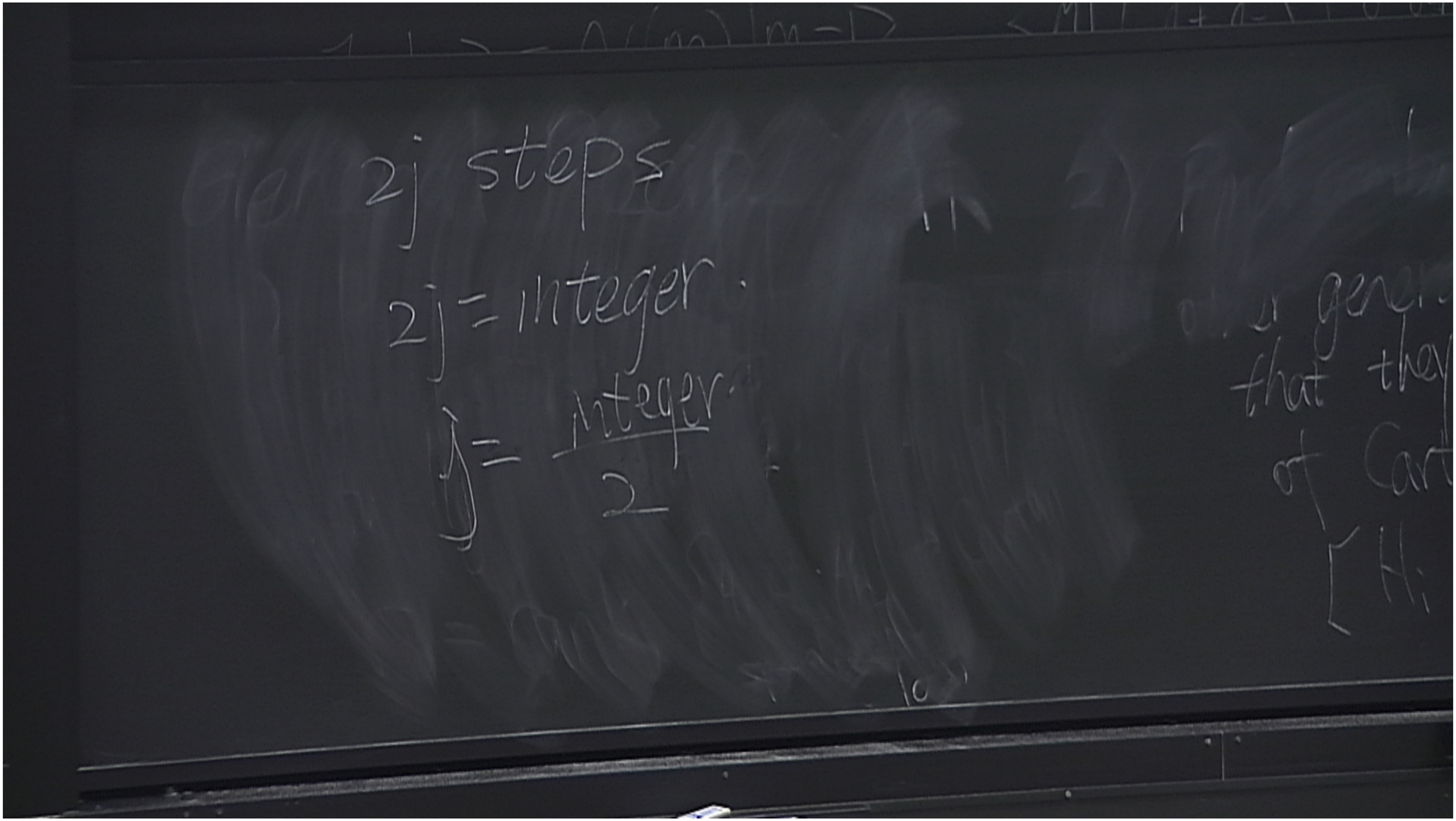
$$N(j+1) = 0$$

$$\begin{cases} N^2(j) = j + N^2(j+1) \\ N^2(j-1) = j-1 + N^2(j) \\ N^2(j-2) = j-2 + N^2(j-1) \\ \vdots \\ N^2(m) = j-m + N^2(m+1) \end{cases}$$


N²(m) = j+j-1+...+m = (j+m)(j-m+1)/2
J+
|j⟩
0
J+|j⟩ = 0 = N(j+1)|j+1⟩
N(j+1) = 0
N²(j) = j + N²(j+1)
N²(j-1) = j-1 + N²(j)
N²(j-2) = j-2 + N²(j-1)
N²(m) = m + N²(m+1)


Resolve
N²(m) = m + N²(m+1)
N²(m) = j+j
0 |−j⟩ |m⟩ |m+1⟩ |m+2⟩ ...... |j⟩
J+
J−


$N(m) = \sqrt{\frac{(j+m)(j-m+1)}{2}}$
$m = j+1 \quad N(j+1) = 0$
$m = -j \quad N(-j) = 0$
$J_- |-j\rangle = 0$
$T_a^\dagger = T_a$
$(T_a^\dagger)_{bc} = (T_a)^*_{cb}$
$= (-i f_{acb})^*$
$= i f_{acb}$
$= -i f_{abc}$


