Title: An algebraic approach to the problem of time

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Abstract: <p>An algebraic view allows one to compute properties of physical states without having to introduce an explicit Hilbert space. In relational systems without absolute time, a state cannot be completely positive once a choice for a (possibly local) internal time has been made. A more general class of `almost-positive' states is introduced in this talk, together with a discussion of their constrained dynamics.</p>

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An algebraic approach to the problem of time

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Relational time: Classical formulation

Single constraint $C(q,p,\phi,p_\phi)=0$.

Example: $C=H(q,p)^2-p_\phi^2$ with $H(q,p)\geq 0$.

Solve $C=0$ for $p_\phi(q,p)=-H(q,p)$. Gauge flow

$$\frac{\mathrm{d}q}{\mathrm{d}\epsilon}=\{q,C\}=2H\{q,H\} \quad , \quad \frac{\mathrm{d}p}{\mathrm{d}\epsilon}=\{p,C\}=2H\{p,H\}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\epsilon}=\{\phi,C\}=-2p_\phi \quad , \quad \frac{\mathrm{d}p_\phi}{\mathrm{d}\epsilon}=\{p_\phi,C\}=0$$

equivalent to Hamiltonian equations of motion

$$\frac{\mathrm{d}q}{\mathrm{d}\phi}=\{q,H\}=\frac{\mathrm{d}q/\mathrm{d}\epsilon}{\mathrm{d}\phi/\mathrm{d}\epsilon} \quad , \quad \frac{\mathrm{d}p}{\mathrm{d}\phi}=\{p,H\}=\frac{\mathrm{d}p/\mathrm{d}\epsilon}{\mathrm{d}\phi/\mathrm{d}\epsilon}$$

as long as $\mathrm{d}\phi/\mathrm{d}\epsilon\neq 0$. Local time if H depends on ϕ.

Relational time: Quantum problems

- Quantize $\hat{H}(q,p)$ on, say, $L^2(\mathbb{R}, \mathrm{d}q)$. Hamiltonian equations replaced by Schrödinger flow in ϕ. *Physical Hilbert space.*

  Or: Quantize $\hat{C}(q,p,\phi,p_\phi)$ on, say, $L^2(\mathbb{R}^2, \mathrm{d}q\mathrm{d}\phi)$. Constraint equation $\hat{C}\psi = 0$ re-interpreted as Schrödinger equation on "positive-frequency" solutions. *Kinematical Hilbert space.*
- Need Global time ϕ without turning points ($p_\phi \neq 0$) for unitarity.
- Technical problems with square root of $\widehat{H(q,p)}^2$ for many interesting systems. H may not be positive definite.
- Physical Hilbert space known only in special cases with global time. Here: $L^2(\mathbb{R}, \mathrm{d}q)$ obtained from $L^2(\mathbb{R}^2, \mathrm{d}q\mathrm{d}\phi)$ by eliminating internal time ϕ as an operator.

Strategy

Difficult to define suitable transformations between physical Hilbert spaces obtained with different choices of internal times.

Necessary if time is local, and to demonstrate covariance.

Here:

→ Keep time as an observable: use kinematical algebra $\mathcal{A} \ni \hat{\phi}$.

→ Introduce generalized class of states which "treat $\hat{\phi}$ as a number":
Contain physical states but defined on kinematical algebra.

→ Hope that it will be easier to transform generalized states than entire Hilbert spaces.

Non-positive states

If $\phi = \langle \hat{\phi} \rangle$ treated as time in Schrödinger flow, must be factorizable in expectation values: For any operator $\hat{O}$,

$$\langle \hat{\phi}\hat{O} \rangle = \langle \hat{\phi} \rangle \langle \hat{O} \rangle$$

→ For $\hat{O} = \hat{\phi}$: No time fluctuations, $(\Delta\phi)^2 = \langle \hat{\phi}^2 \rangle - \langle \hat{\phi} \rangle^2 = 0$.

→ For $\hat{O} = \hat{p}_\phi$: Complex covariance
$\Delta(\phi p_\phi) = \frac{1}{2}\langle \hat{\phi}\hat{p}_\phi + \hat{p}_\phi\hat{\phi} \rangle - \langle \hat{\phi} \rangle \langle \hat{p}_\phi \rangle = -\frac{1}{2}i\hbar$.

Formally consistent with uncertainty relation

$$(\Delta\phi)^2(\Delta p_\phi)^2 - \Delta(\phi p_\phi)^2 \geq \frac{\hbar^2}{4}$$

but cannot belong to states as positive linear functionals on the kinematical algebra of observables. Not a physical state.

Algebraic approach

States as positive linear functionals $\omega\colon \mathcal{A} \to \mathbb{C}$ on unital $*$-algebra $\mathcal{A}$: ω linear with $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$.

