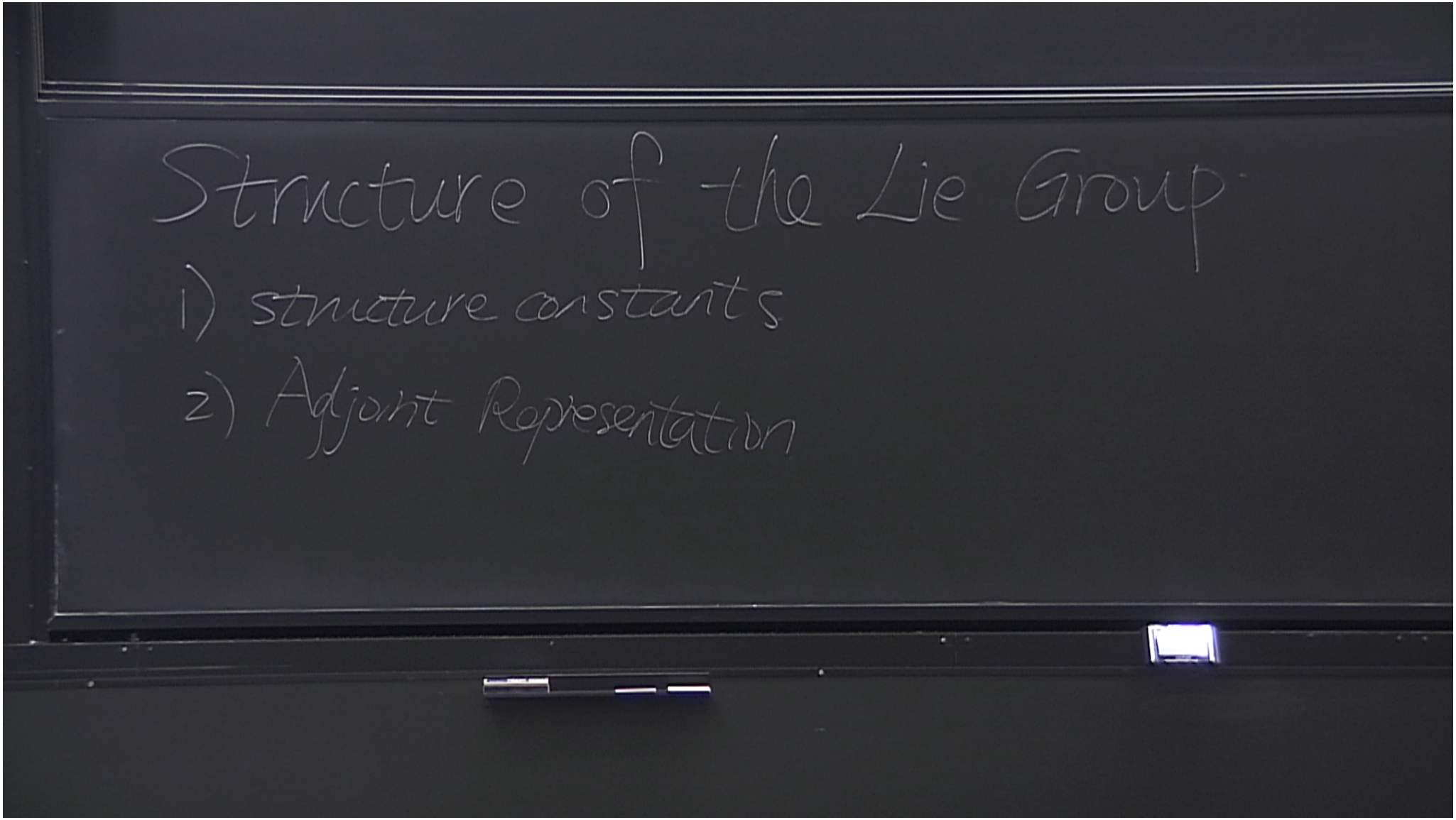
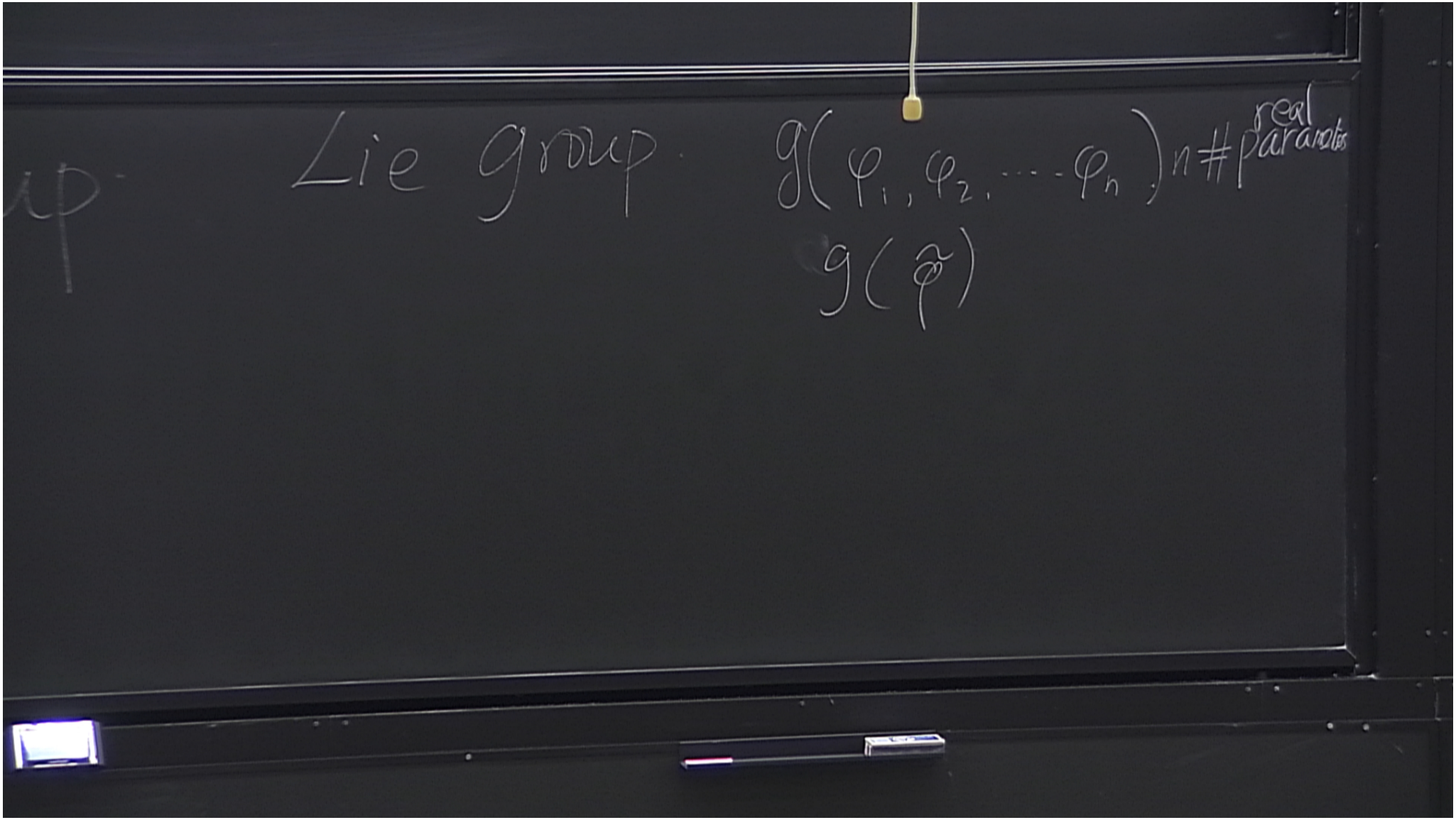
Title: PSI 2015/2016 Lie Groups and Algebras - Gang Xu -3

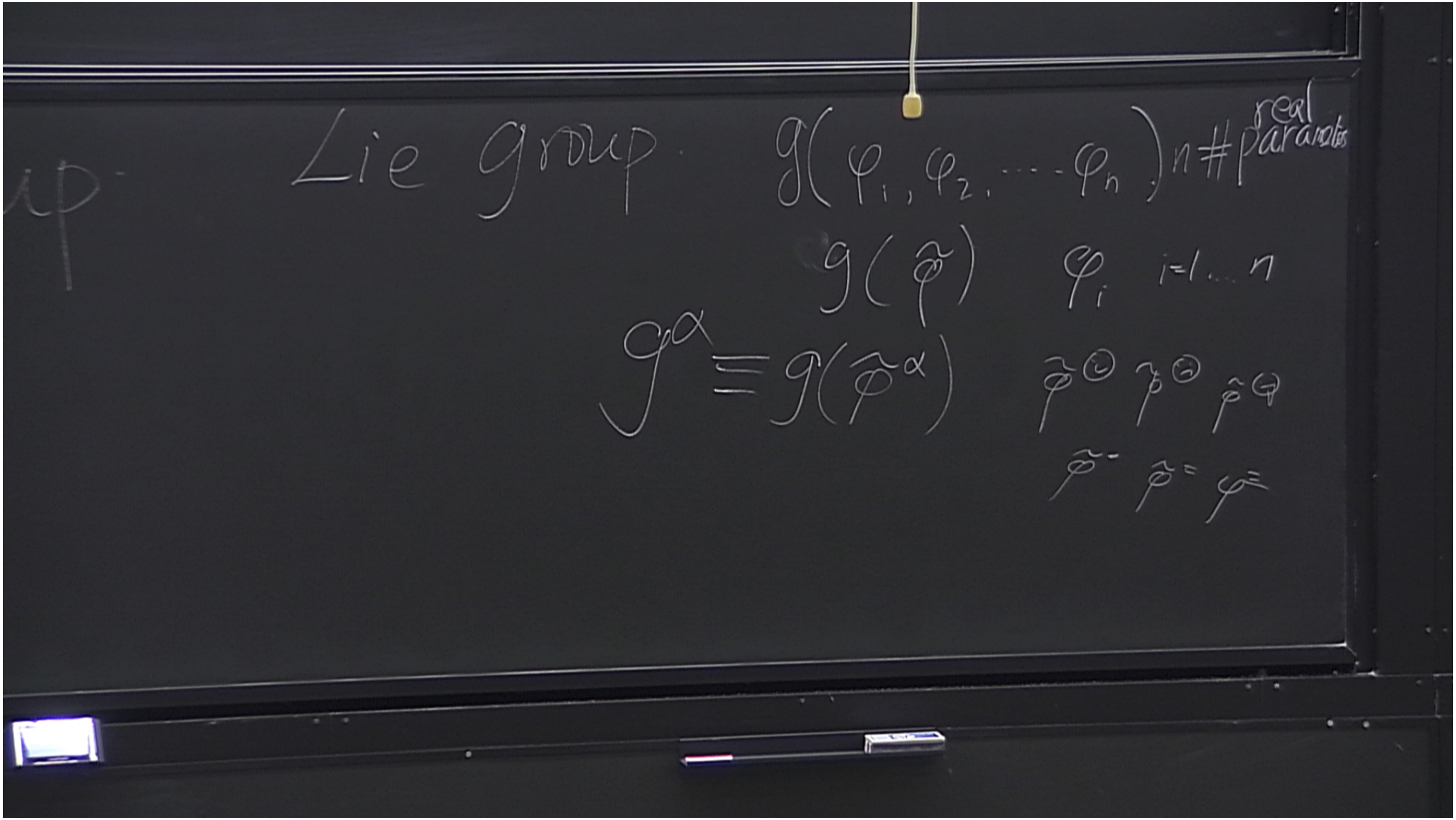
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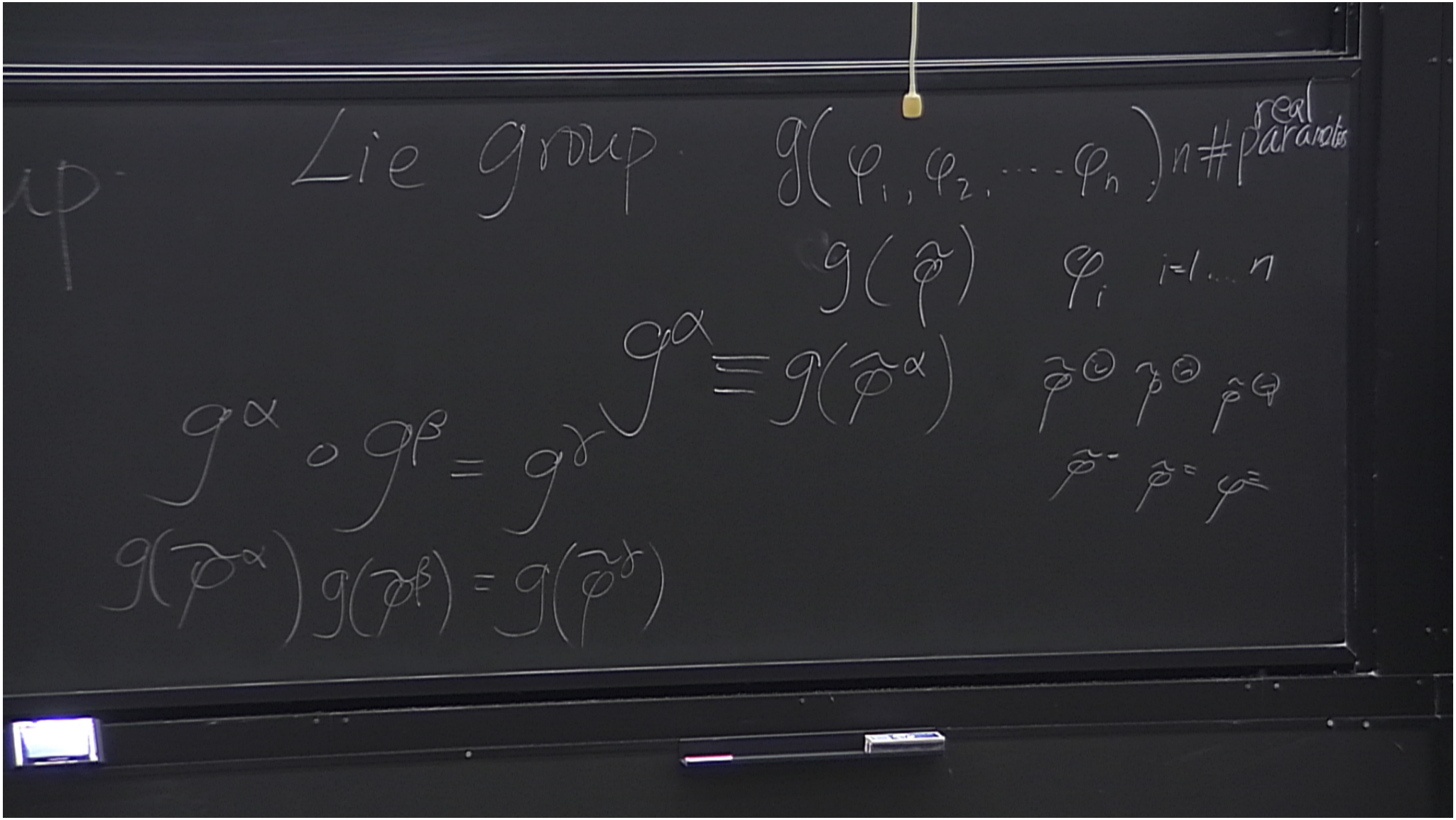
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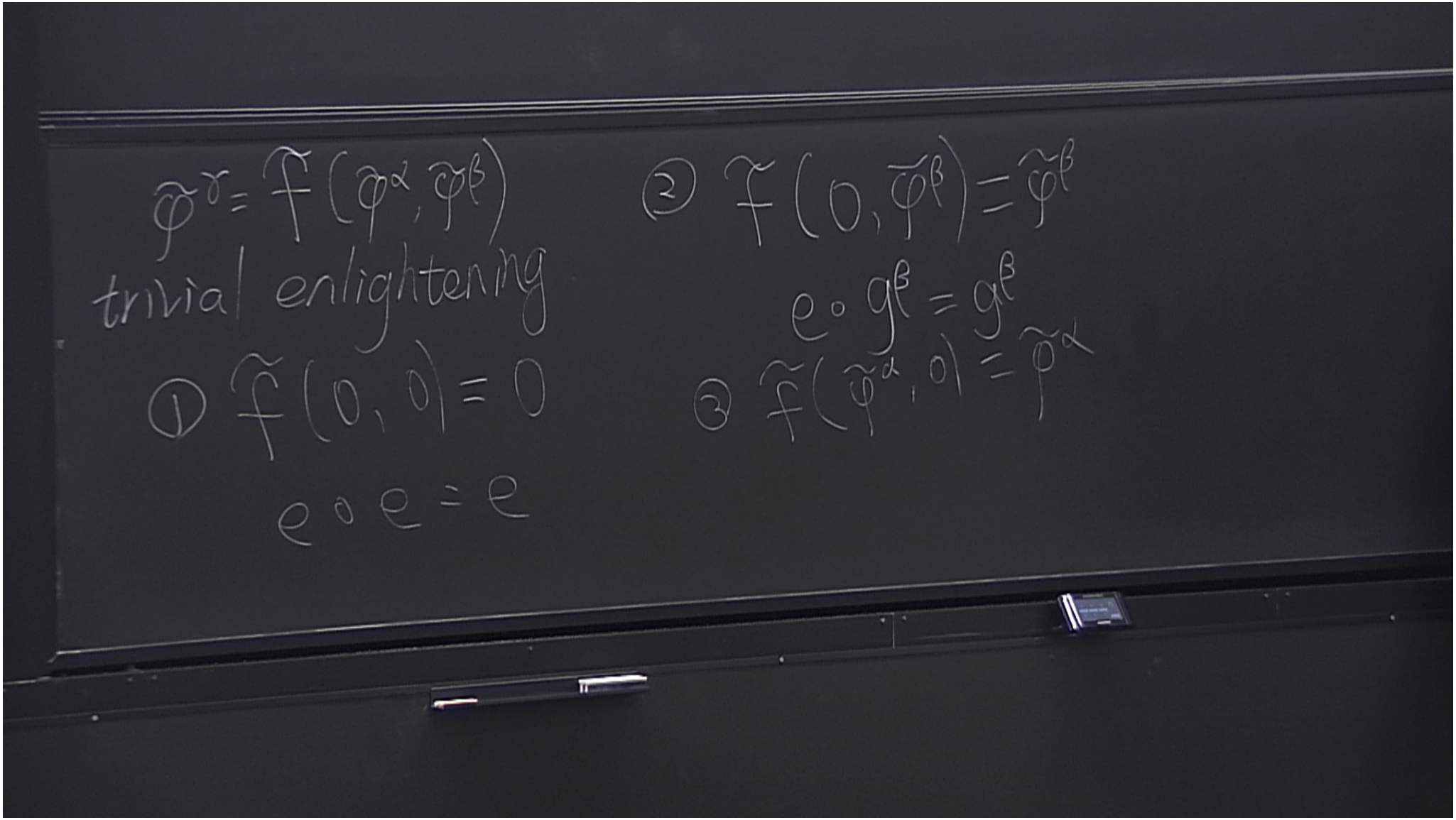
Abstract:

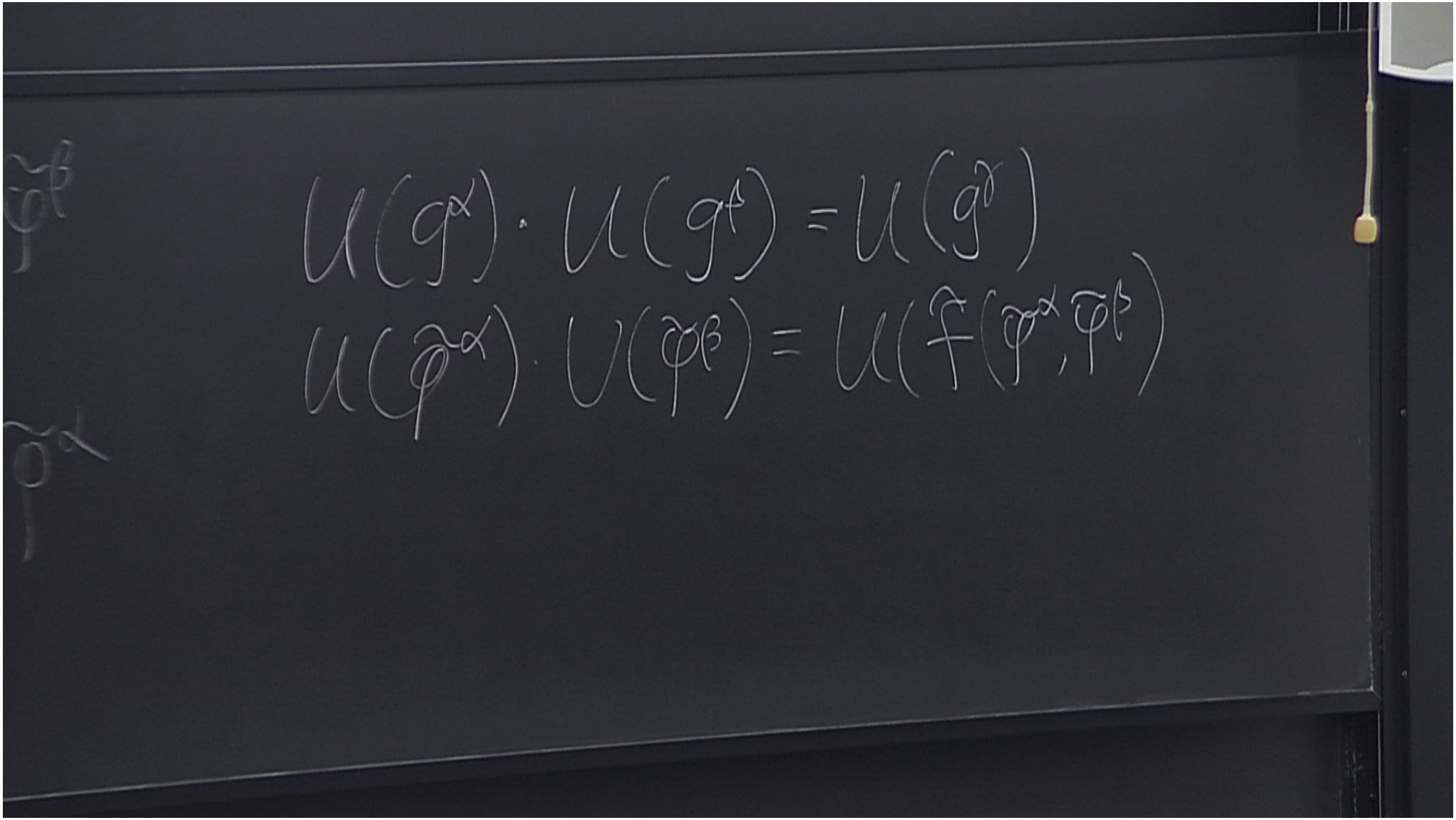

Structure of the Lie Group
1) structure constants
2) Adjoint Representation

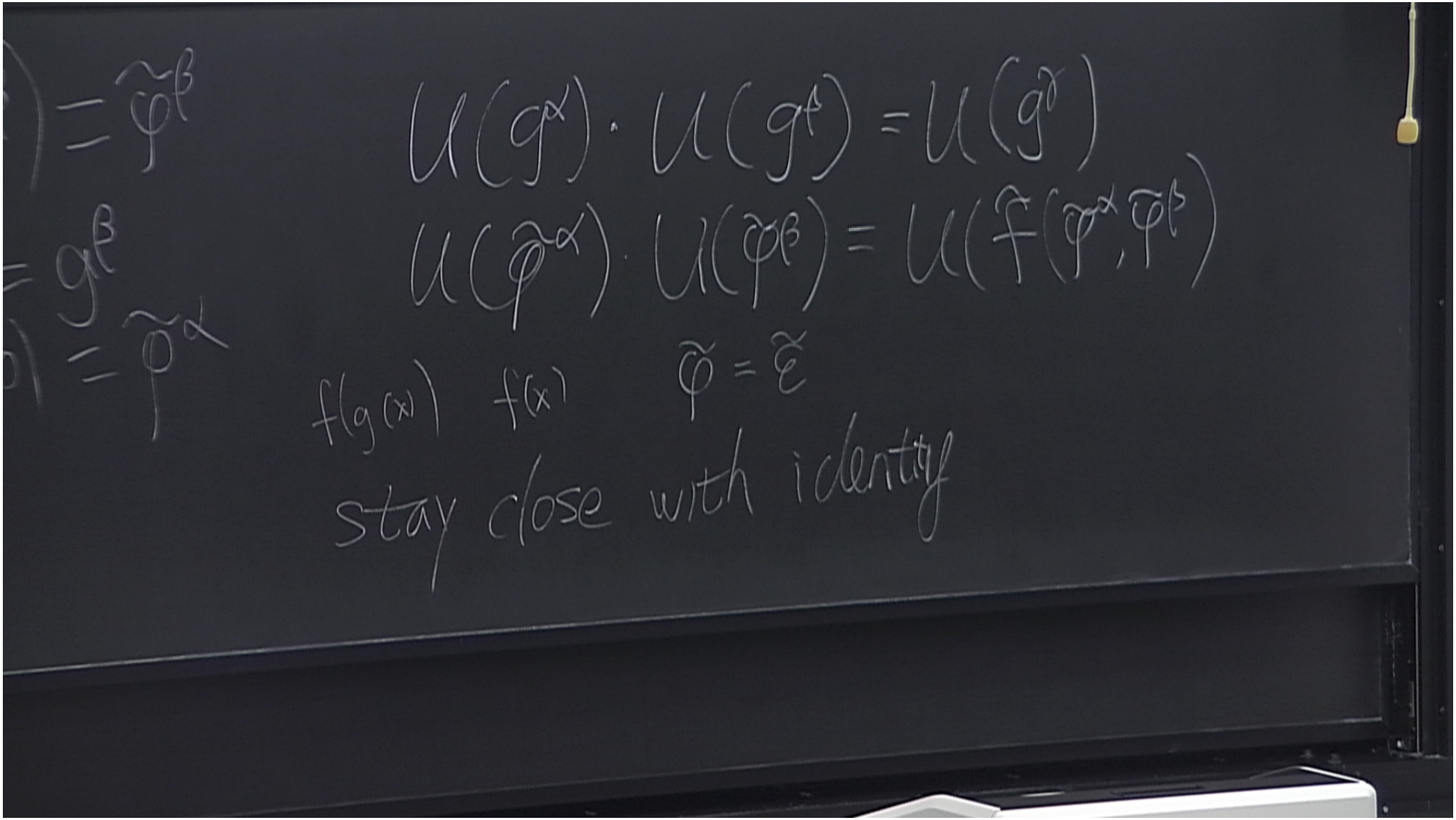

Lie group
$g(\varphi_1, \varphi_2, \cdots \varphi_n)$ n # real parameters
$g(\vec{\varphi})$