2j steps
2j = integer
j = integer / 2
that they
of Cart

$$\rangle = 0 = N(\hat{j}+1)\,|\hat{j}+1\rangle$$

$$N(\hat{j}+1) = 0$$

$$N^2(\hat{j}) = \hat{j} + N^2(\hat{j}+1)$$

$$N^2(\hat{j}-1) = \hat{j}-1 + N^2(\hat{j})$$

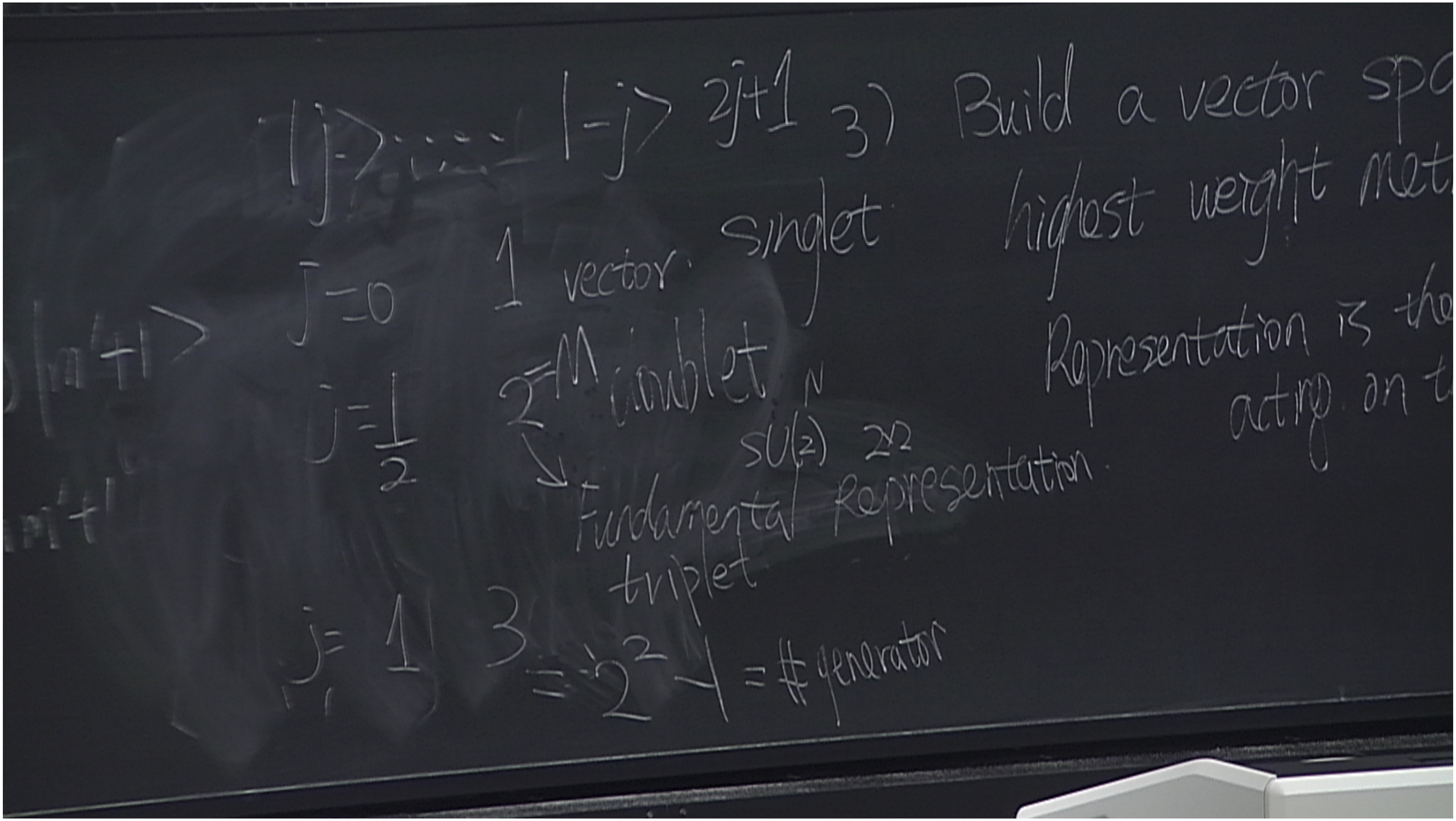
$$N^2(\hat{j}-2) = \hat{j}-2 + N^2(\hat{j}-1)$$

$$\vdots$$

$$N^2(m) = \hat{m} + N^2(m+1)$$

$$\{|m\rangle\}$$

$$|m\rangle = \left.\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\} 2j+1$$


|j> ...... |-j> 2j+1
3) Build a vector spa
highest weight met
singlet
j=0 1 vector
Representation is the
j=1/2 2=N doublet
acting on t
SU(2) 2×2
Fundamental Representation
triplet
j=1 3 = 2^2 - 1 = # generator