Positivity implies reality $\omega(A^*) = \overline{\omega(A)}$ and Cauchy–Schwarz inequality

$$\omega(A^*A)\omega(B^*B) \geq |\omega(A^*B)|^2$$

from which uncertainty relations can be derived.

Internal time: Positivity respected only on physical Hilbert space from which time as an operator has been eliminated.

Hard to generalize to local internal time.

Formal saturation of uncertainty relation suggests that more general kinematical setting may exist.

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For comparison: Canonical effective theory

Parameterize state by expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ of basic operators and moments

$$\Delta(q^a p^b \phi^c p_\phi^d) = \langle(\hat{q}-\langle\hat{q}\rangle)^a(\hat{p}-\langle\hat{p}\rangle)^b(\hat{\phi}-\langle\hat{\phi}\rangle)^c(\hat{p}_\phi-\langle\hat{p}_\phi\rangle)^d\rangle_{\rm symm}$$

with totally symmetric ordering.

Commutator of operators determines Poisson bracket of moments.

$$\{\langle\hat{A}\rangle,\langle\hat{B}\rangle\} = \frac{\langle[\hat{A},\hat{B}]\rangle}{i\hbar}$$

Expand phase-space functions:

$$\langle\hat{C}\rangle(\langle\hat{\bullet}\rangle,\Delta(\cdots)) = \langle C(\langle\hat{q}\rangle + (\hat{q}-\langle\hat{q}\rangle),\ldots)\rangle$$

$$= \quad C(\langle\hat{\bullet}\rangle) + \sum_{a,b,c,d}\frac{1}{a!b!c!d!}\frac{\partial^{a+b+c+d}C(\langle\hat{\bullet}\rangle)}{\partial\langle\hat{q}\rangle^a\partial\langle\hat{p}\rangle^b\langle\hat{\phi}\rangle^c\langle\hat{p}_\phi\rangle^d}\Delta(q^a p^b \phi^c p_\phi^d)$$

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Effective constraints

Constraint equation $\hat{C}\psi = 0$ implies

$$\begin{aligned} \langle\hat{C}\rangle(\langle\hat{\bullet}\rangle, \Delta(\cdots)) &= 0 \\ \langle\widehat{\mathrm{pol}}\hat{C}\rangle(\langle\hat{\bullet}\rangle, \Delta(\cdots)) &= 0 \end{aligned}$$

for "operators" $\widehat{\mathrm{pol}}$ polynomial in $\hat{q} - \langle\hat{q}\rangle, \ldots$

Semiclassical expansion: $\Delta(q^a p^b \phi^c p_\phi^d) \sim O(\hbar^{(a+b+c+d)/2})$.

Only finitely many moments and effective constraints relevant to finite order in $\hbar$.

Constrained system on Poisson manifold, not always symplectic: Need to be careful with counting degrees of freedom and classification into constraints of first and second class.

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An effective approach to the problem of time

[with Philipp Höhn, Artur Tsobanjan]

Applied to systems without global internal time, first order in $\hbar$ (second-order moments). Results:

- → If ϕ chosen as internal time, $\Delta(\phi^2) = \Delta(p_\phi^2) = 0$ good gauge fixing of first-order constraints with linear $\widehat{\mathrm{pol}}$.
  On constraint surface, $\Delta(\phi p_\phi) = -\frac{1}{2}i\hbar$ imaginary.
- → "Time expectation value" $\langle\hat{\phi}\rangle$ with imaginary part, proportional to $\langle\hat{p}_\phi\rangle^{-1}$.
- → Constraint $\langle\hat{C}\rangle = 0$ can be solved for Hamiltonian generating evolution by $\mathrm{Re}\langle\hat{\phi}\rangle$.
- → Gauge transformation maps evolving expectation values and moments from one local time choice to another.

Higher orders? Algebraic view?

Almost-positive functionals

Kinematical $*$-algebra $\mathcal{A}$, constraint $C \in \mathcal{A}$.
Find functionals $\omega: \mathcal{A} \to \mathbb{C}$ such that $\omega(\mathcal{A}C) = 0$.

Local time: Choose Zeitgeist $Z = Z^* \in \mathcal{A}$, $[Z, C] \neq 0$.

Linear functional $\omega: \mathcal{A} \to \mathbb{C}$ *almost-positive with respect to Z* if

$$\omega(ZO) = \omega(Z)\omega(O)$$

for all $O \in Z'$ (commutant $Z' = \{A \in \mathcal{A} : [A, Z] = 0\}$) and ω is positive on $\mathcal{F} := Z'/\mathcal{Z}$ with center $\mathcal{Z}$ in Z'.