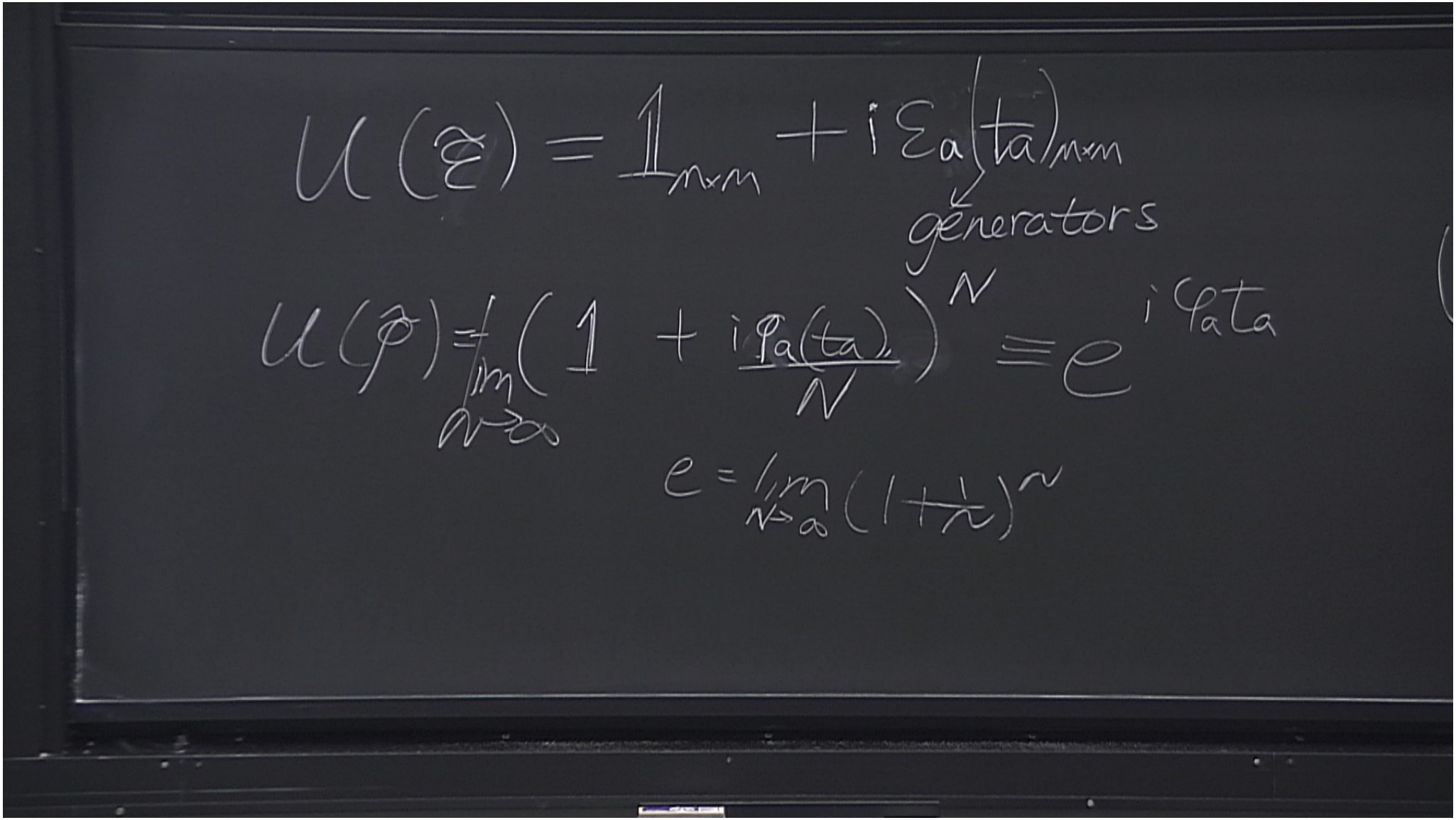

Lie group
$g(\varphi_1, \varphi_2, \ldots \varphi_n)$ n # real parameters
$g(\tilde{\varphi})$ φ_i $i=1 \ldots n$
$g^\alpha \equiv g(\tilde{\varphi}^\alpha)$


Lie group
$g(\varphi_1, \varphi_2, \cdots \varphi_n)$ n # real parameters
$g(\tilde{\varphi})$ φ_i $i=1 \ldots n$
$g^\alpha \cdot g^\beta = g^\gamma$
$g^\alpha \equiv g(\tilde{\varphi}^\alpha)$
$g(\tilde{\varphi}^\alpha)\, g(\tilde{\varphi}^\beta) = g(\tilde{\varphi}^\gamma)$


$\tilde{\varphi}^\gamma = \tilde{f}(\varphi^\alpha, \varphi^\beta)$
trivial enlightening
① $\tilde{f}(0,0) = 0$
$e \circ e = e$
② $\tilde{f}(0, \tilde{\varphi}^\beta) = \tilde{\varphi}^\beta$
$e \circ g^\beta = g^\beta$
③ $\tilde{f}(\tilde{\varphi}^\alpha, 0) = \tilde{\varphi}^\alpha$


$U(\varphi^\alpha) \cdot U(\varphi^\beta) = U(\varphi^\gamma)$
$U(\varphi^\alpha) \cdot U(\varphi^\beta) = U(f(\varphi^\alpha, \varphi^\beta))$


$U(g^\alpha) \cdot U(g^\beta) = U(g^\gamma)$
$U(\tilde{\varphi}^\alpha) \cdot U(\varphi^\beta) = U(f(\varphi^\alpha, \varphi^\beta))$
$f(g(x))$ $f(x)$ $\tilde{\varphi} = \tilde{\varepsilon}$
stay close with identity

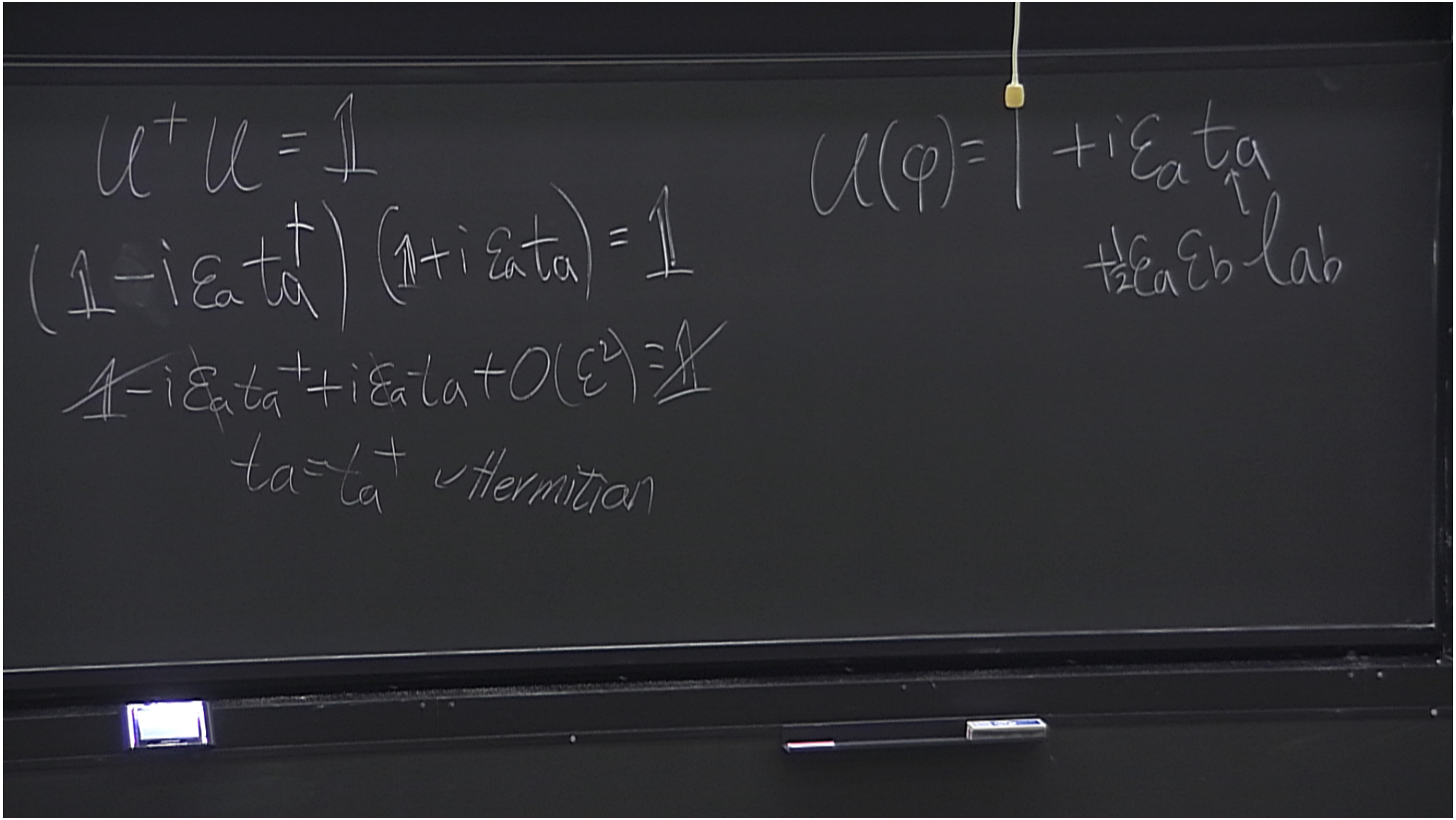

$U(\varepsilon) = 1_{n\times m} + i\varepsilon_a (t_a)_{n\times m}$
generators
$U(\varphi) = \lim_{N\to\infty} \left(1 + i\frac{\varphi_a (t_a)}{N}\right)^N \equiv e^{i\varphi_a t_a}$
$e = \lim_{N\to\infty} \left(1 + \frac{1}{N}\right)^N$

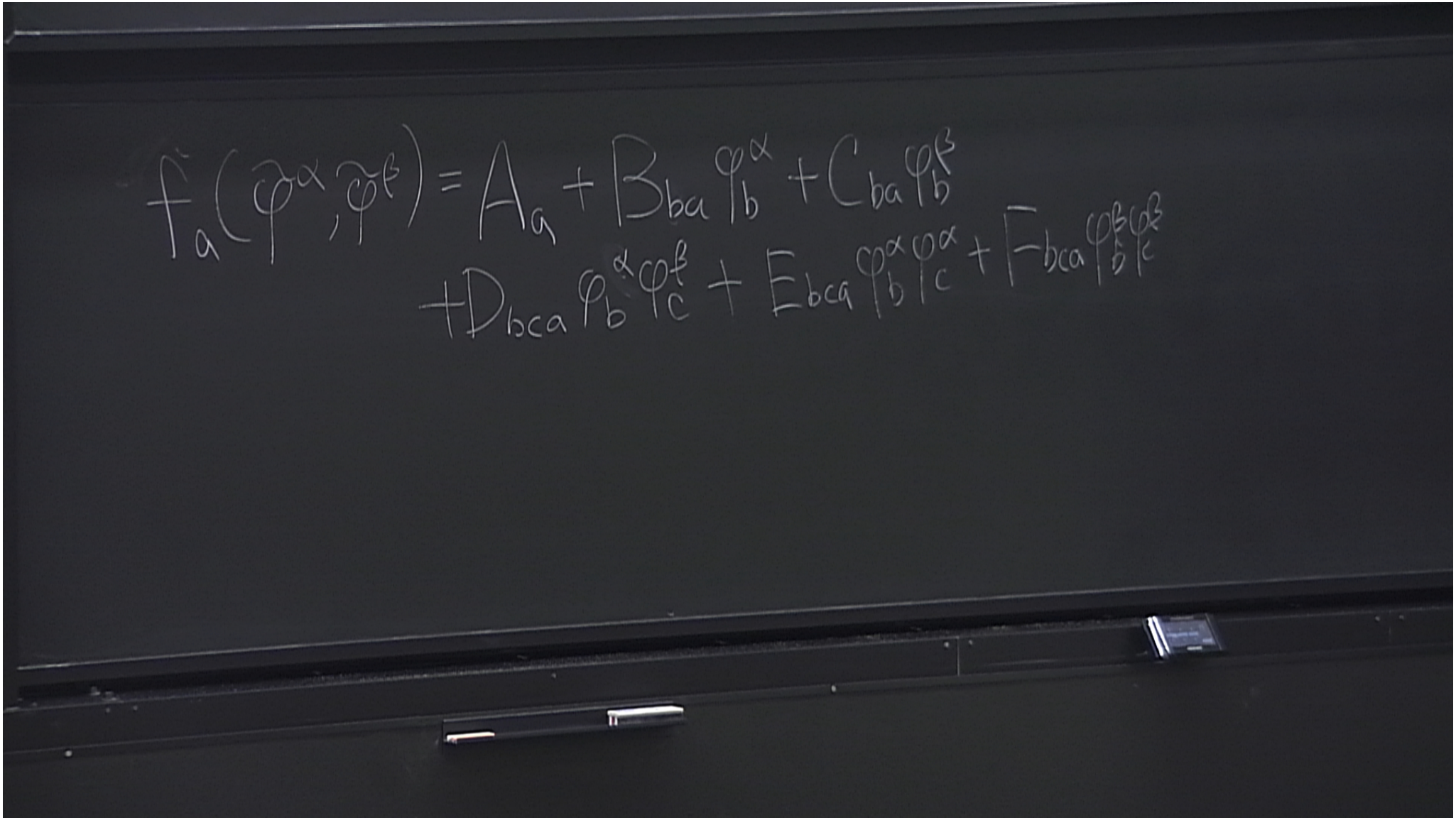
$$U^\dagger U = \mathbb{1}$$

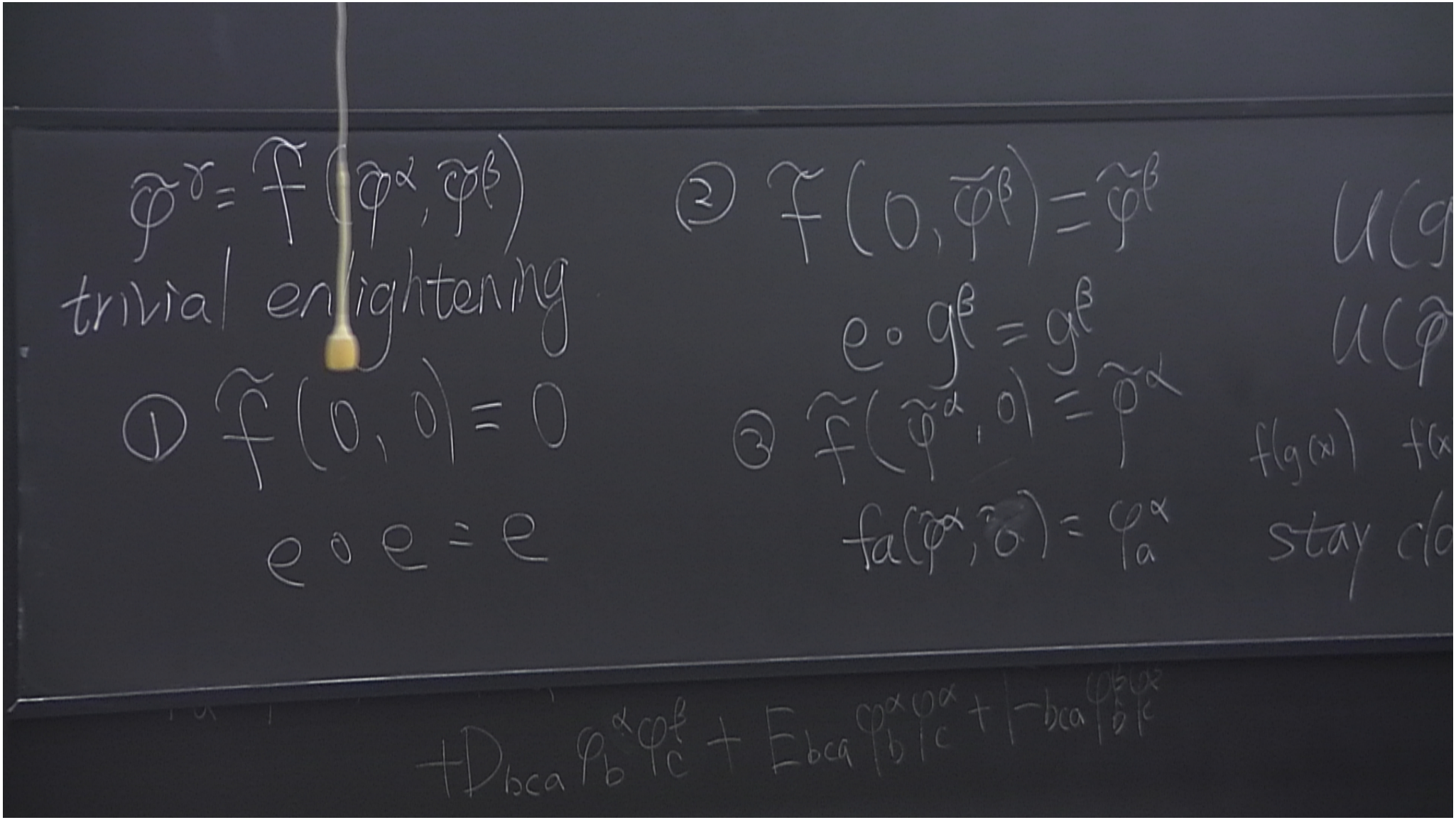
$$(\mathbb{1} - i\varepsilon_a t_a^\dagger)(\mathbb{1} + i\varepsilon_a t_a) = \mathbb{1}$$

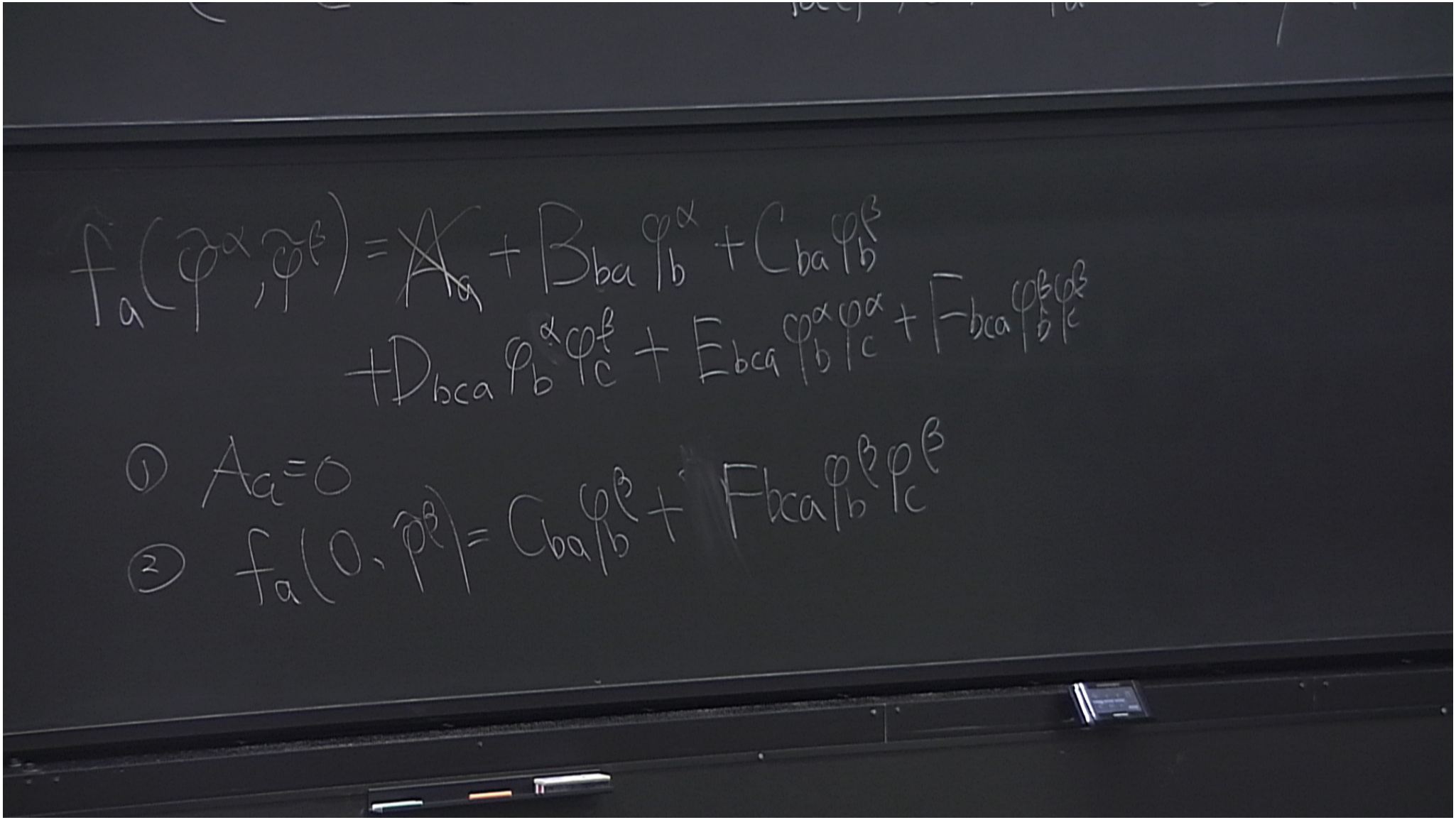
$$\mathbb{1} - i\varepsilon_a t_a^\dagger + i\varepsilon_a t_a + O(\varepsilon^2) = \mathbb{1}$$