Canonical example:
$Z = \hat{\phi}$. Z' excludes $\hat{p}_\phi$. $\mathcal{F}$ excludes $\hat{\phi}$ and $\hat{p}_\phi$.
$\mathcal{A}$ represented on kinematical Hilbert space.
$\mathcal{F}$ represented on physical Hilbert space.
ω not contained in either Hilbert space.

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Properties

Almost Cauchy–Schwarz:

$$|\omega(Z^*O)|^2 = |\omega(Z)|^2|\omega(O)|^2 \leq |\omega(Z)|^2\omega(O^*O)$$

Would imply Cauchy–Schwarz if ω real ($\omega(Z^*) = \overline{\omega(Z)}$), but not possible in general. Therefore, ω cannot be positive.

Constraint generates gauge flow

$$\frac{\mathrm{d}\omega_z(A)}{\mathrm{d}z} = \frac{\omega_z([A, NC])}{i\hbar}$$

with some $N \in \mathcal{A}$ ("lapse function").

Local-time evolution by z well-defined if flow preserves almost-positivity with respect to Z.

Preserving almost-positivity

$\longrightarrow$ $\omega(ZO) = \omega(Z)\omega(O)$ preserved by flow if

$$\begin{aligned}\omega([AB,NC]) &= \omega(A[B,NC]) + \omega([A,NC]B)\\ &= \omega(A)\omega([B,NC]) + \omega([A,NC])\omega(B)\end{aligned}$$

for all $A \in \mathcal{Z}$ and $B \in Z'$.

Sufficient condition:
$[B,NC] \in Z'$ for all $B \in Z'$ and $[A,NC] \in \mathcal{Z}$ for all $A \in \mathcal{Z}$.

Example: NC linear in E if $[T,E] = i\hbar$.

$\longrightarrow$ Positivity on $\mathcal{F}$ preserved if $[A,NC]^* = -[A^*,NC]$.

Sufficient condition: $(NC)^* = NC$.

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Global time

Example: $C = E^2 - H^2$ with $[E, H] = 0$ and T exists such that $[T, E] = i\hbar$.

Factorize $C = (E + H)(E - H)$, choose $Z = T$ and $N = (E + H)^{-1}$.

Formal, state-dependent power series:
Introduce $\Delta_\omega A := A - \omega(A)$ for any $A \in \mathcal{A}$, almost-positive ω.

$$E + H = \omega(E) + \omega(H) + \Delta_\omega E + \Delta_\omega H \approx 2\omega(E) + \Delta_\omega E + \Delta_\omega H$$

if $\omega(NC) = \omega(E - H) = 0$. Expand

$$N = (E + H)^{-1} = \frac{1}{2\omega(E)} \sum_{n=0}^{\infty} \left(-\frac{\Delta_\omega E + \Delta_\omega H}{2\omega(E)} \right)^n$$

Agrees with moment expansion in effective treatment.

Time-dependent potential

$C = E^2 - H^2 - V(Z)$. Still $[E, H] = 0$, $[Z, E] = i\hbar$.

$[E, V(Z)] \neq 0$: time-dependent classical Hamiltonian $\sqrt{H^2 + V}$.

Z not a global internal time: turning points where $[Z, C] = 2E = 0$, $E(z)$ no longer constant.

No simple factorization:

$$\left(E + \sqrt{H^2 + V}\right)\left(E - \sqrt{H^2 + V}\right) \neq E^2 - H^2 - V$$

if $[E, V] \neq 0$.

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Complex local time

Approximate factorization:

$$C = \left(E + \sqrt{H^2 + V} + X\right)\left(E - \sqrt{H^2 + V} - X\right)$$

to first order in $\hbar$ and $V \ll H^2$ if

$$X = \frac{1}{4} i\hbar \frac{V'(Z)}{H^2} + O(\hbar^2) + O(V^2)$$

$\longrightarrow$ Choose $N = (E + \sqrt{H^2 + V} + X)^{-1}$.
$NC \neq (NC)^*$ if $Z = Z^*$.
$\longrightarrow$ Require $\omega(NC)$ real: $\omega(V(Z) + \frac{1}{2} i\hbar H^{-1} V'(Z))$ real.
$\longrightarrow$ Implies

$$\mathrm{Im}\omega(Z) = -\frac{1}{2}\hbar\omega(H^{-1}) + O(\hbar^2)$$

Agrees with effective result.

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Summary

New generalization of states:
Almost-positive linear functionals in presence of internal time.

Several effective results reproduced quantitatively:

- → Imaginary part of $\omega(Z)$ for local internal time Z.
- → Some complex moments.
- → Moment expansion of multiplier N provides correct combination of effective gauge flows preserving Zeitgeist. Easier to compute.

Open questions:

- → Gauge flow changing local internal time. Cannot be based solely on almost-positive functionals.
- → Existence of almost-positive functionals? Generalized GNS construction?