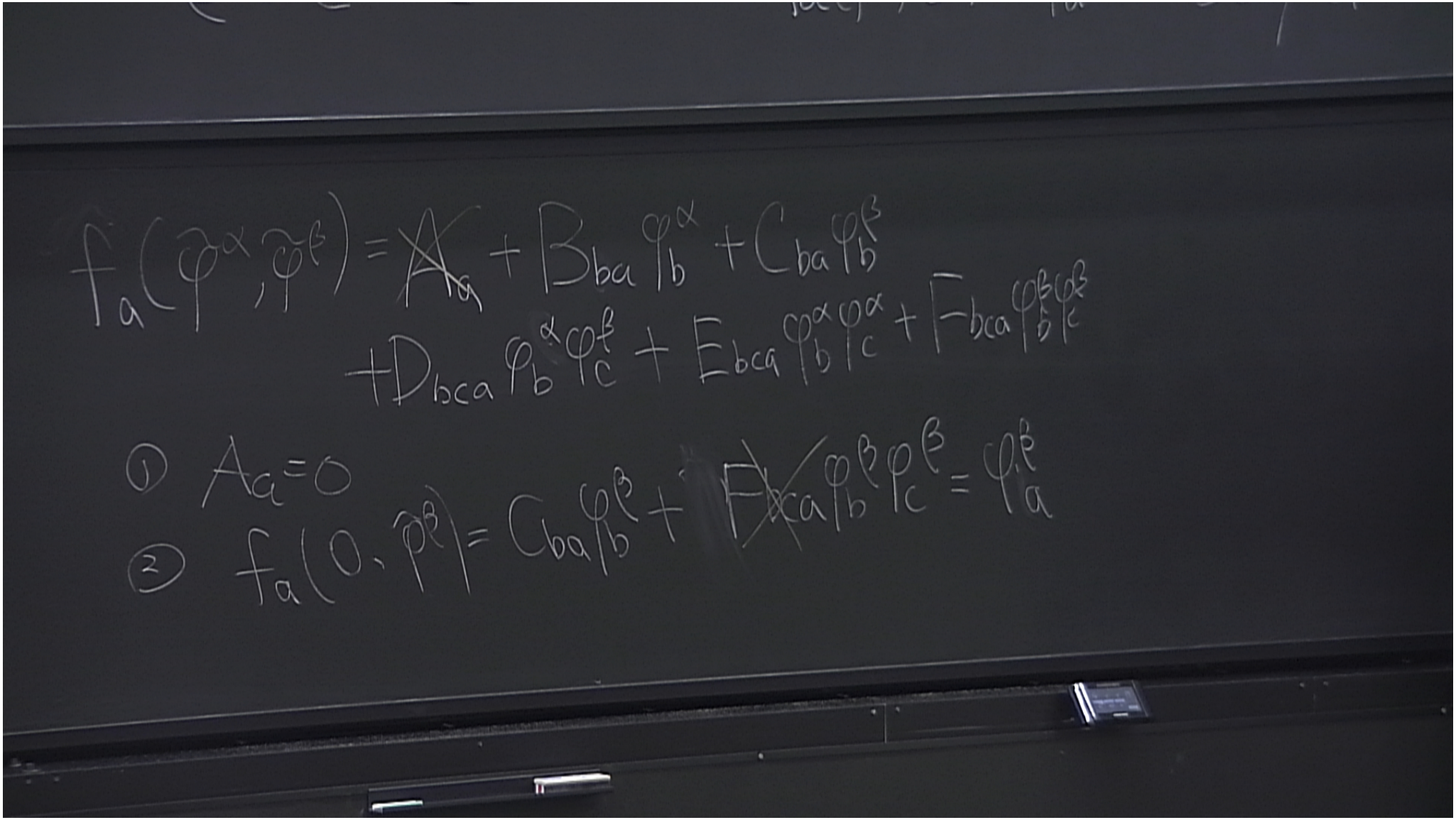
$t_a = t_a^\dagger$ ✓ Hermitian

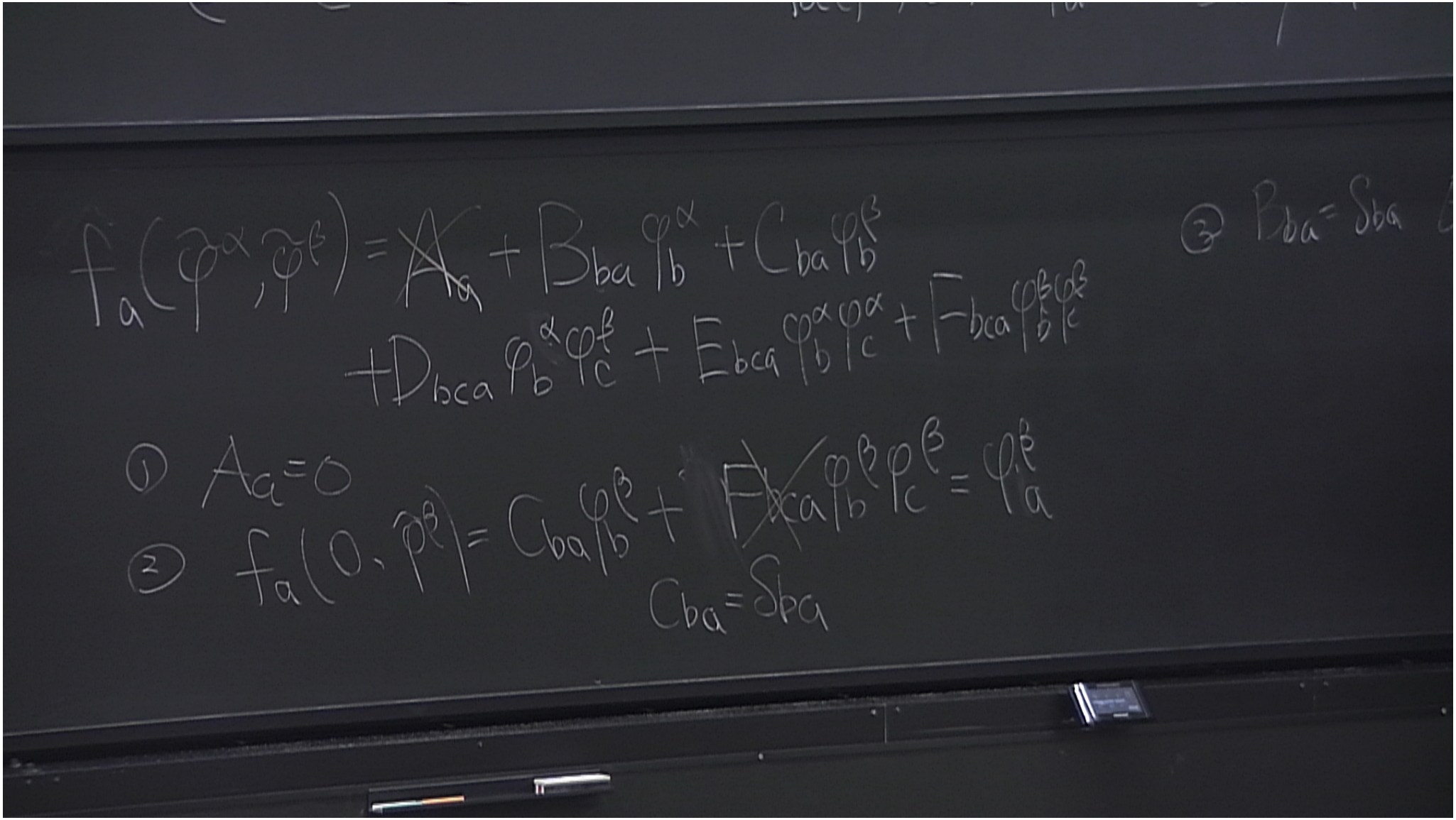

$U^+ U = \mathbb{1}$
$(\mathbb{1} - i\varepsilon_a t_a^\dagger)(\mathbb{1} + i\varepsilon_a t_a) = \mathbb{1}$
$\mathbb{1} - i\varepsilon_a t_a^+ + i\varepsilon_a t_a + O(\varepsilon^2) = \mathbb{1}$
$t_a = t_a^+$ ↳ Hermitian
$U(\varphi) = \mathbb{1} + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a \varepsilon_b t_{ab}$

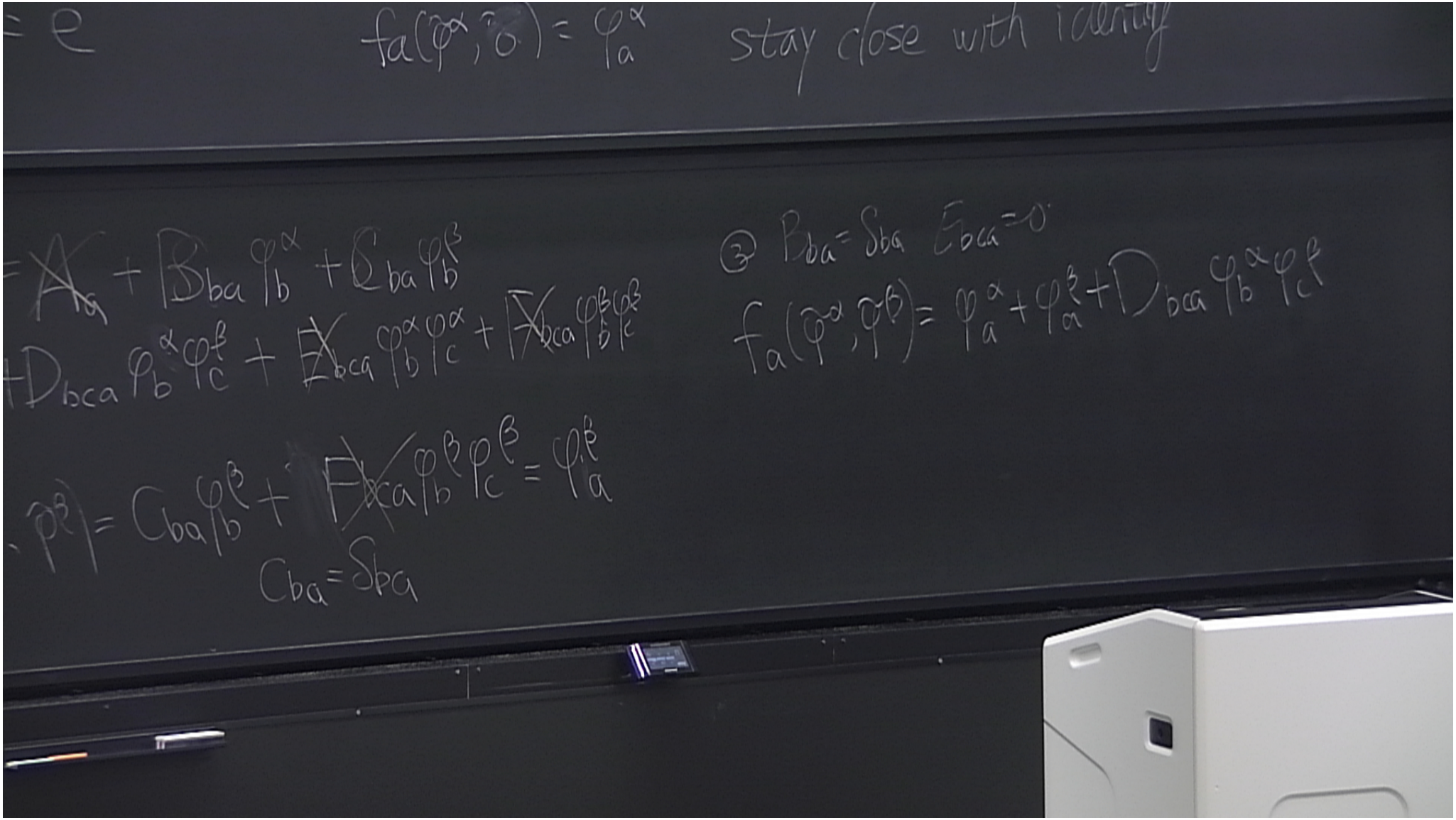

$\tilde{f}_a(\varphi^\alpha, \tilde{\varphi}^\beta) = A_a + B_{ba}\varphi_b^\alpha + C_{ba}\varphi_b^\beta$
$+ D_{bca}\varphi_b^\alpha\varphi_c^\beta + E_{bca}\varphi_b^\alpha\varphi_c^\alpha + \bar{F}_{bca}\varphi_b^\beta\varphi_c^\beta$

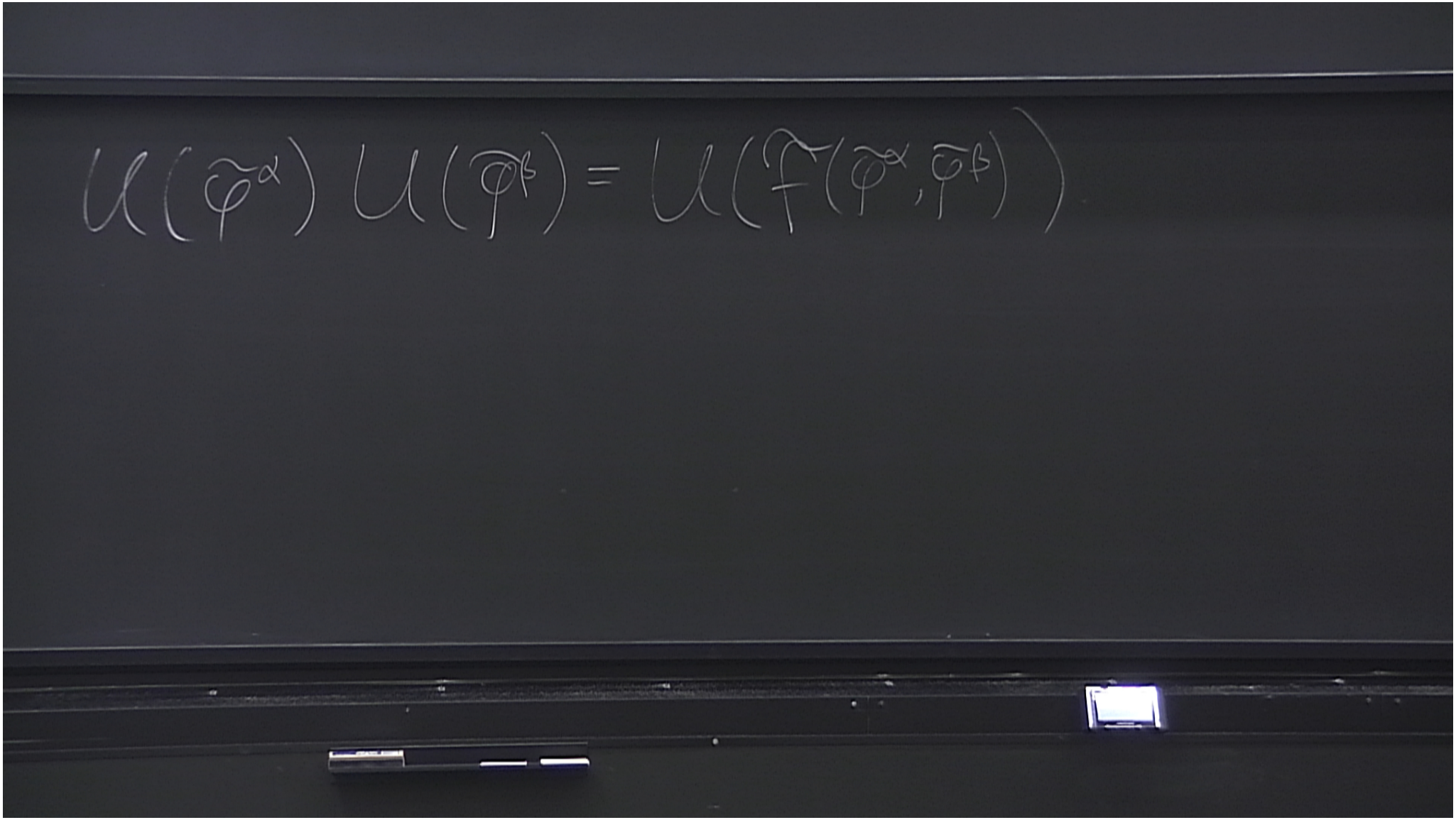

$\tilde{\varphi}^\gamma = \tilde{f}(\varphi^\alpha, \varphi^\beta)$
trivial enlightening
① $\tilde{f}(0,0) = 0$
$e \circ e = e$
② $\tilde{f}(0, \varphi^\beta) = \varphi^\beta$
$e \circ g^\beta = g^\beta$
③ $\tilde{f}(\varphi^\alpha, 0) = \varphi^\alpha$
$f_a(\varphi, 0) = \varphi^\alpha_a$
$u(g$
$u(\varphi$
$f(g(x)$ $f(x$
stay cl
$+D_{bca}\varphi_b^\alpha\varphi_c^\beta + E_{bca}\varphi^\alpha_{b\,c}\varphi^\alpha + \dots - b_{ca}\varphi^{\alpha\alpha}_{b\,c}$

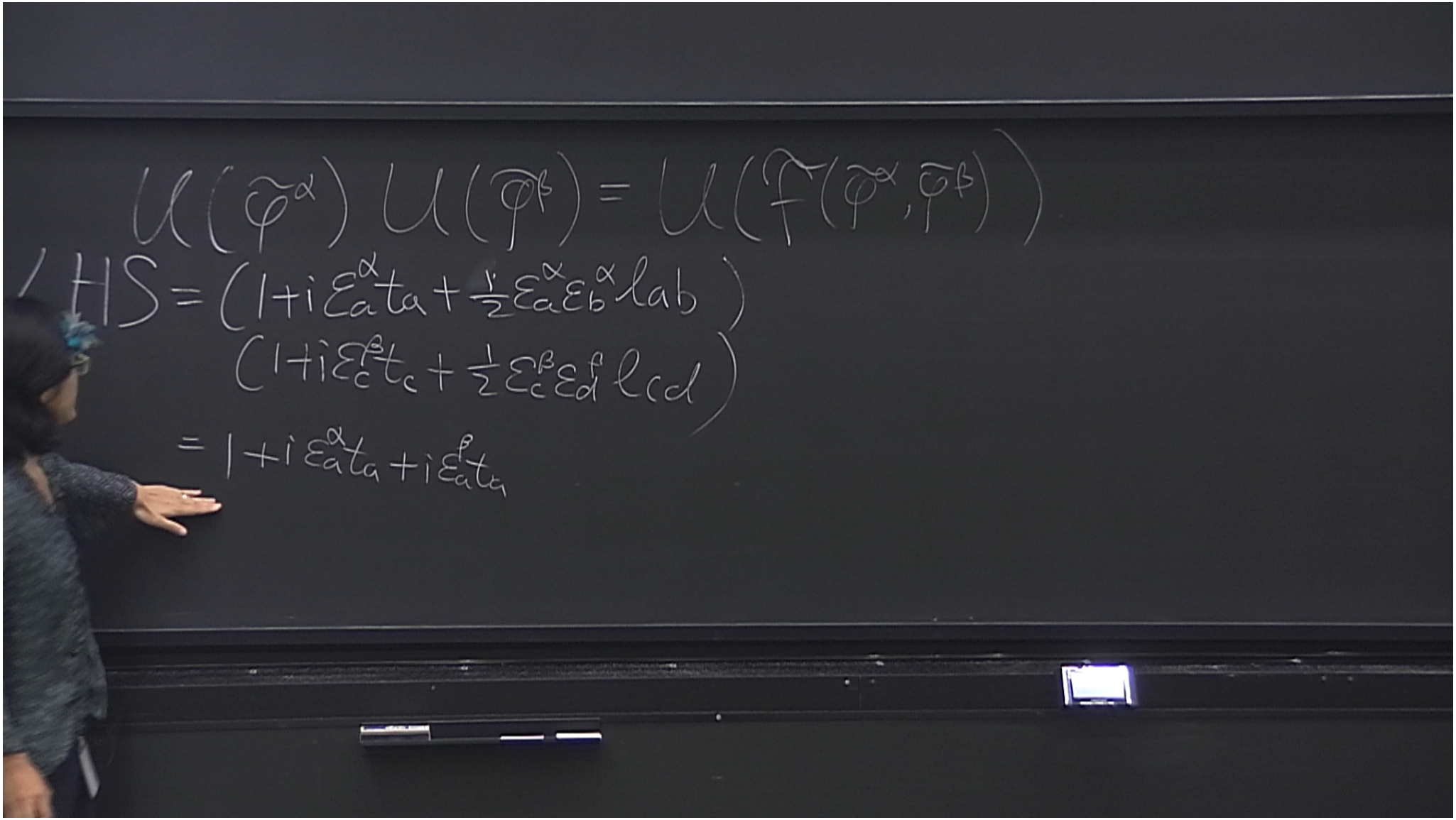

$f_a(\varphi^\alpha, \varphi^\beta) = A_a + B_{ba}\varphi_b^\alpha + C_{ba}\varphi_b^\beta$
$+ D_{bca}\varphi_b^\alpha\varphi_c^\beta + E_{bca}\varphi_b^\alpha\varphi_c^\alpha + F_{bca}\varphi_b^\beta\varphi_c^\beta$
① $A_a = 0$
② $f_a(0, \varphi^\beta) = C_{ba}\varphi_b^\beta + F_{bca}\varphi_b^\beta\varphi_c^\beta$


$f_a(\varphi^\alpha, \tilde{\varphi}^\beta) = A_a + B_{ba}\varphi_b^\alpha + C_{ba}\varphi_b^\beta$
$+ D_{bca}\varphi_b^\alpha\varphi_c^\beta + E_{bca}\varphi_b^\alpha\varphi_c^\alpha + F_{bca}\varphi_b^\beta\varphi_c^\beta$
① $A_a = 0$
② $f_a(0, \tilde{\varphi}^\beta) = C_{ba}\varphi_b^\beta + F_{bca}\varphi_b^\beta\varphi_c^\beta = \varphi_a^\beta$


$f_a(\varphi^\alpha, \varphi^\beta) = A_a + B_{ba}\varphi_b^\alpha + C_{ba}\varphi_b^\beta$
$+ D_{bca}\varphi_b^\alpha\varphi_c^\beta + E_{bca}\varphi_b^\alpha\varphi_c^\alpha + F_{bca}\varphi_b^\beta\varphi_c^\beta$
③ $B_{ba} = S_{ba}$
① $A_a = 0$
② $f_a(0, \varphi^\beta) = C_{ba}\varphi_b^\beta + F_{bca}\varphi_b^\beta\varphi_c^\beta = \varphi_a^\beta$
$C_{ba} = S_{ba}$


= e
$f_a(\varphi^\alpha, 0) = \varphi_a^\alpha$
stay close with identity
$= A_a + B_{ba}\varphi_b^\alpha + C_{ba}\varphi_b^\beta$
$+ D_{bca}\varphi_b^\alpha\varphi_c^\beta + E_{bca}\varphi_b^\alpha\varphi_c^\alpha + F_{bca}\varphi_b^\beta\varphi_c^\beta$
$\varphi^\beta) = C_{ba}\varphi_b^\beta + F_{bca}\varphi_b^\beta\varphi_c^\beta = \varphi_a^\beta$
$C_{ba} = S_{ba}$
③ $B_{ba} = S_{ba}$ $E_{bca} = 0$
$f_a(\varphi^\alpha, \varphi^\beta) = \varphi_a^\alpha + \varphi_a^\beta + D_{bca}\varphi_b^\alpha\varphi_c^\beta$

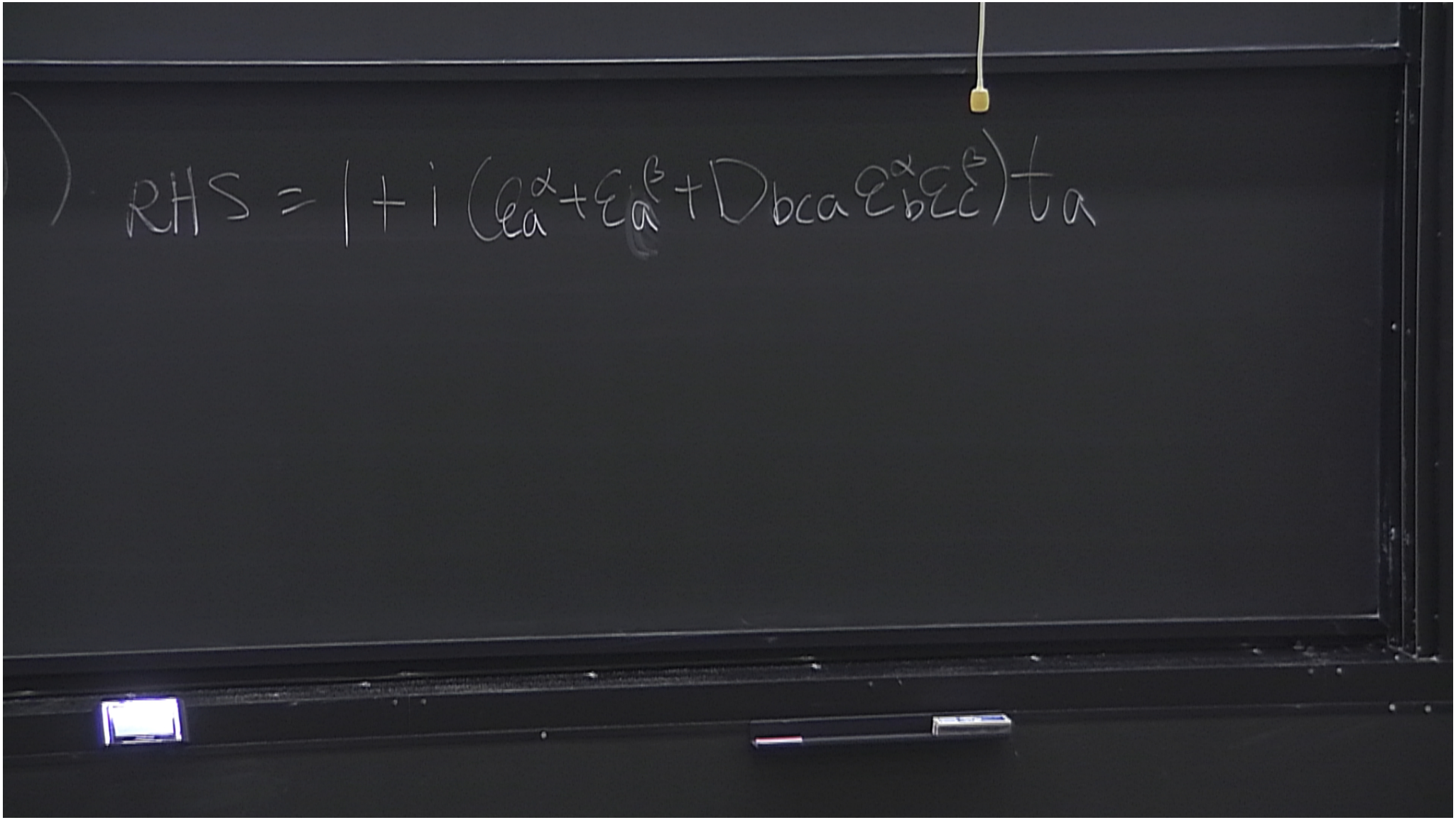

$U(\tilde{\varphi}^\alpha)\, U(\tilde{\varphi}^\beta) = U\left(\tilde{f}(\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta)\right)$


$U(\tilde{\varphi}^\alpha)\,U(\varphi^\beta) = U(f(\varphi^\alpha, \varphi^\beta))$
$HS = (1 + i\varepsilon_a^\alpha t_a + \frac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab})$
$(1 + i\varepsilon_c^\beta t_c + \frac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd})$
$= 1 + i\varepsilon_a^\alpha t_a + i\varepsilon_a^\beta t_a$

$$U(\tilde{\varphi}^\alpha)\,U(\varphi^\beta) = U\big(\tilde{f}(\varphi^\alpha,\varphi^\beta)\big) \qquad \text{RHS}$$

$$\text{LHS} = \left(1 + i\varepsilon_a^\alpha t_a + \tfrac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab}\right)\left(1 + i\varepsilon_c^\beta t_c + \tfrac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd}\right)$$

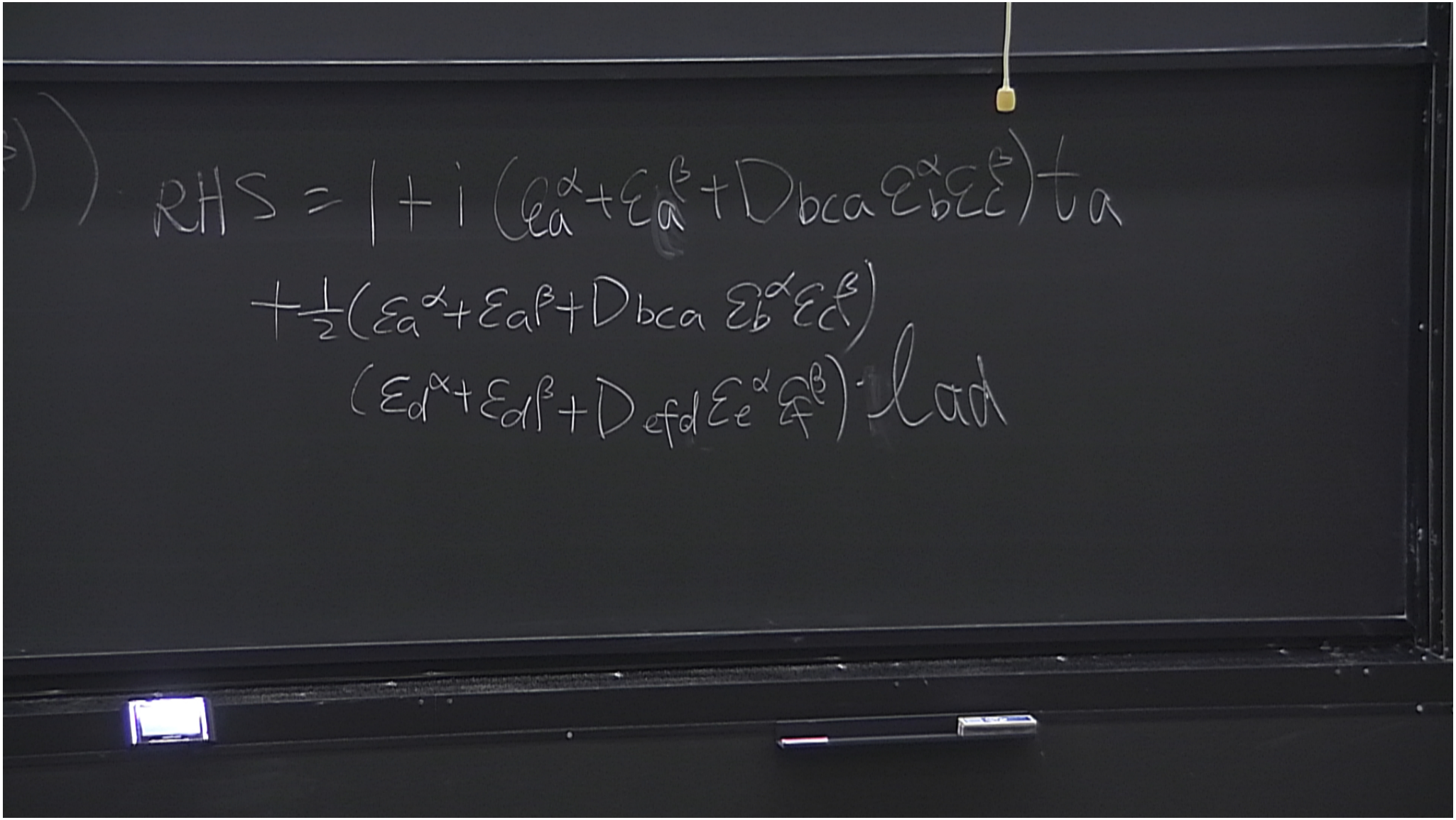
$$= 1 + i\varepsilon_a^\alpha t_a + i\varepsilon_a^\beta t_a - \varepsilon_a^\alpha \varepsilon_b^\beta t_a t_b + \tfrac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab} + \tfrac{1}{2}\varepsilon_a^\beta \varepsilon_b^\beta l_{ab} + \mathcal{O}(\varepsilon^3)$$

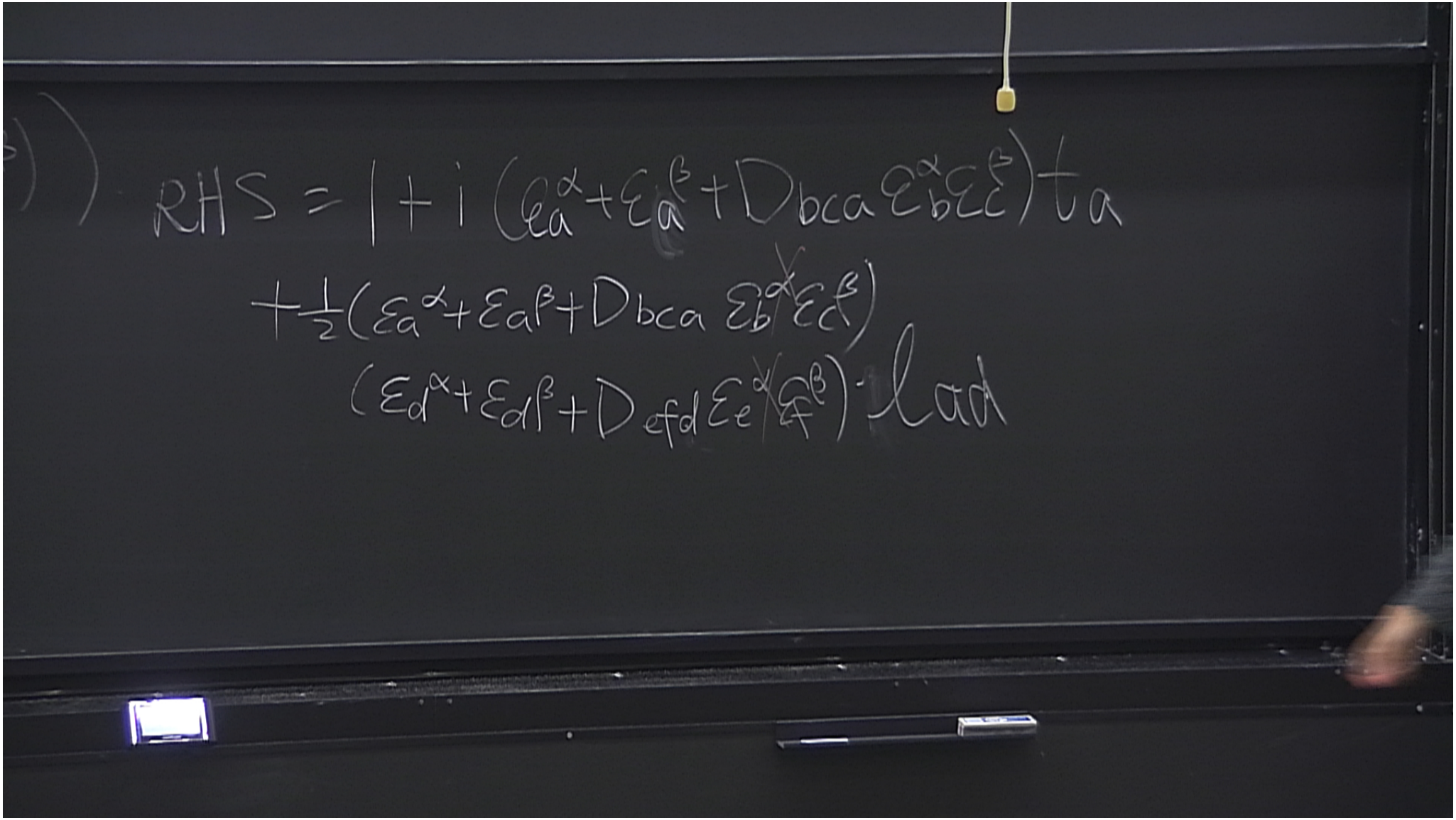

RHS = 1 + i (ε_a^α + ε_a^β + D_bca ε_b^α ε_c^β) t_a

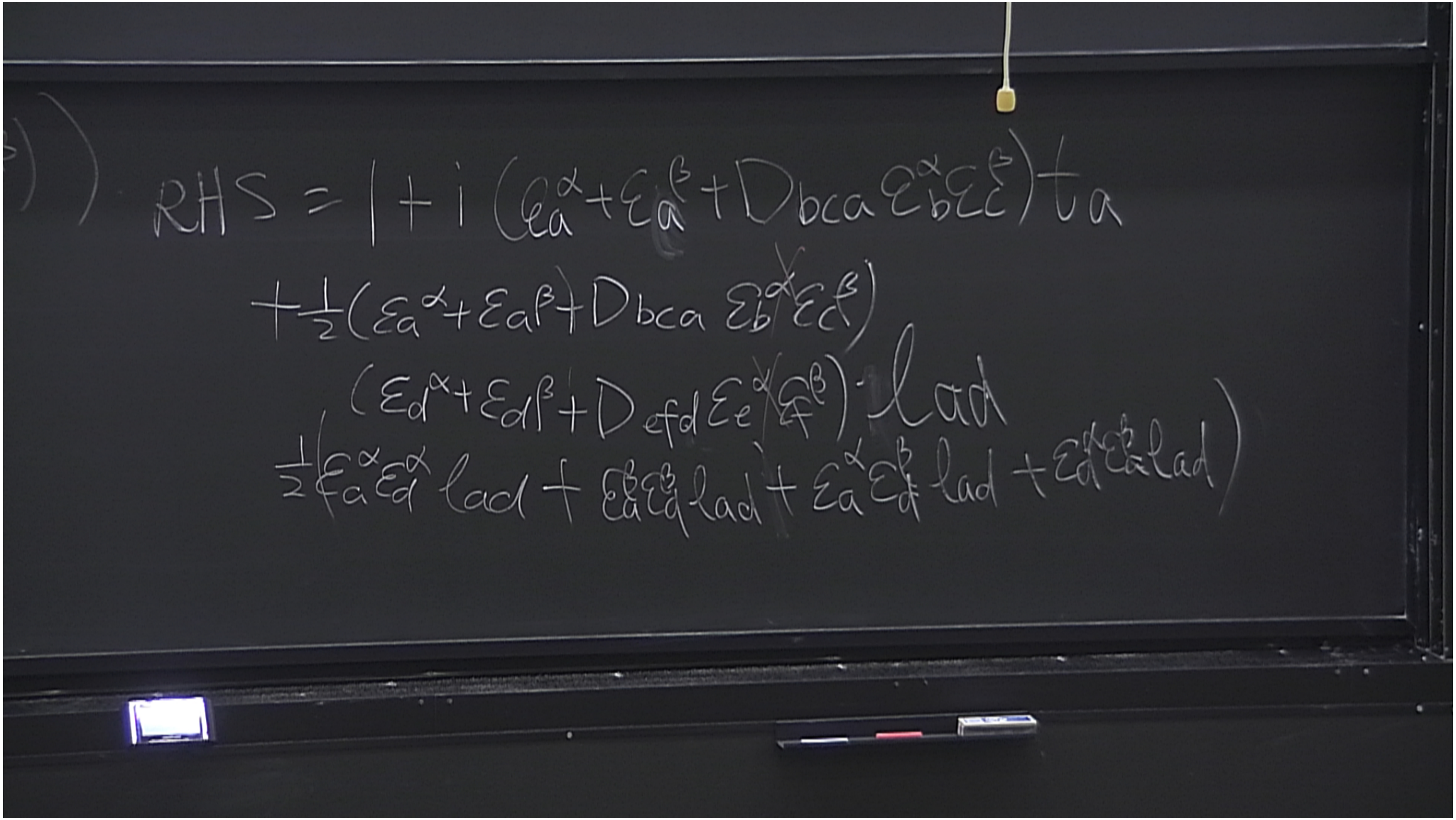
$$RHS = 1 + i\left(\varepsilon_a^{\alpha} + \varepsilon_a^{\beta} + D_{bca}\,\varepsilon_b^{\alpha}\varepsilon_c^{\beta}\right) t_a$$

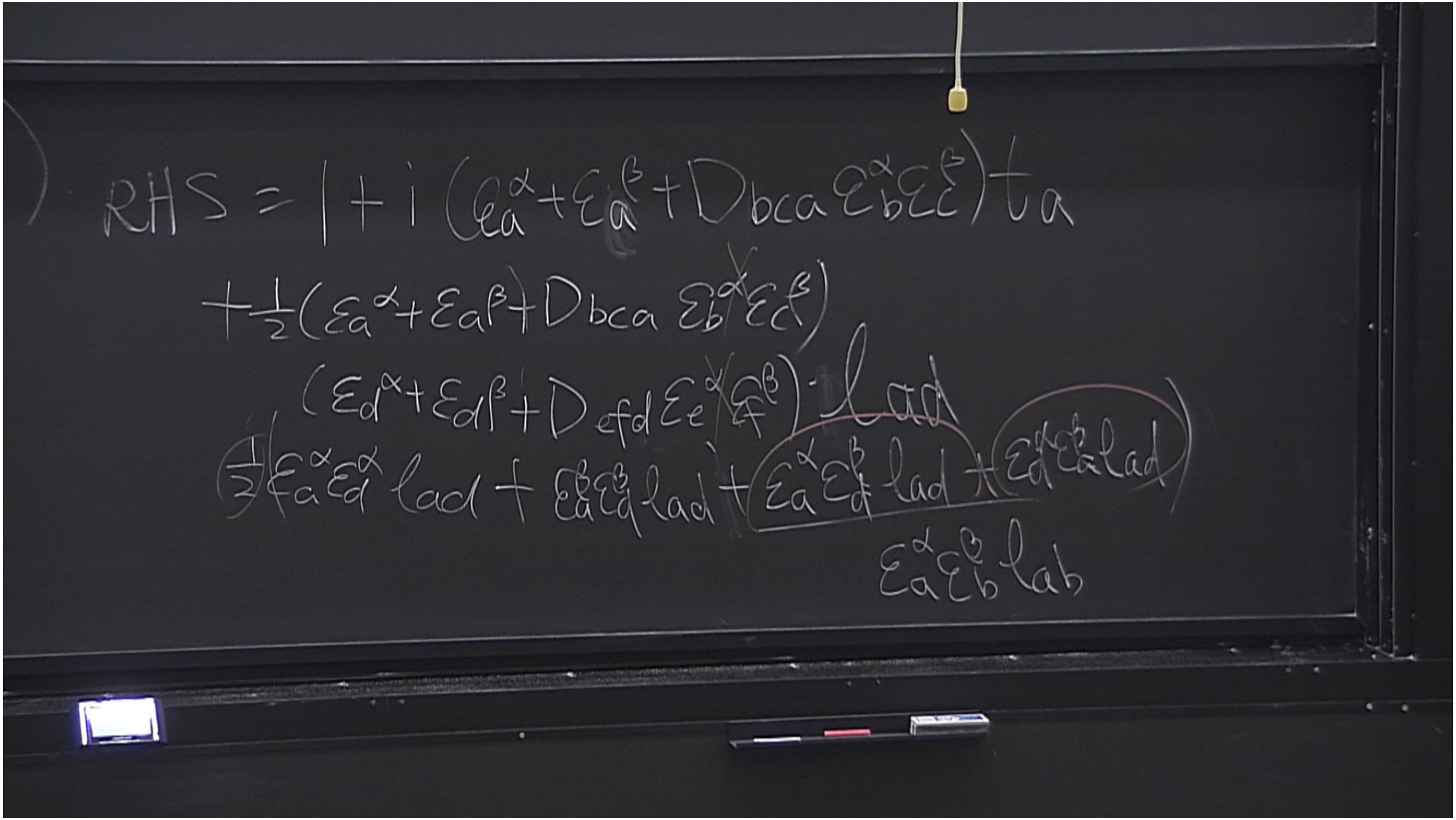
$$+ \frac{1}{2}\left(\varepsilon_a^{\alpha} + \varepsilon_a^{\beta} + D_{bca}\,\varepsilon_b^{\alpha}\varepsilon_c^{\beta}\right)$$

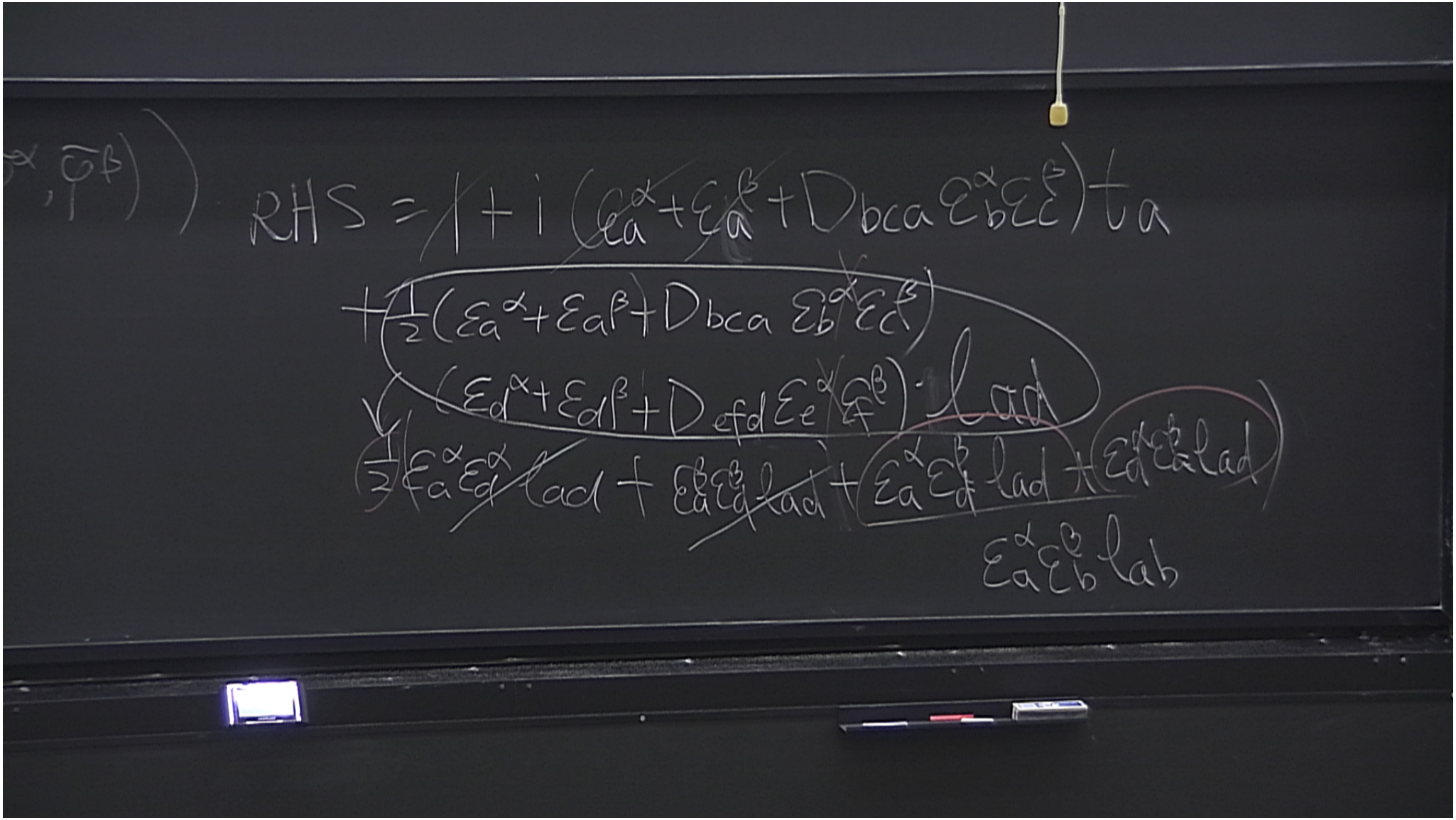
$$\left(\varepsilon_d^{\alpha} + \varepsilon_d^{\beta} + D_{efd}\,\varepsilon_e^{\alpha}\varepsilon_f^{\beta}\right)$$

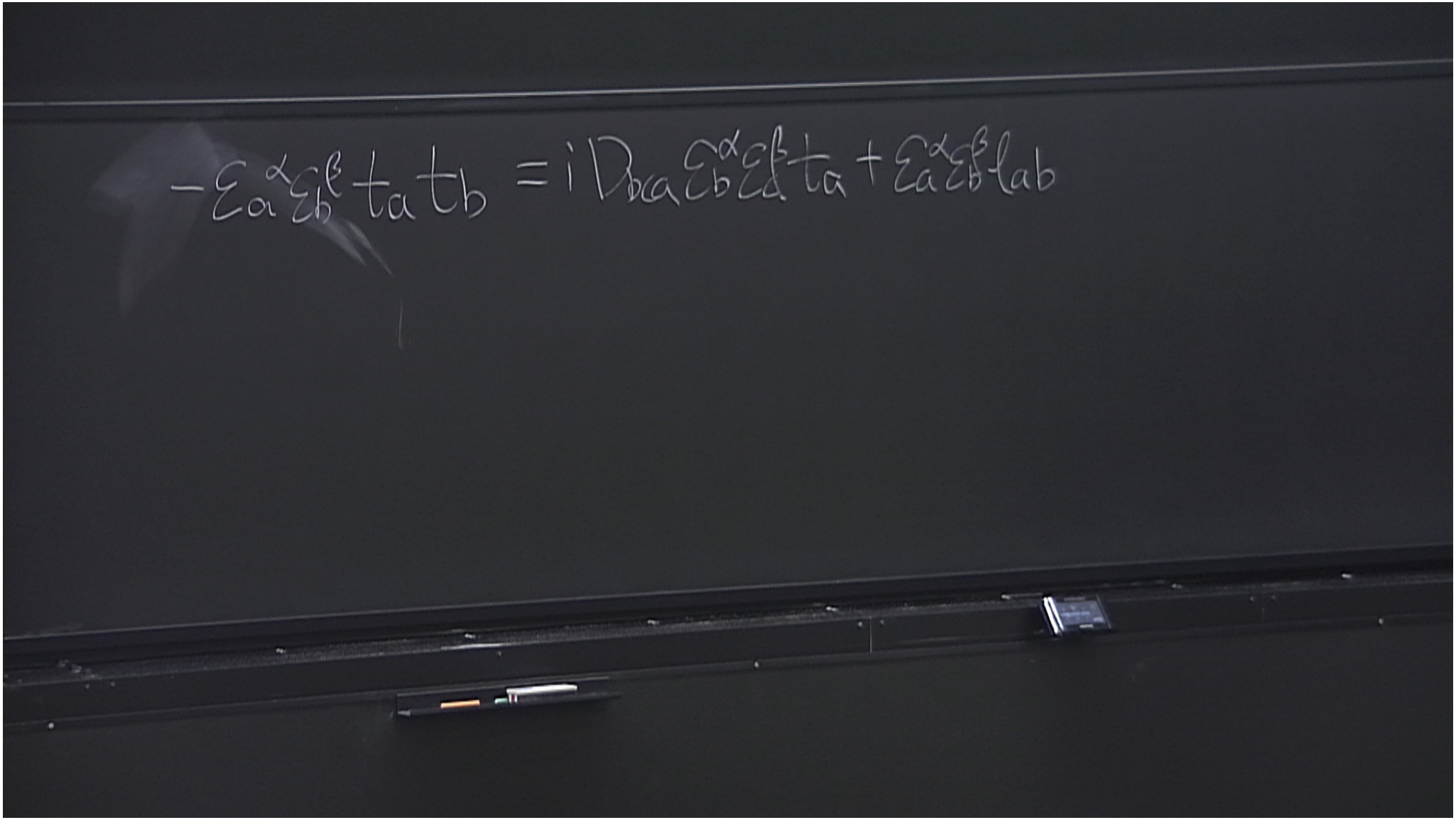

RHS = 1 + i (ε_a^α + ε_a^β + D_bca ε_b^α ε_c^β) t_a
+ 1/2 (ε_a^α + ε_aβ + D_bca ε_b^α ε_cβ)
(ε_d^α + ε_dβ + D_efd ε_e^α ε_f^β) t_a t_d

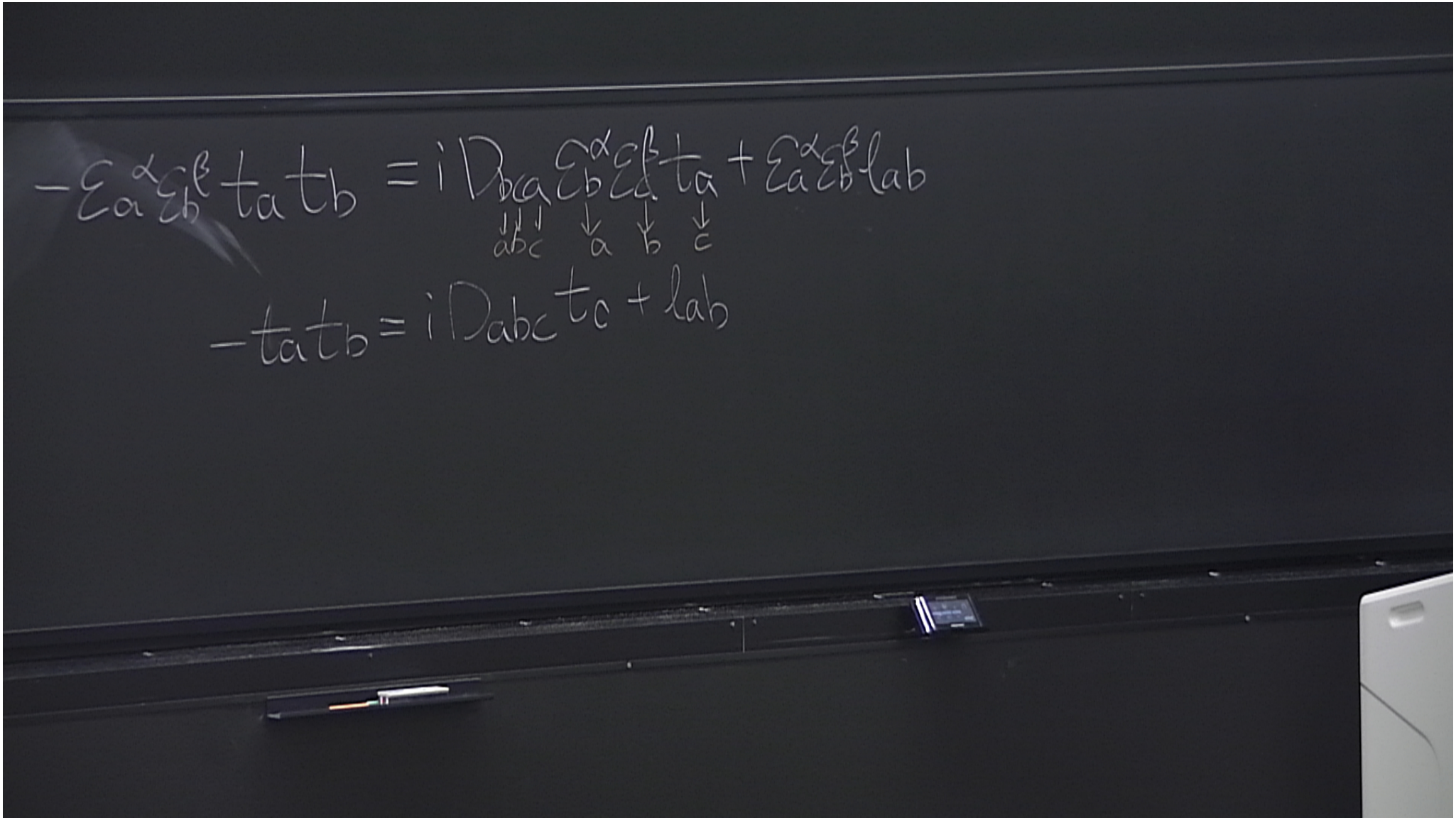

RHS = 1 + i (ε_a^α + ε_a^β + D_bca ε_b^α ε_c^β) t_a
+ 1/2 (ε_a^α + ε_aβ + D_bca ε_b^α ε_c^β)
(ε_d^α + ε_dβ + D_efd ε_e^α ε_f^β) t_a t_d

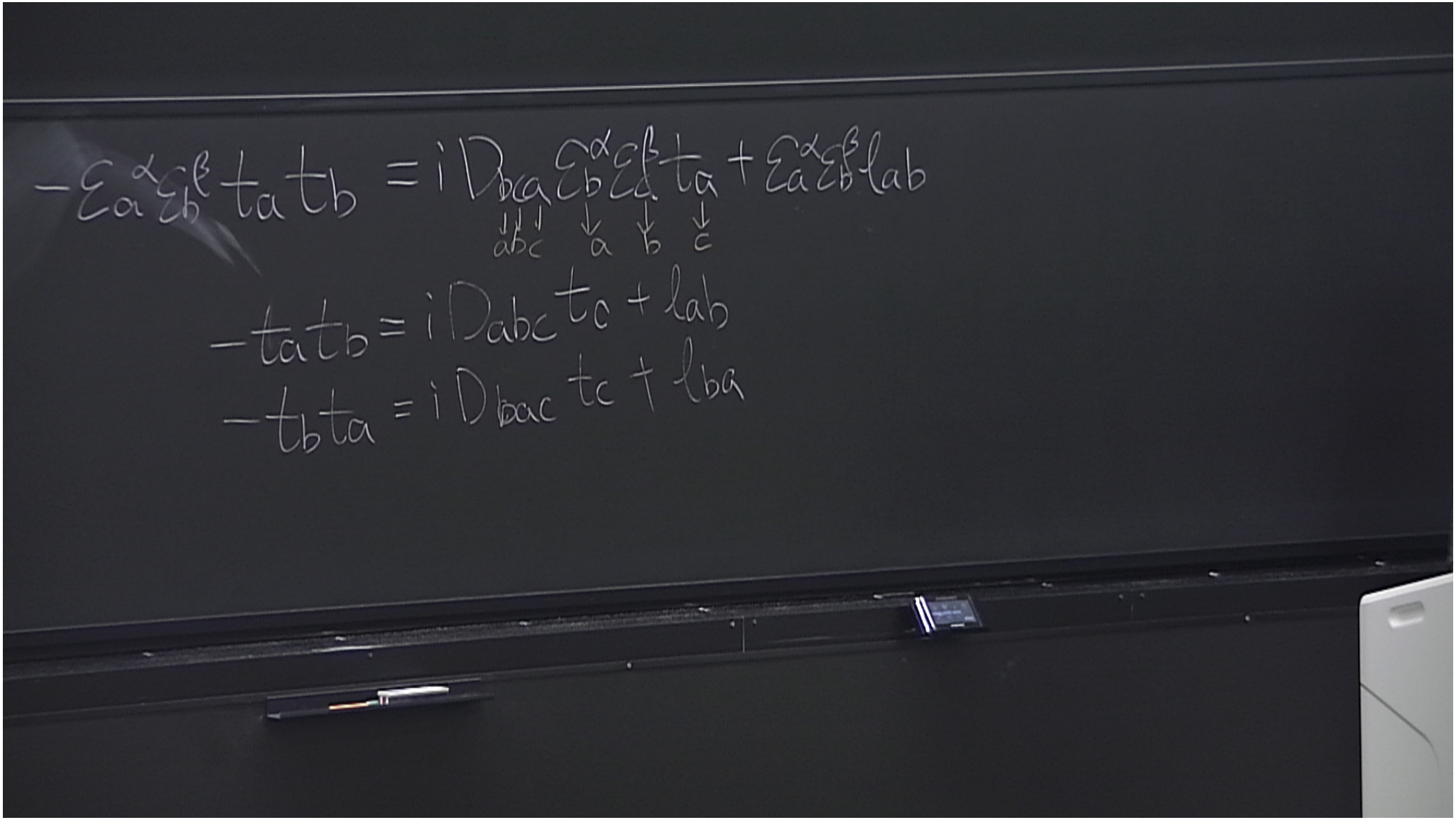

RHS = 1 + i (ε_a^α + ε_a^β + D_bca ε_b^α ε_c^β) t_a
+ 1/2 (ε_a^α + ε_a^β + D_bca ε_b^α ε_c^β)
(ε_d^α + ε_d^β + D_efd ε_e^α ε_f^β) l_ad
1/2 (ε_a^α ε_d^α l_ad + ε_a^β ε_d^β l_ad + ε_a^α ε_d^β l_ad + ε_d^α ε_a^β l_ad)

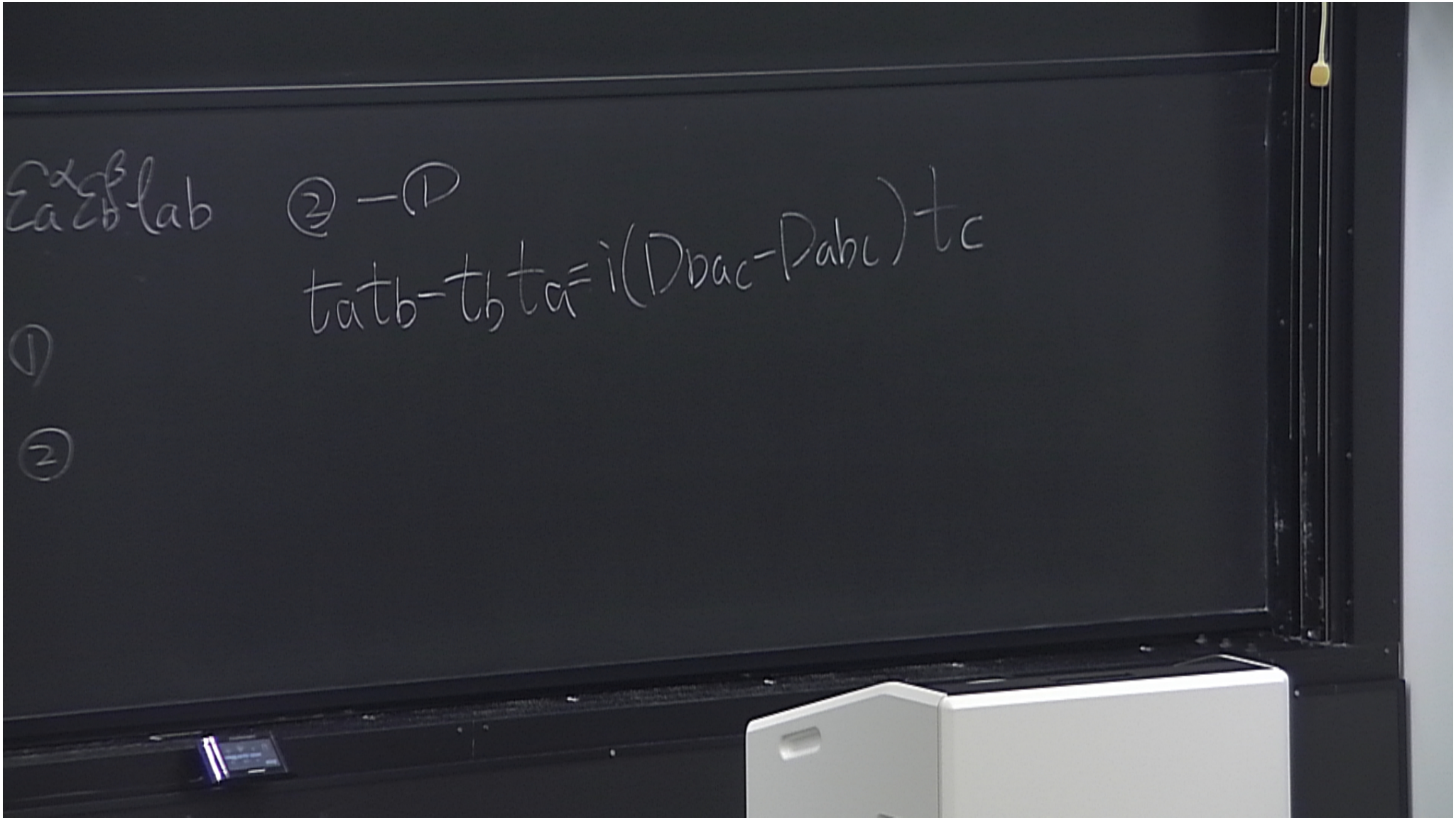

RHS = 1 + i (ε_a^α + ε_a^β + D_bca ε_b^α ε_c^β) t_a
+ 1/2 (ε_a^α + ε_a^β + D_bca ε_b^α ε_c^β)
(ε_d^α + ε_d^β + D_efd ε_e^α ε_f^β) · l_ad
1/2 (ε_a^α ε_d^α l_ad + ε_a^β ε_d^β l_ad + ε_a^α ε_d^β l_ad + ε_a^β ε_d^α l_ad)
ε_a^α ε_b^β l_ab

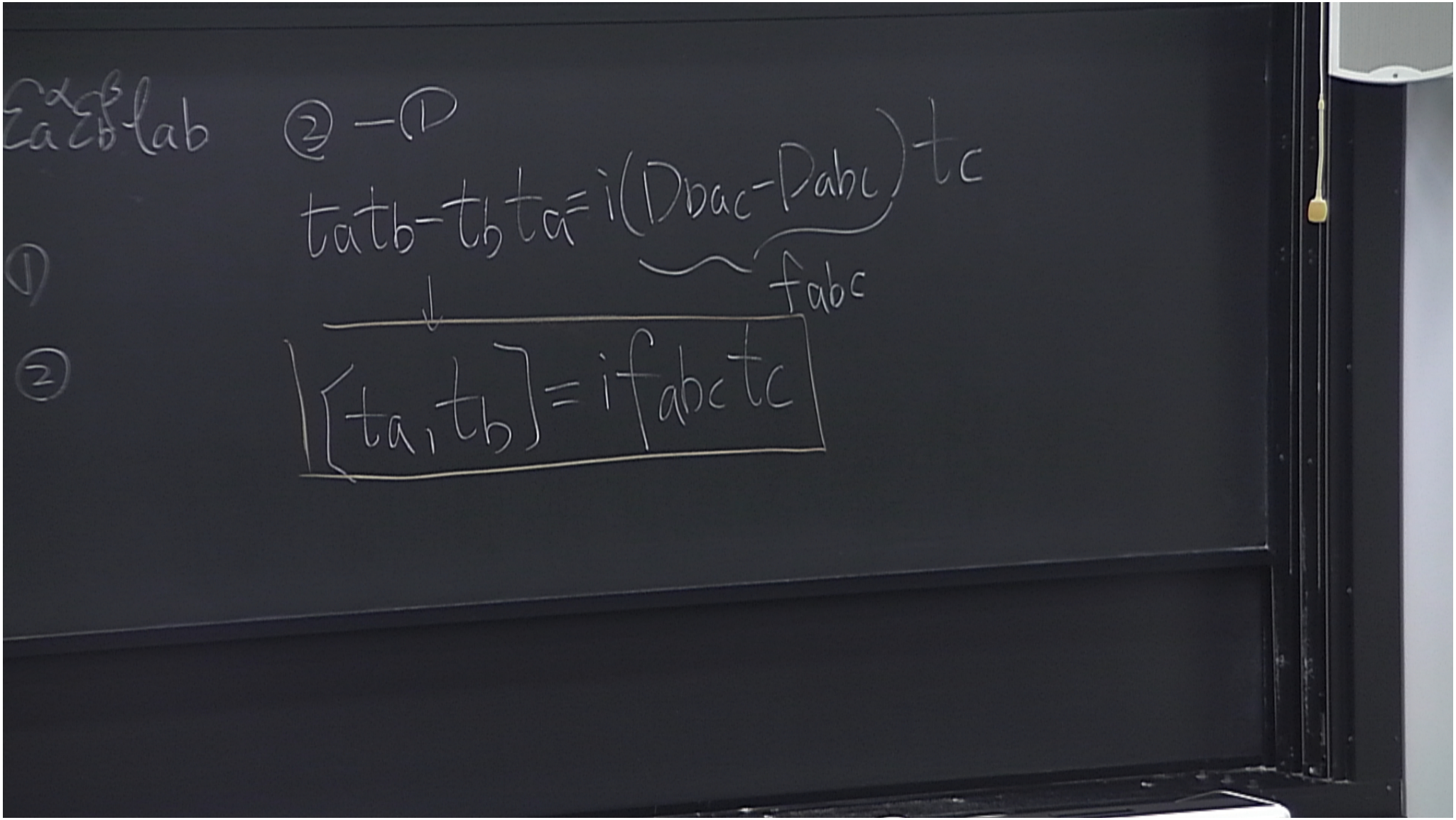

RHS = 1 + i (εa^α + εa^β + Dbca εb^α εc^β) ta
+ 1/2 (εa^α + εa^β + Dbca εb^α εc^β)
(εd^α + εd^β + Defd εe^α εf^β) lad
1/2 εa^α εd^α lad + εa^β εd^β lad + εa^α εd^β lad + εa^β εd^α lad
εa^α εb^β lab

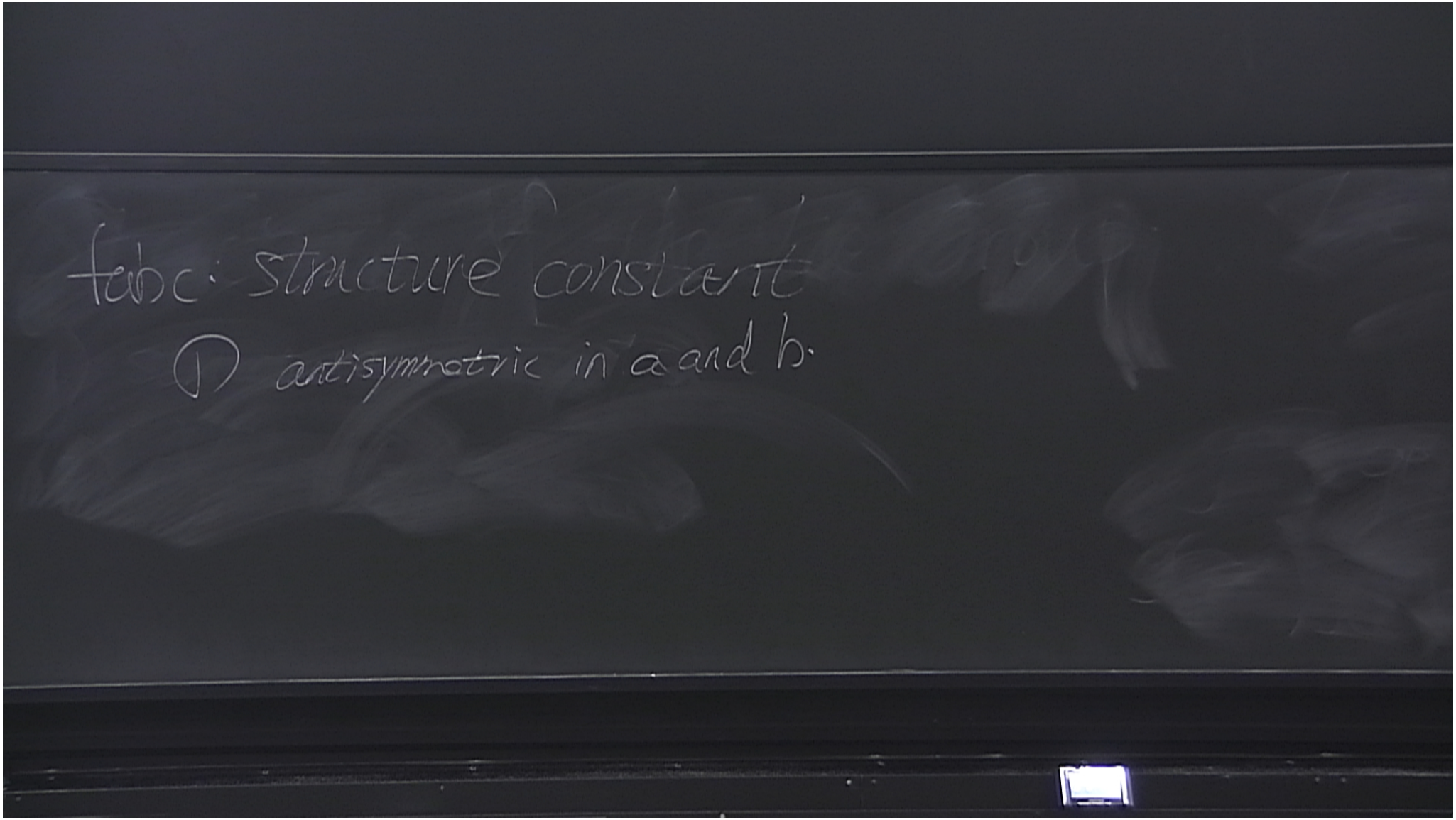

$-\varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} t_a t_b = i D_{bca} \varepsilon_b{}^{\alpha}\varepsilon_d{}^{\beta} t_a + \varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} l_{ab}$


$-\varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta}t_a t_b = iD_{abc}\varepsilon_b{}^{\alpha}\varepsilon_d{}^{\beta}t_a + \varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta}l_{ab}$
abc a b c
$-t_a t_b = iD_{abc}t_c + l_{ab}$


$-\varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} t_a t_b = i D_{bca}\, \varepsilon_b{}^{\alpha}\varepsilon_c{}^{\beta} t_a + \varepsilon_a{}^{\alpha}\varepsilon_b{}^{\beta} l_{ab}$
abc a b c
$-t_a t_b = i D_{abc} t_c + l_{ab}$
$-t_b t_a = i D_{bac} t_c + l_{ba}$


$\varepsilon^{\alpha\beta}\varepsilon_{ab}$ lab
② − ①
$t_a t_b - t_b t_a = i(D_{bac} - D_{abc})\, t_c$
①
②


$\varepsilon^{\alpha\beta}_{ab}$
② − ①
$t_a t_b - t_b t_a = i(D_{bac} - D_{abc})\, t_c$
①
②
f_{abc}
$[t_a, t_b] = i f_{abc} t_c$


fabc: structure constant
① antisymmetric in a and b.

f_{abc}: structure constant

$f_{abc}^* = f_{abc}$

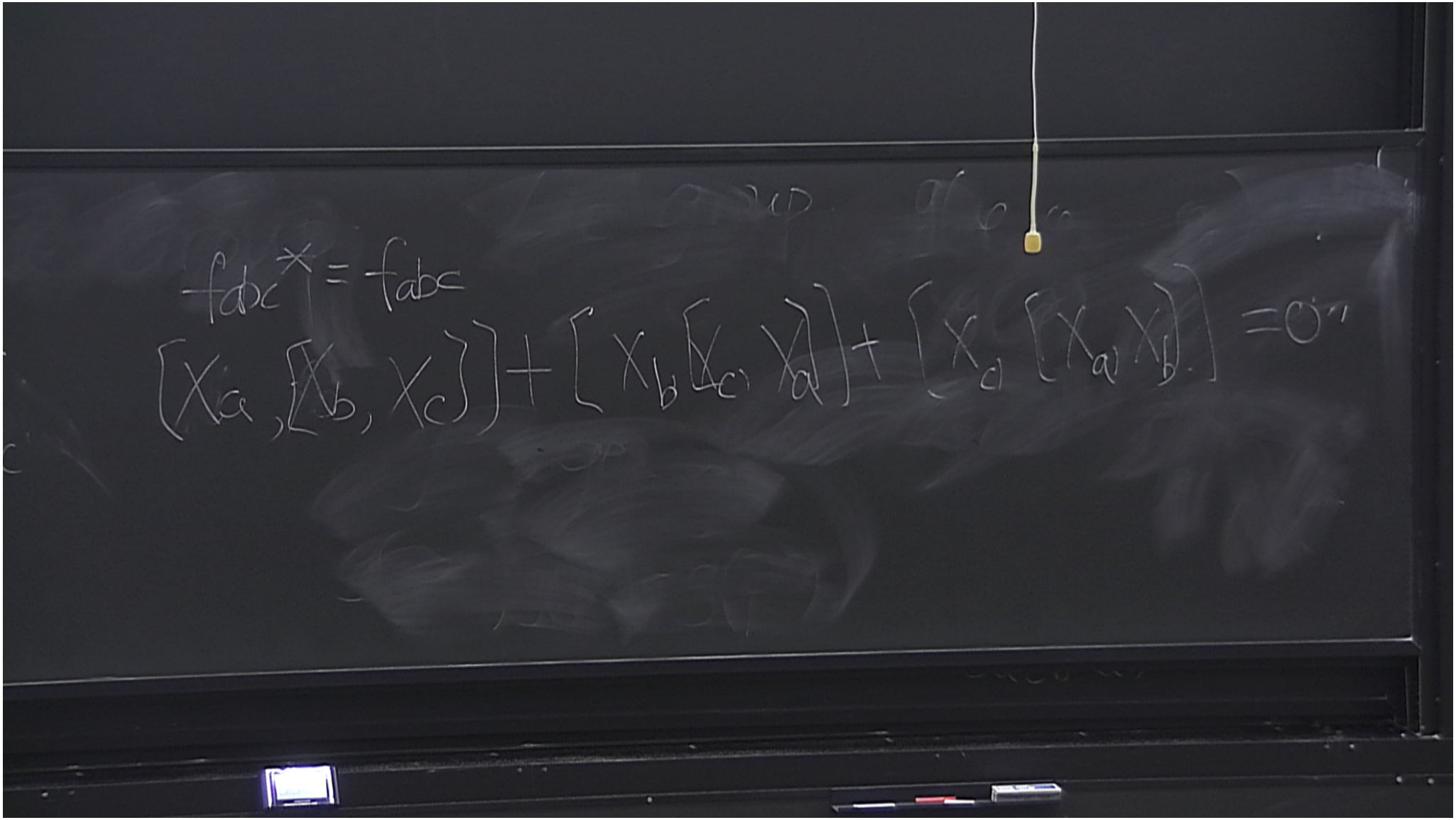
① antisymmetric in a and b.

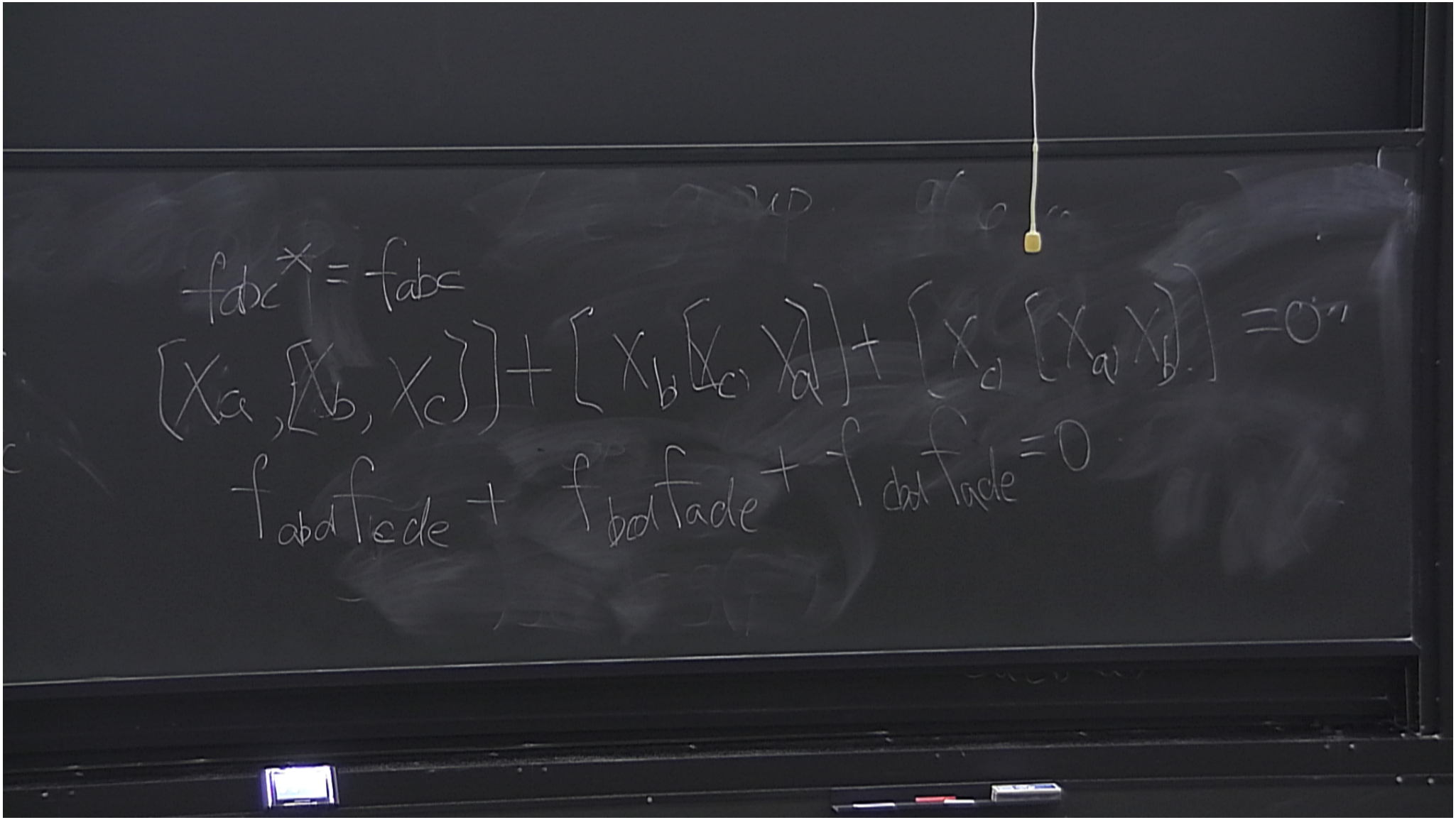
$(AB)^\dagger = B^\dagger A^\dagger$

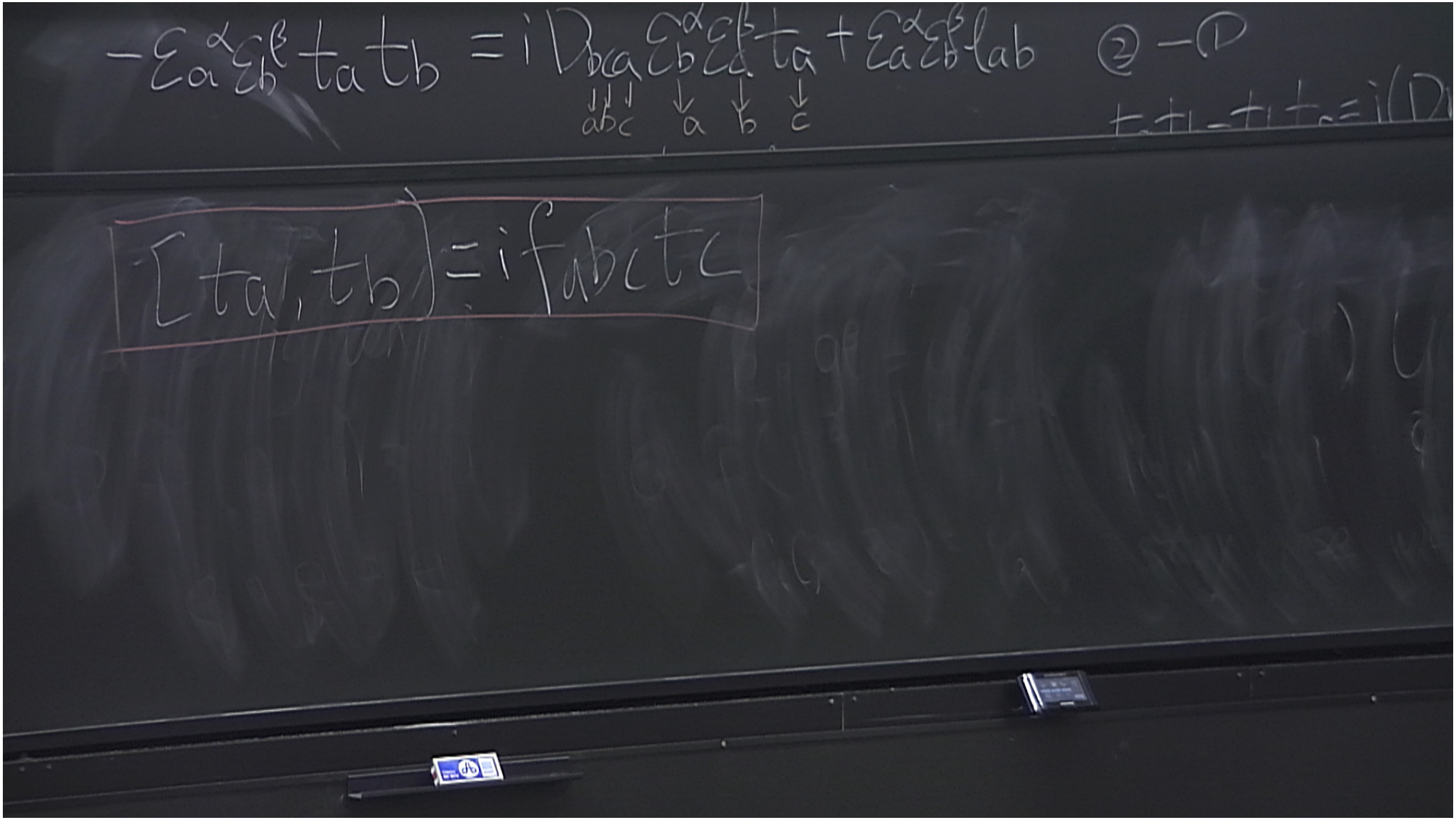
$[t_a, t_b]^\dagger = -i f_{abc}^* t_c$

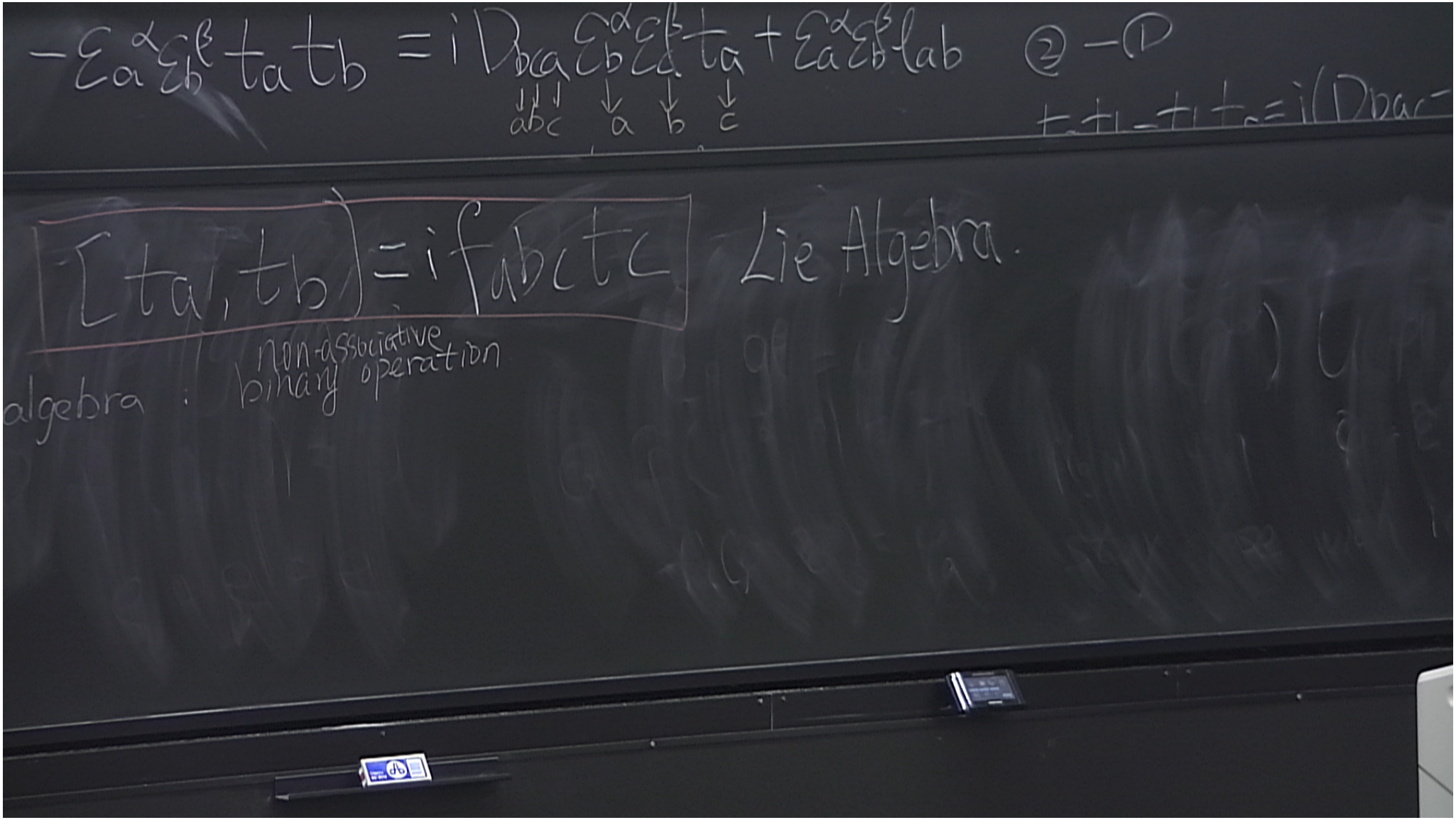
$\| \quad [t_b, t_a] = i f_{bac} t_c$

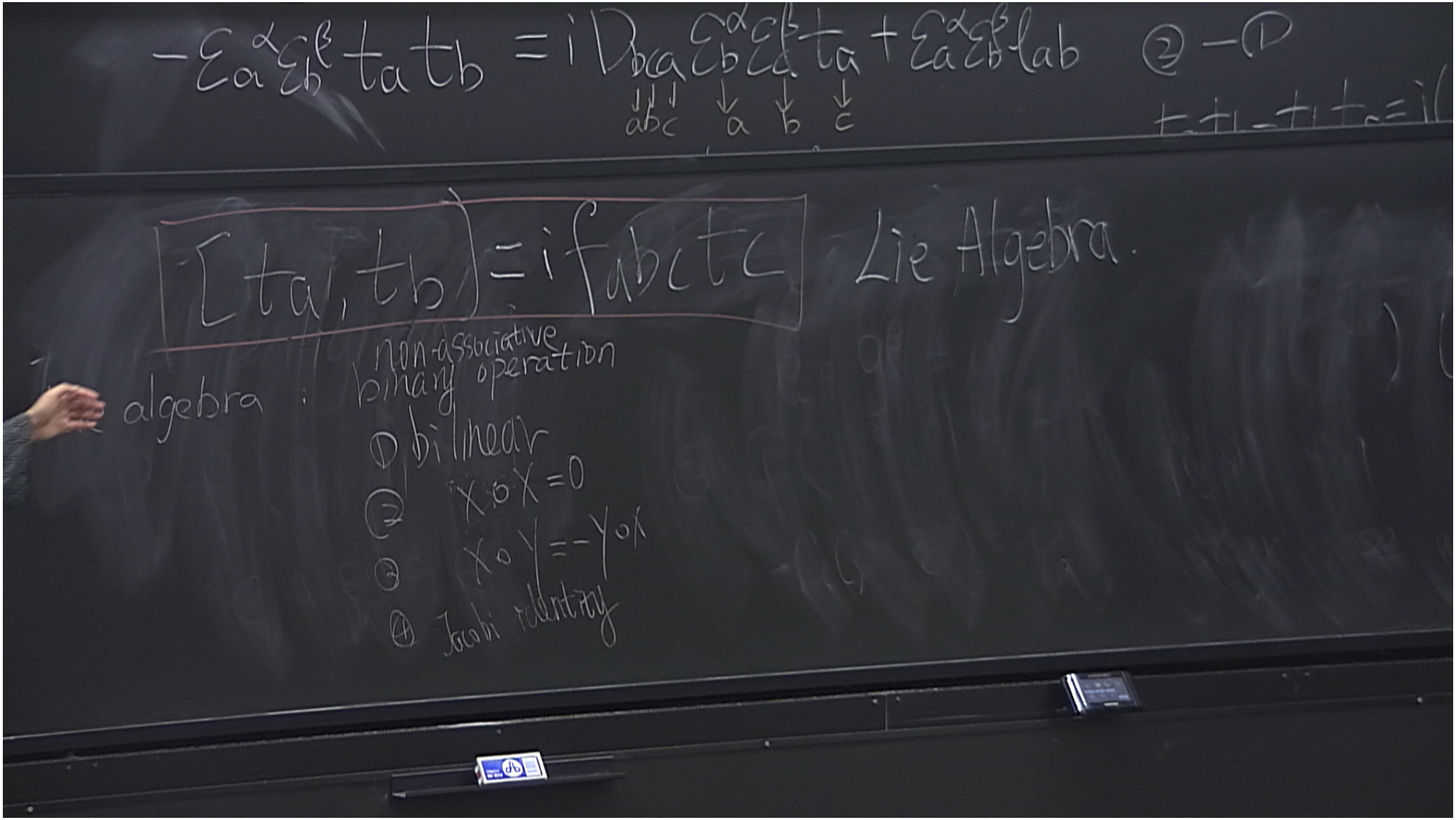
$= -i f_{abc} t_c$

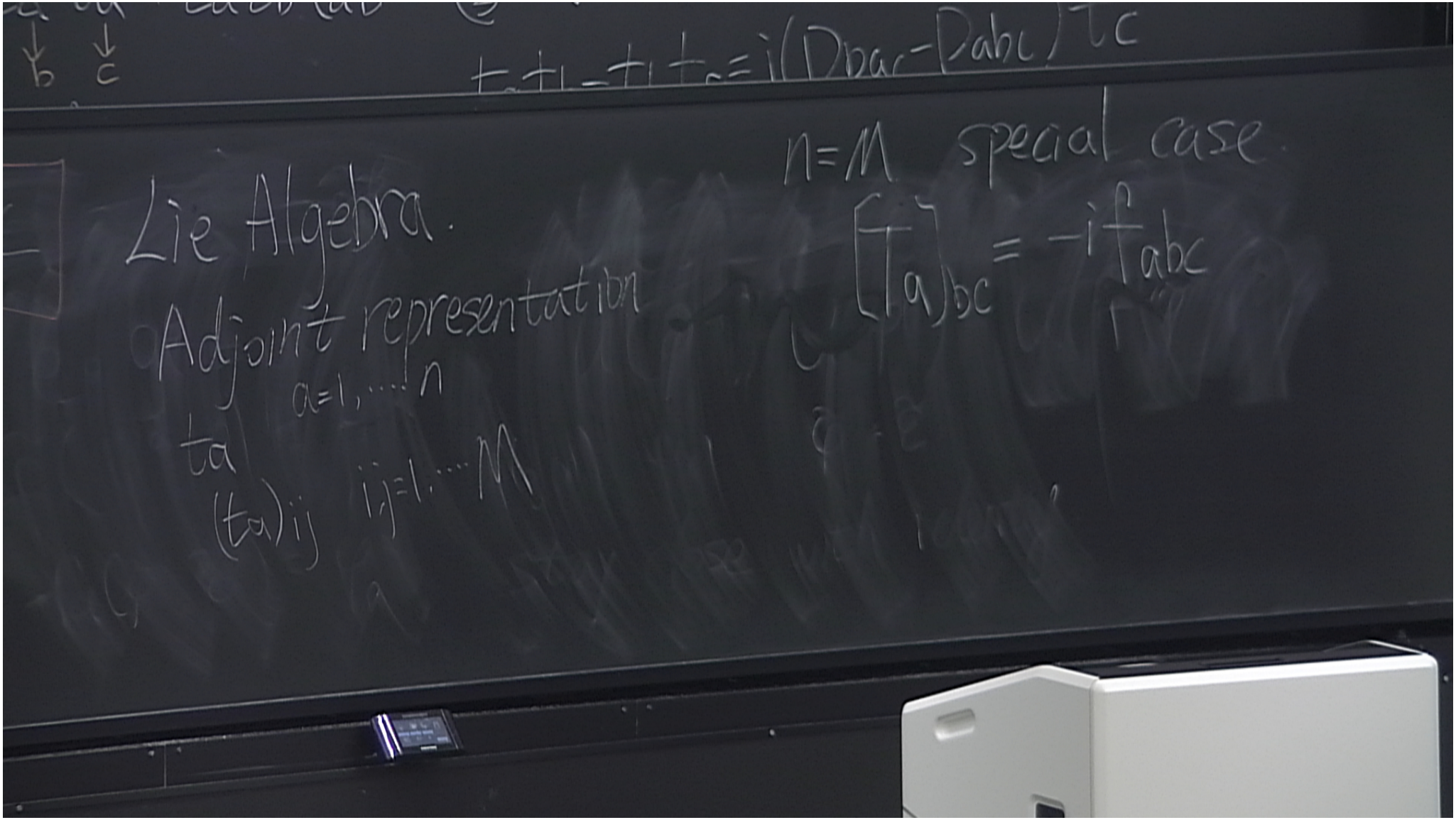

$f_{abc}{}^{*} = f_{abc}$
$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$

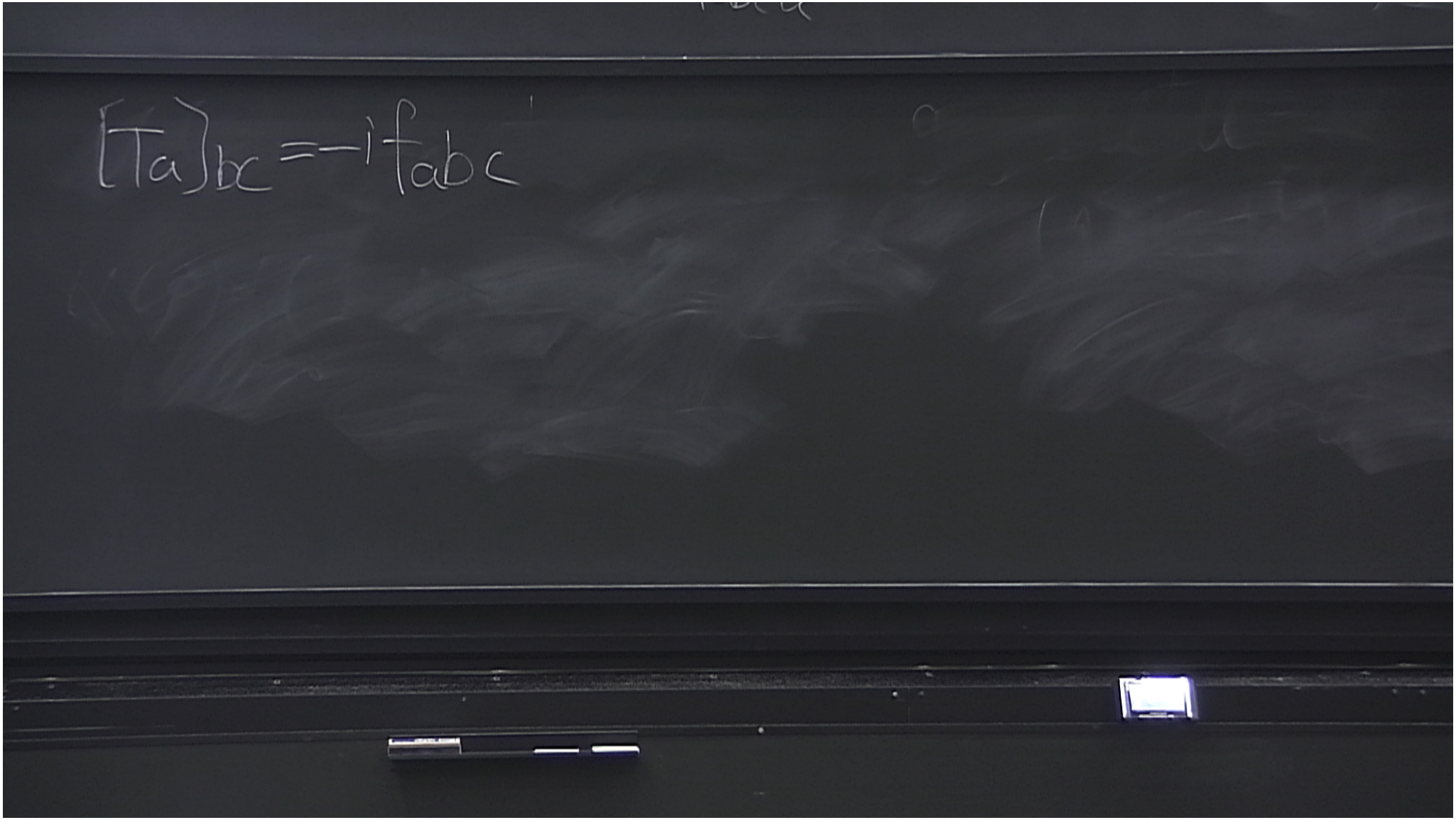

$f_{abc}^{*} = f_{abc}$
$[X_a,[X_b,X_c]]+[X_b,[X_c,X_a]]+[X_c,[X_a,X_b]]=0$
$f_{abd}f_{ecde}+f_{bcd}f_{aede}+f_{cad}f_{bede}=0$

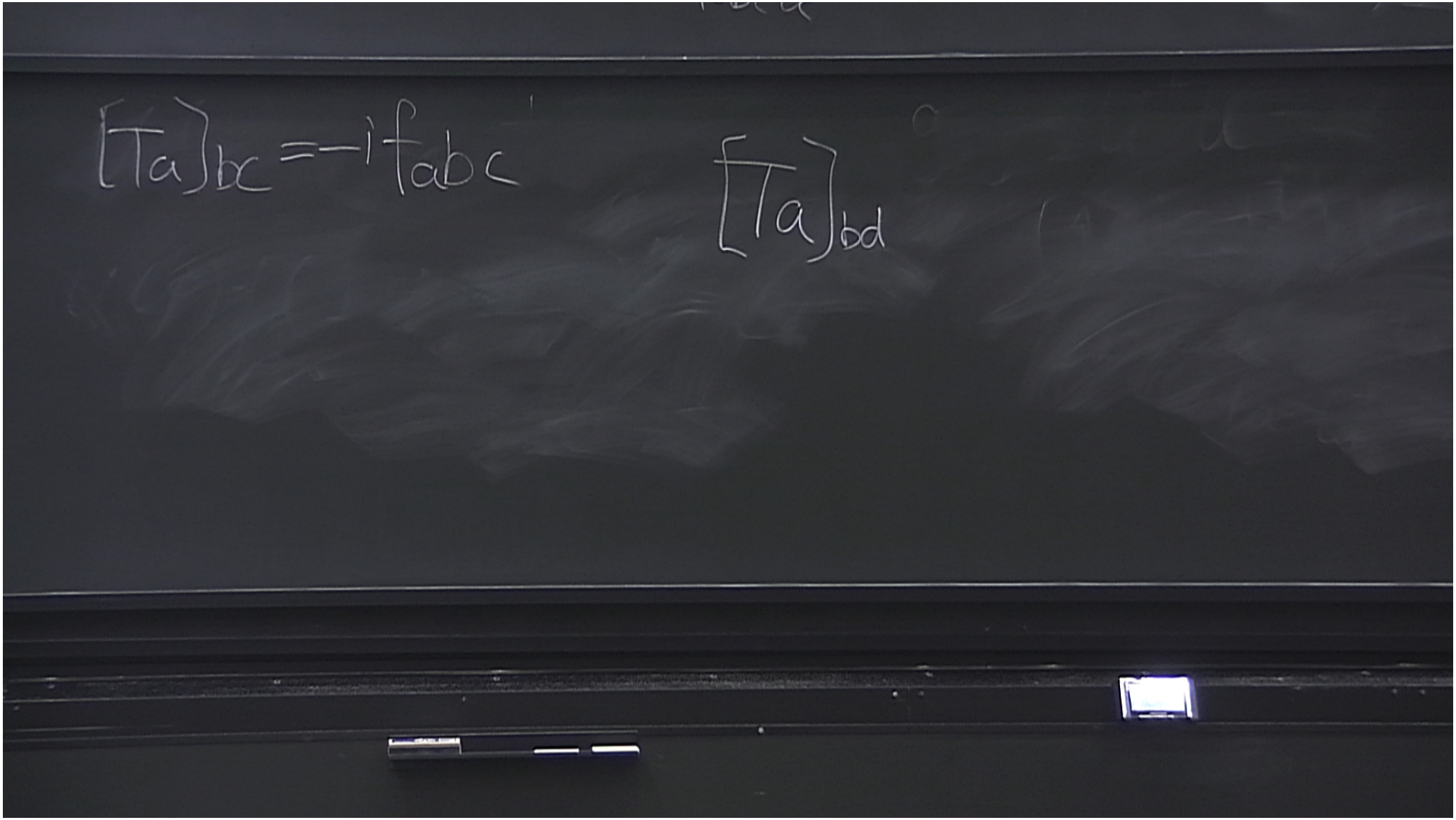

$-\varepsilon_a^{\alpha}\varepsilon_b^{\beta} t_a t_b = i D_{bca}\, \varepsilon_b^{\alpha}\varepsilon_d^{\beta} t_a + \varepsilon_a^{\alpha}\varepsilon_b^{\beta} l_{ab}$
abc a b c
②−①
$[t_a, t_b] = i f_{abc} t_c$

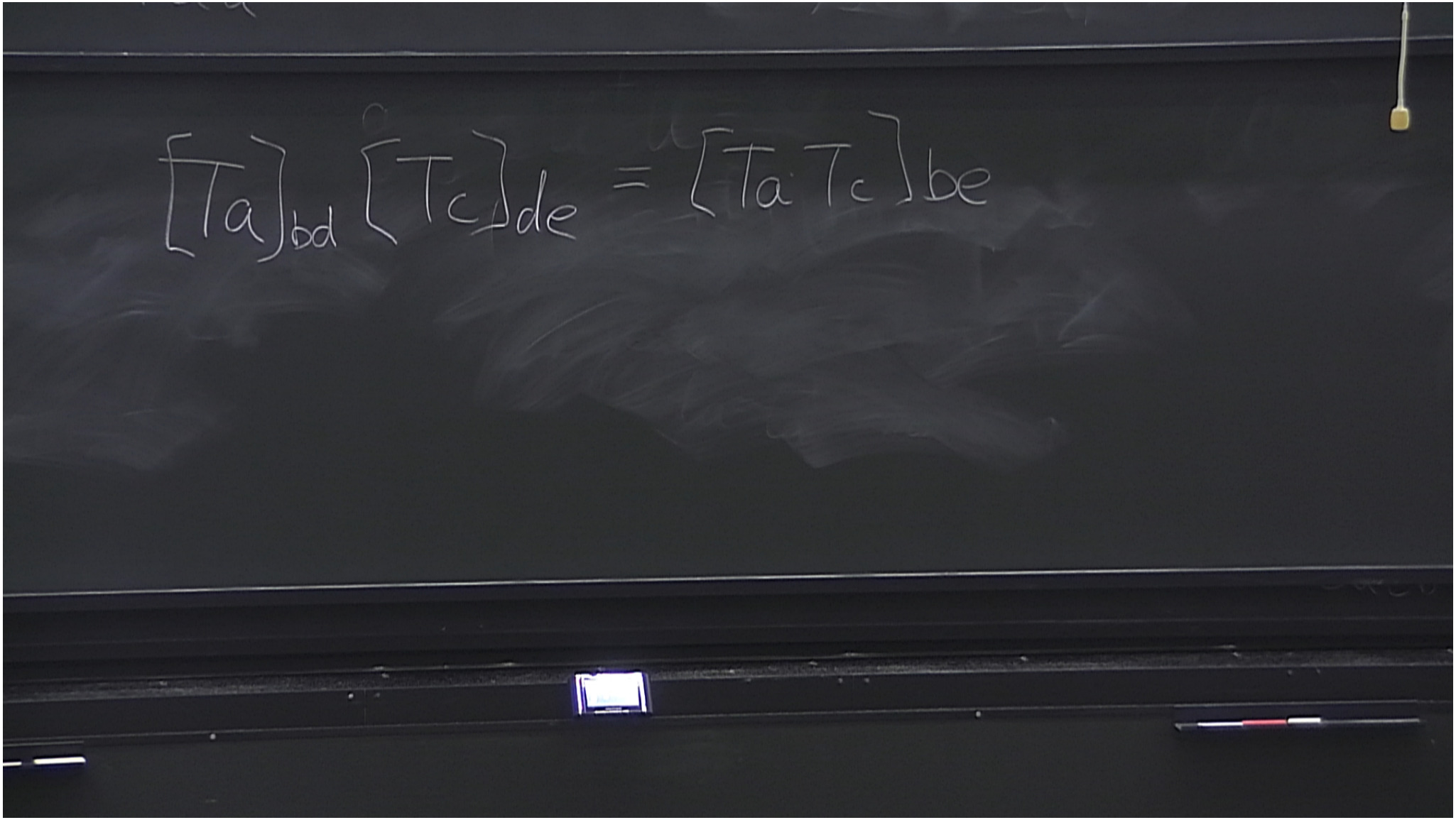

$-\varepsilon_a^{\alpha}\varepsilon_b^{\beta} t_a t_b = i D_{bca}\, \varepsilon_b^{\alpha}\varepsilon_c^{\beta} t_a + \varepsilon_a^{\alpha}\varepsilon_b^{\beta} l_{ab}$
abc a b c
② − ①
$[t_a, t_b] = i f_{abc} t_c$ Lie Algebra.
algebra : non-associative binary operation

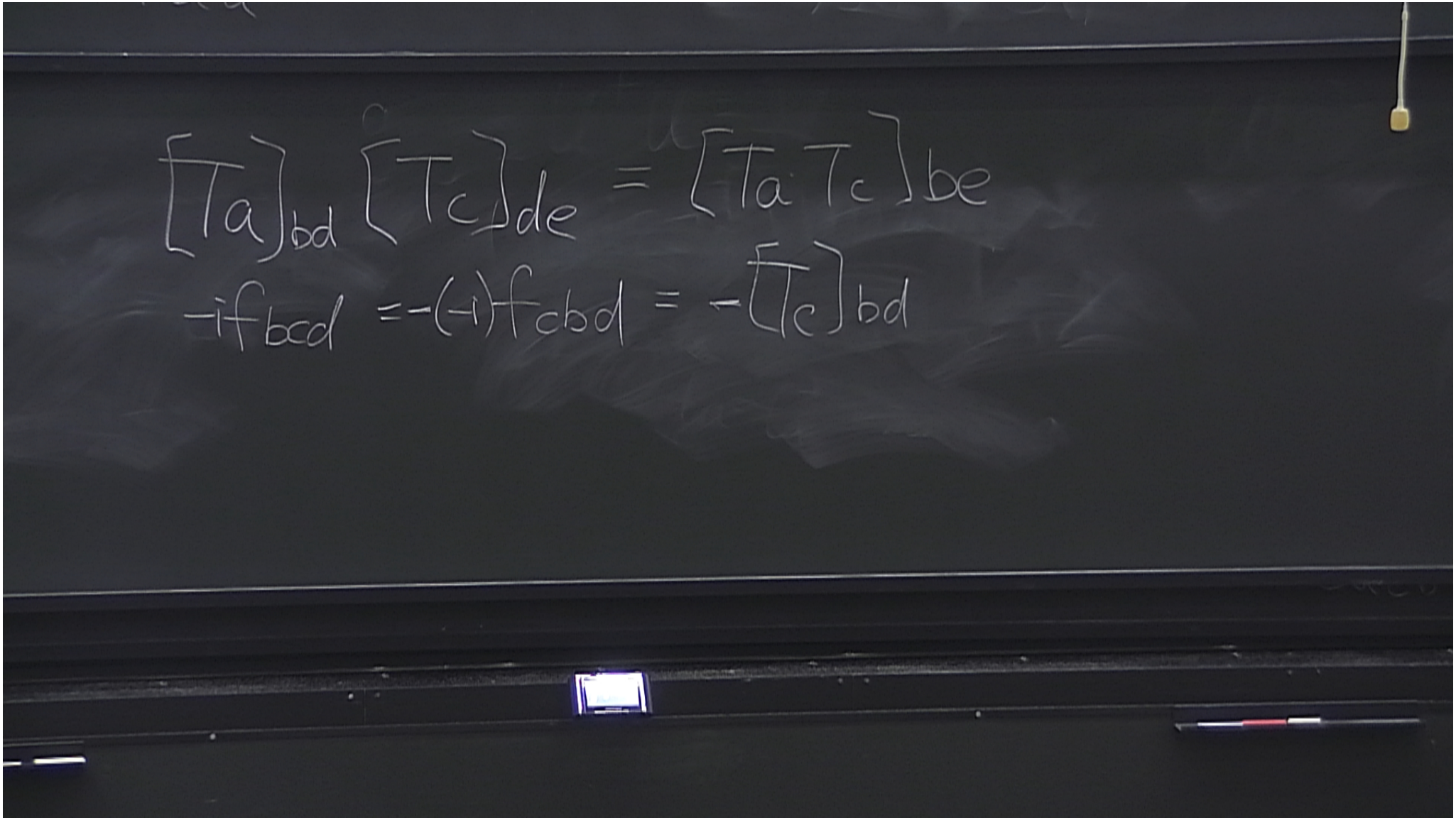

$-\varepsilon_a^{\alpha}\varepsilon_b^{\beta} t_a t_b = i D_{abc}\varepsilon_b^{\alpha}\varepsilon_c^{\beta} t_a + \varepsilon_a^{\alpha}\varepsilon_b^{\beta} i f_{ab}$ ② − ①
$t_a t_b - t_b t_a = i($
$[t_a, t_b] = i f_{abc} t_c$ Lie Algebra.
algebra : non-associative binary operation
① bilinear
② $X \circ X = 0$
③ $X \circ Y = -Y \circ X$
④ Jacobi identity

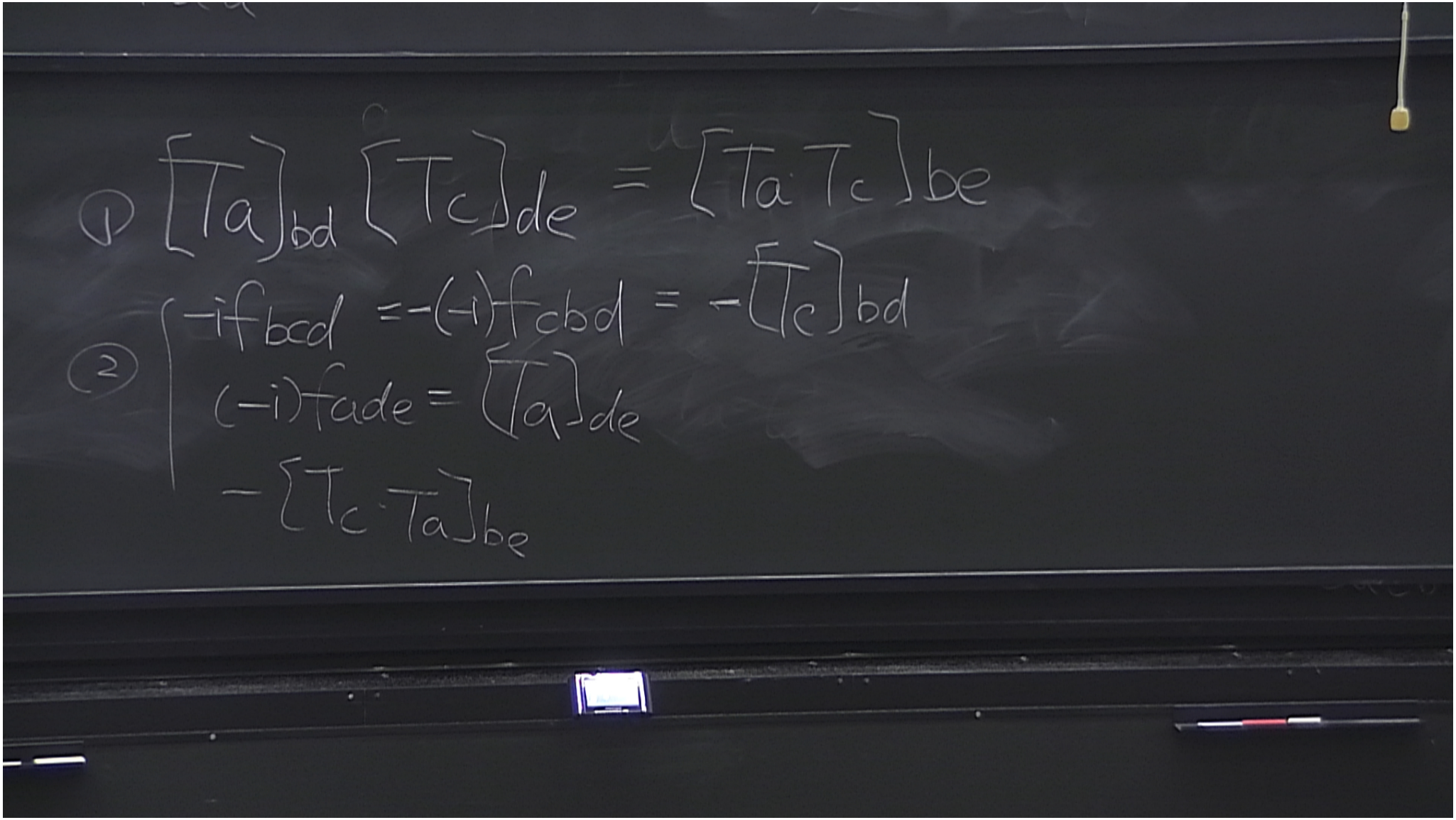

Lie Algebra.
Adjoint representation
a=1, ... n
ta
(ta)ij i,j=1 ... M
n=M speaal case.
[Ta]bc = -i fabc

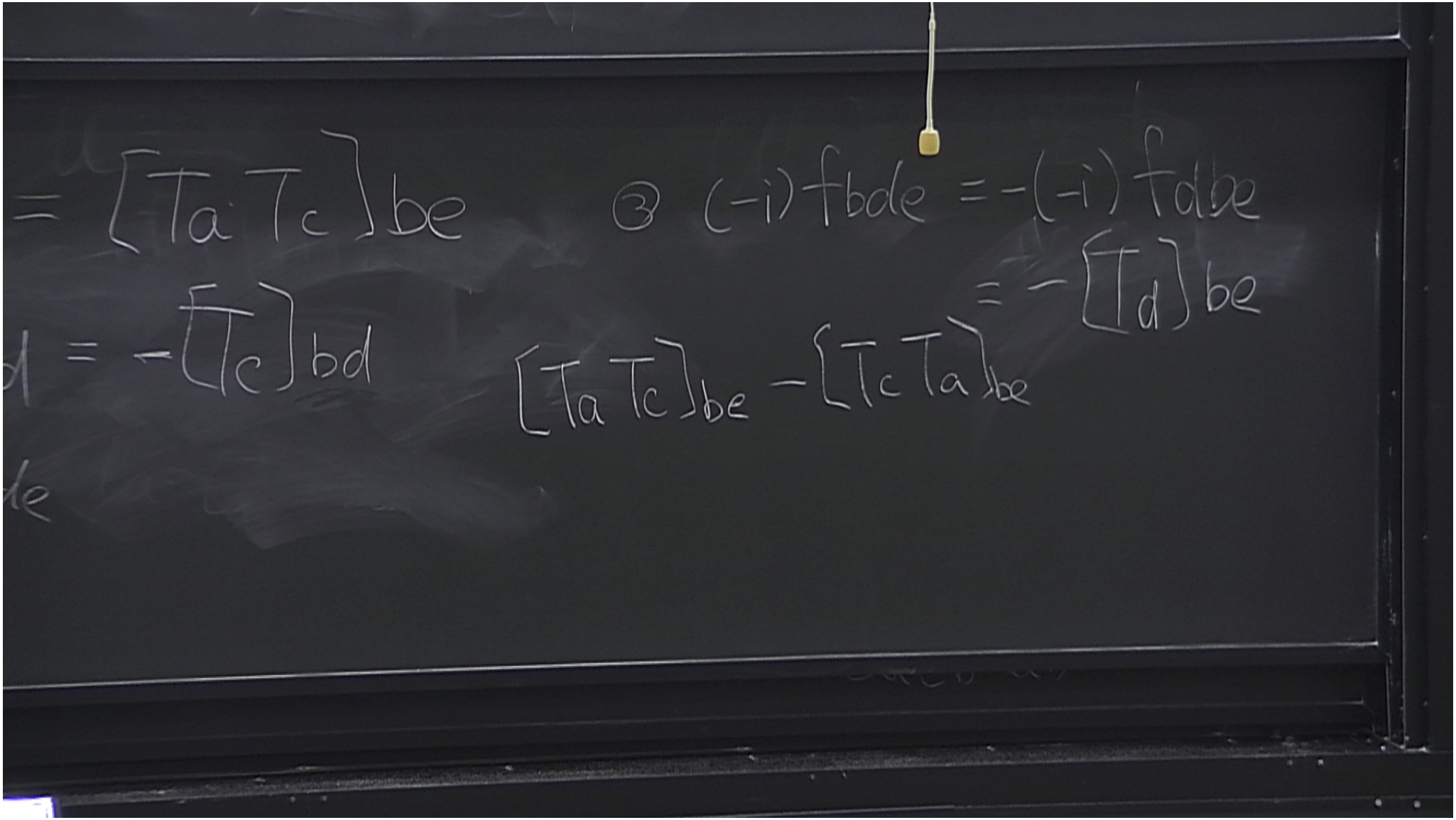

$[T_a]_{bc} = -i f_{abc}$

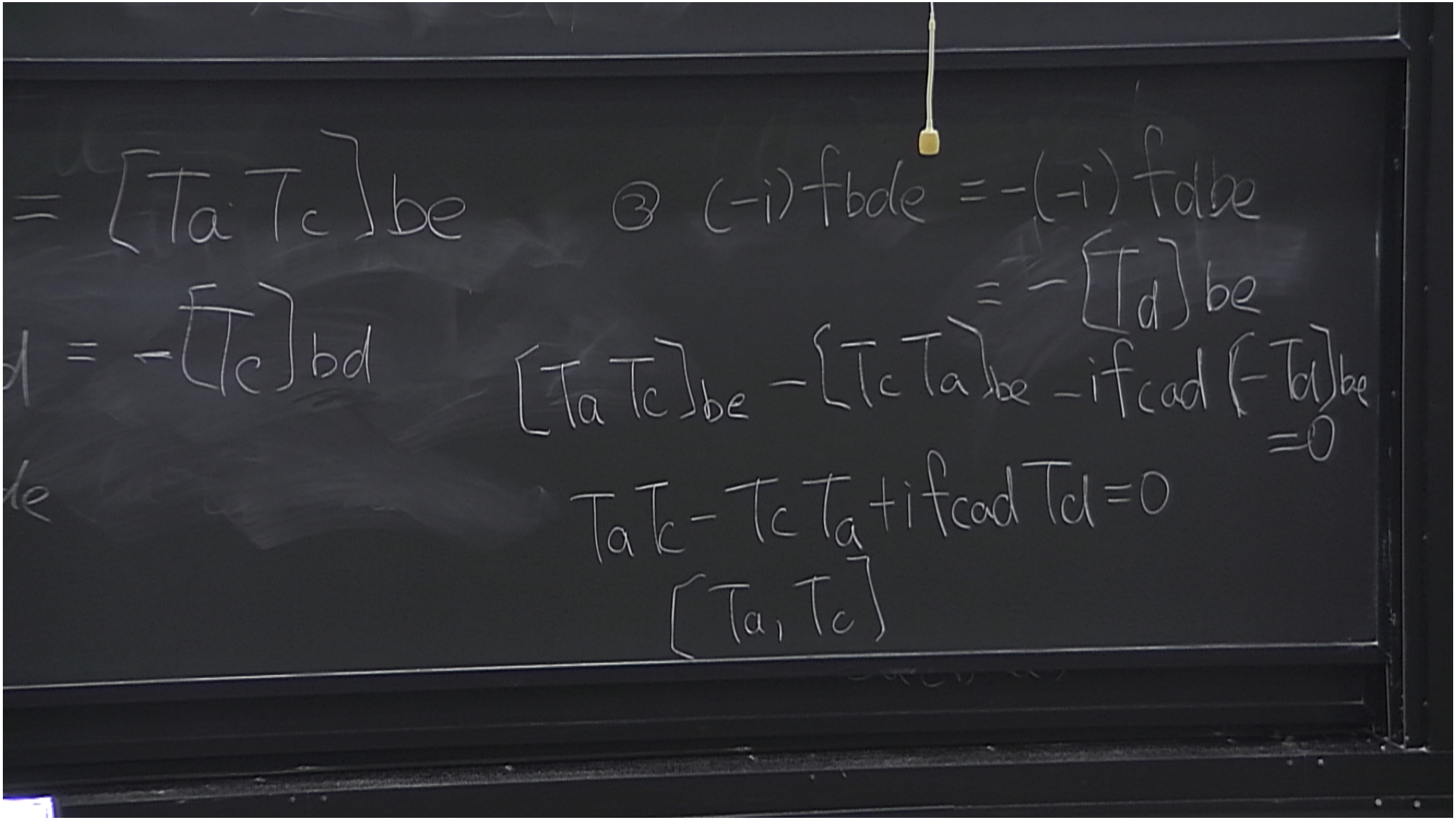

$[T_a]_{bc} = -i f_{abc}$
$[T_a]_{bd}$

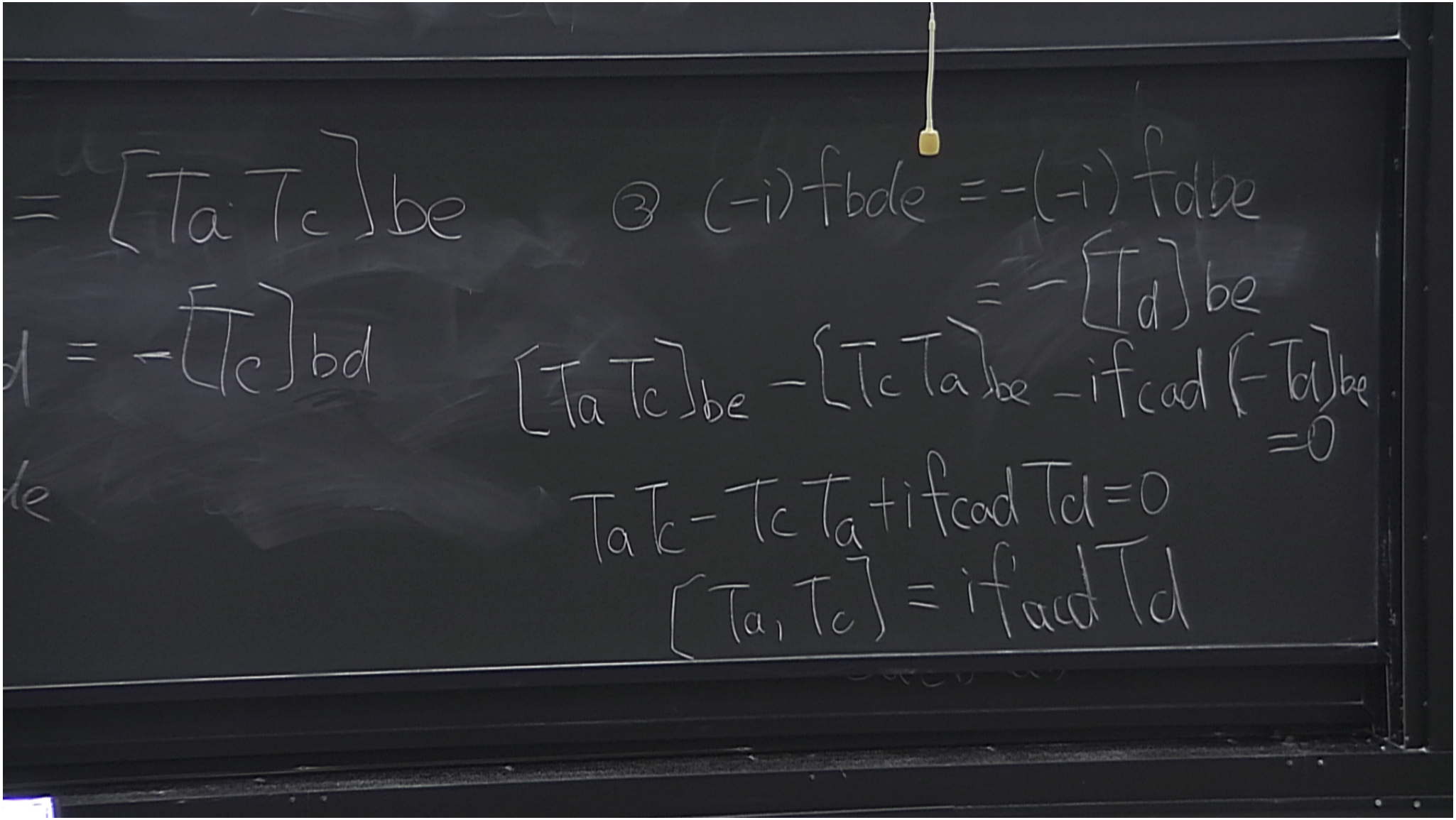

$[T_a]_{bd}\,[T_c]_{de} = [T_a T_c]_{be}$

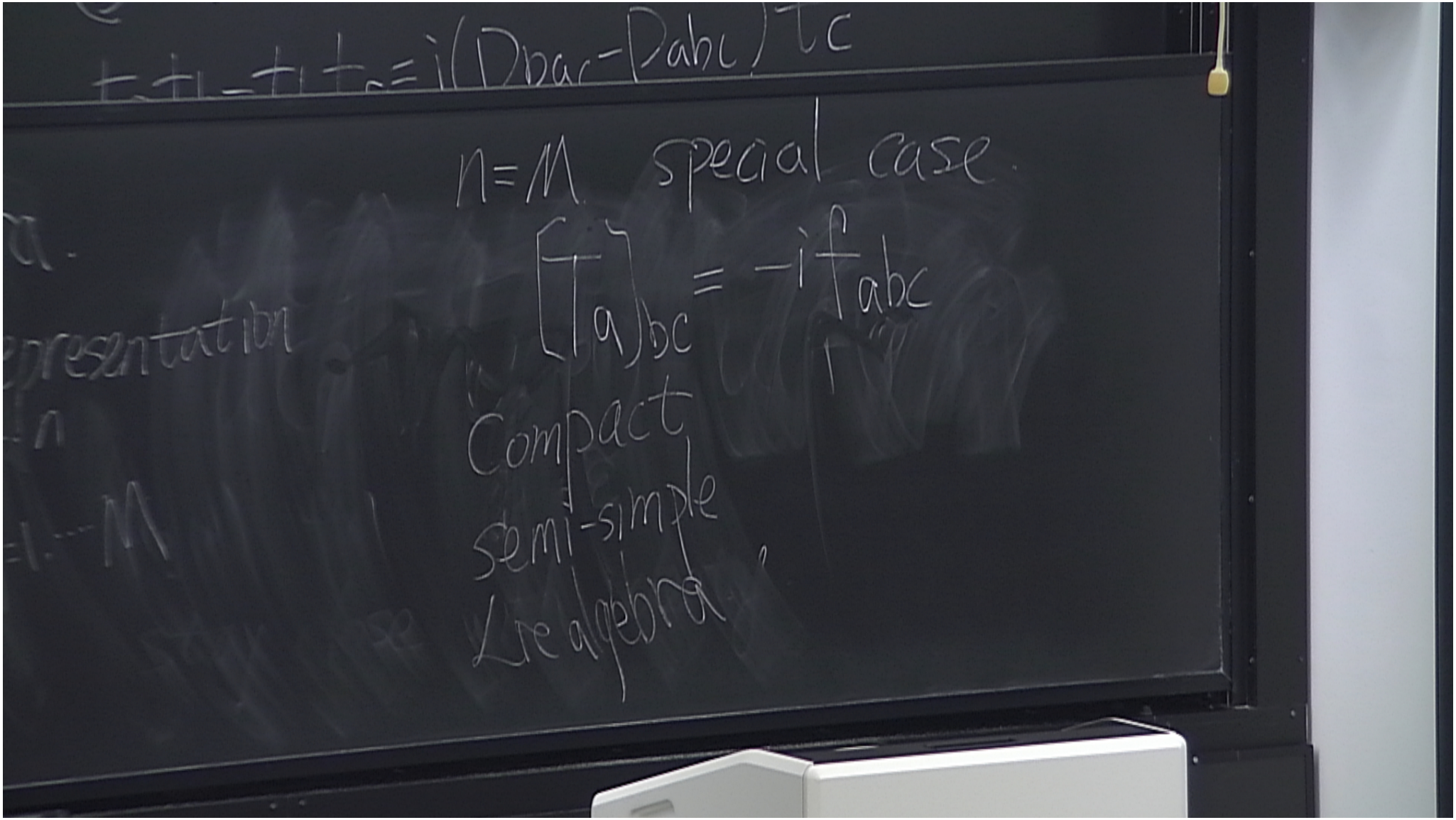

$[T_a]_{bd}[T_c]_{de} = [T_a T_c]_{be}$
$-if_{bcd} = -(-1)f_{cbd} = -[T_c]_{bd}$

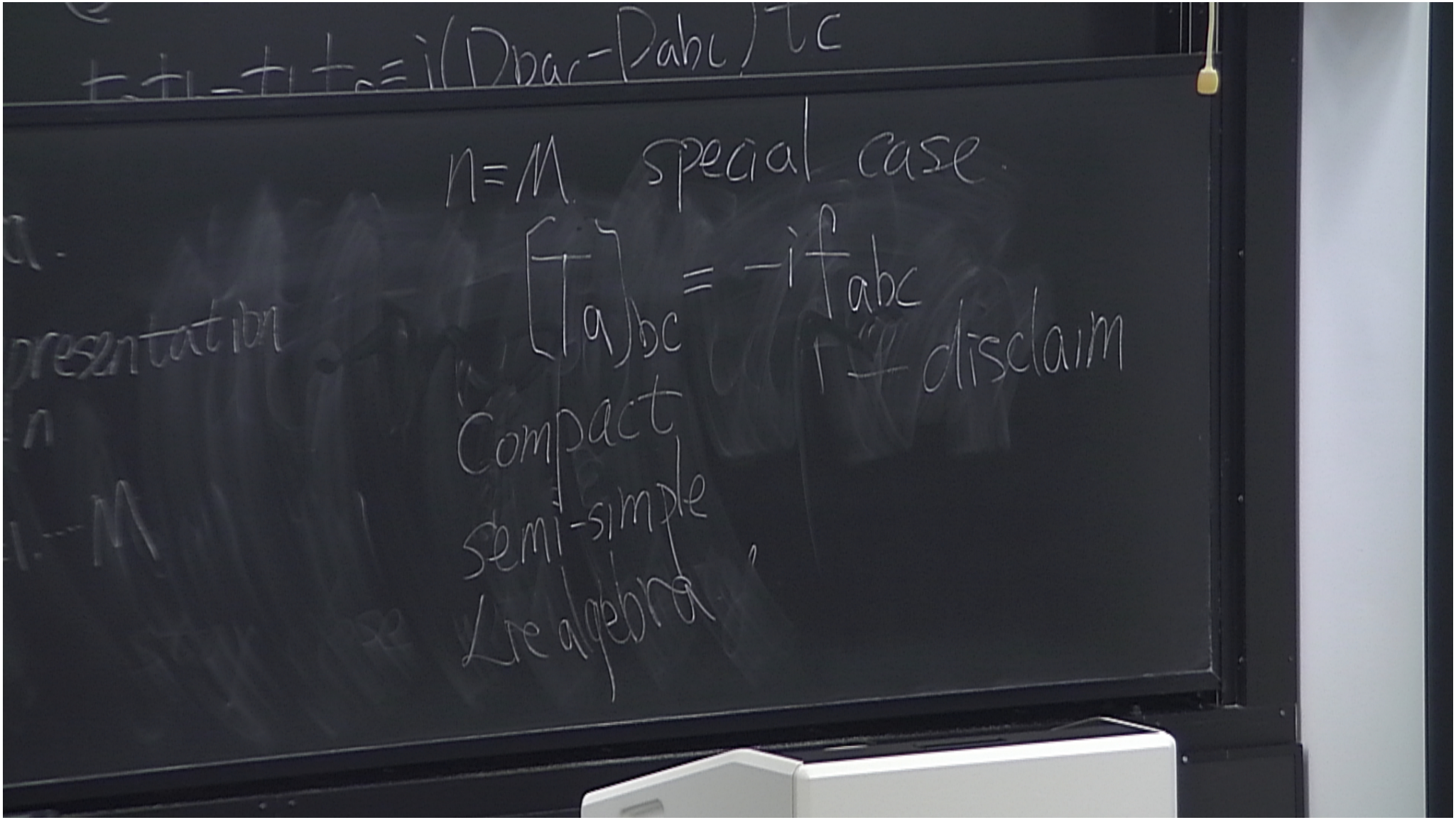

① $[T_a]_{bd}[T_c]_{de} = [T_a T_c]_{be}$
② $-if_{bcd} = -(-i)f_{cbd} = -[T_c]_{bd}$
$(-i)f_{ade} = [T_a]_{de}$
$-[T_c T_a]_{be}$

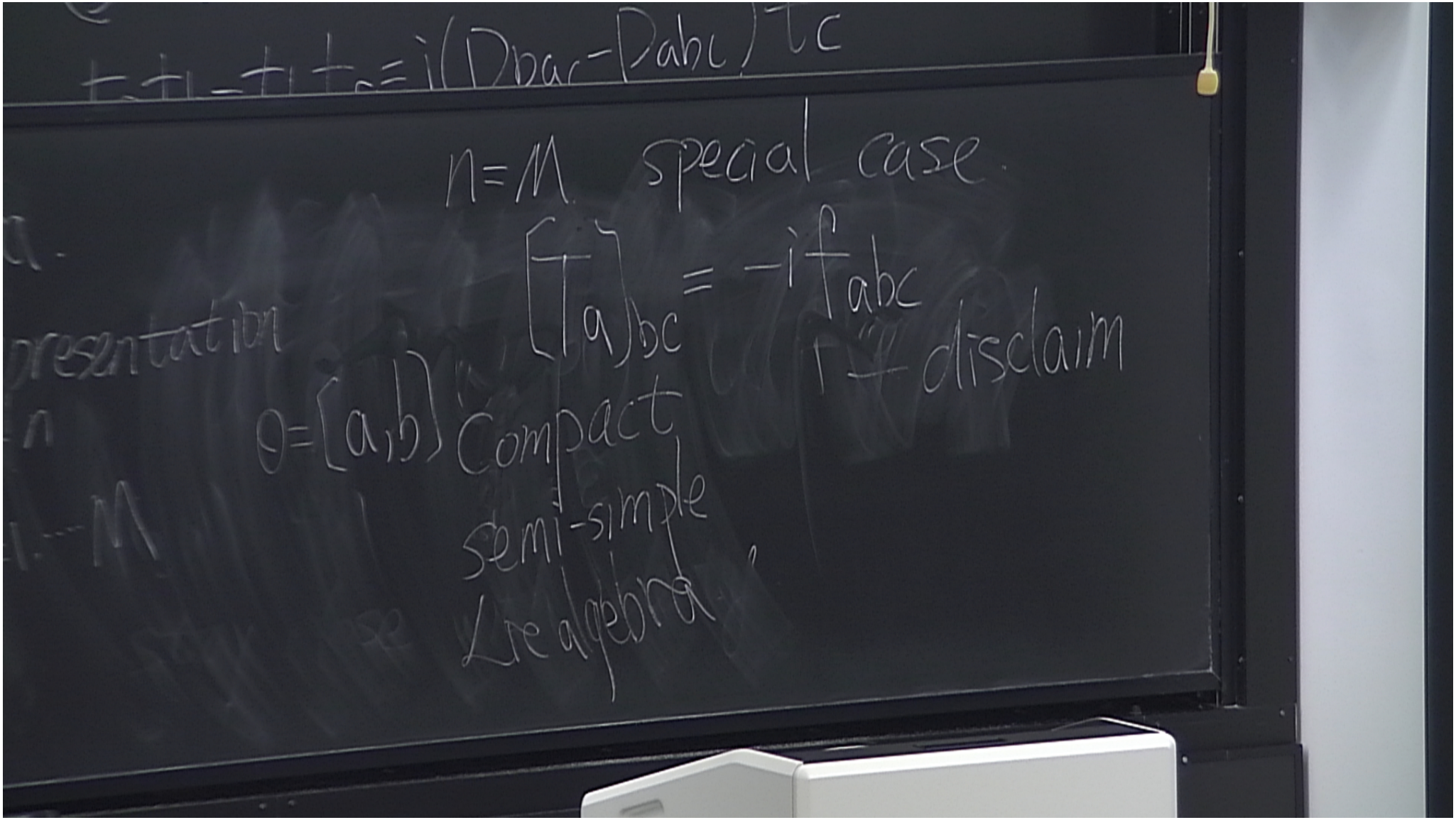

= [Ta Tc] be
d = -[Tc] bd
de
③ (-i) fbde = -(-i) fdbe
= -[Td] be
[Ta Tc]be - [Tc Ta]be

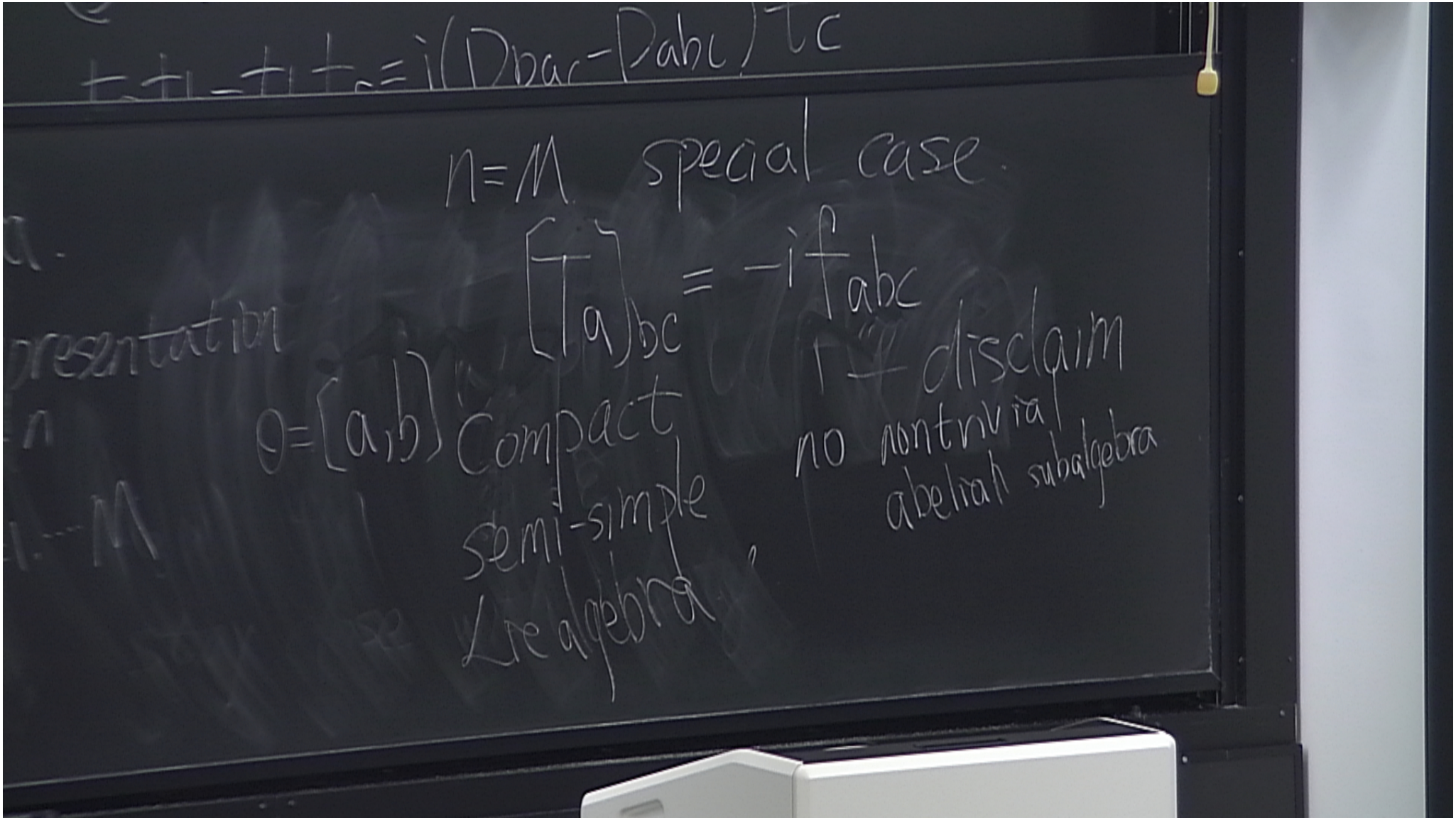

= [Ta Tc] be
d = -[Tc] bd
de
③ (-i) fbde = -(-i) fdbe
= -[Td] be
[Ta Tc]be - [Tc Ta]be - ifcad (-Td)be
=0
Ta Tc - Tc Ta + ifcad Td = 0
[Ta, Tc]

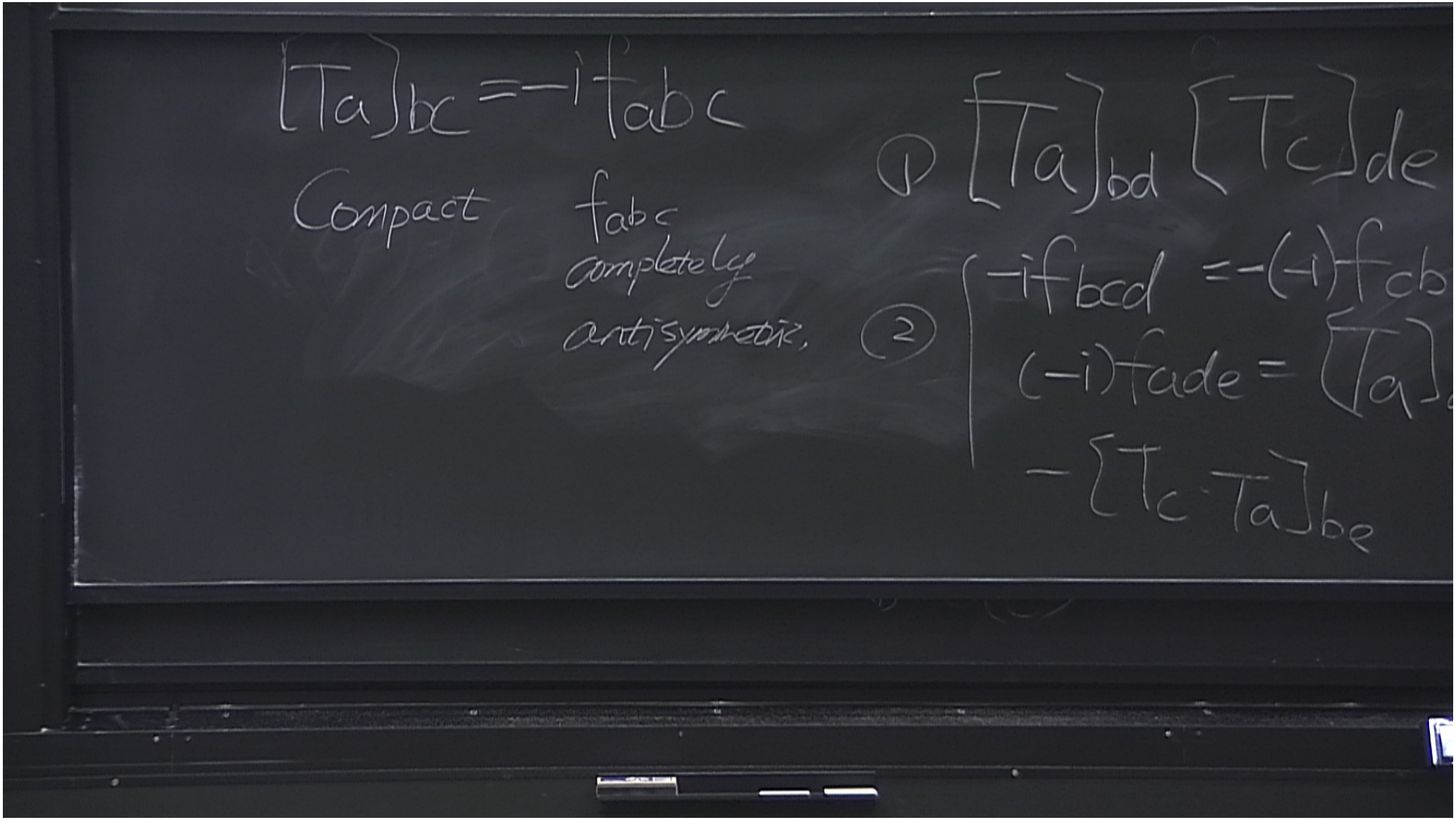

= [Ta Tc] be
= - [Tc] bd
de
③ (-i) fbde = -(-i) fdbe
= - [Td] be
[Ta Tc]be - [Tc Ta]be - ifcad (-Td)be
=0
Ta Tc - Tc Ta + ifcad Td = 0
[Ta, Tc] = ifacd Td


n=M special case.
Compact
semi-simple
Lie algebra


n=M special case.
Compact
semi-simple
Lie algebra
disclaim


$n = M$ special case.
$[T_a]_{bc} = -if_{abc}$
$\theta = [a, b)$ Compact
semi-simple
Lie algebra
disclaim


n=M special case
$[T_a]_{bc} = -if_{abc}$
$\theta=[a,b)$ Compact
semi-simple
Lie algebra
disclaim
no nontrivia
abelian subalgebra


$[T_a]_{bc} = -i f_{abc}$
Compact
f_{abc} completely antisymmetric.
① $[T_a]_{bd} [T_c]_{de}$
②
$-i f_{bcd} = -(-i) f_{cb}$
$(-i) f_{ade} = [T_a]$
$- [T_c, T_a]_{be}$