Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 5

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Abstract: <p>Course Description coming soon.</p>

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on M is an associative algebra, called the Grassmann algebra over M.

- The multiplication in $\Lambda(M)$ is the wedge product: $\wedge : \Lambda_s(M) \times \Lambda_r(M) \to \Lambda_{s+r}(M)$

- The exterior derivative $d : \Lambda(M) \to \Lambda(M)$ is an anti-derivation of degree $k=1$ of the Grassmann algebra $\Lambda(M)$.

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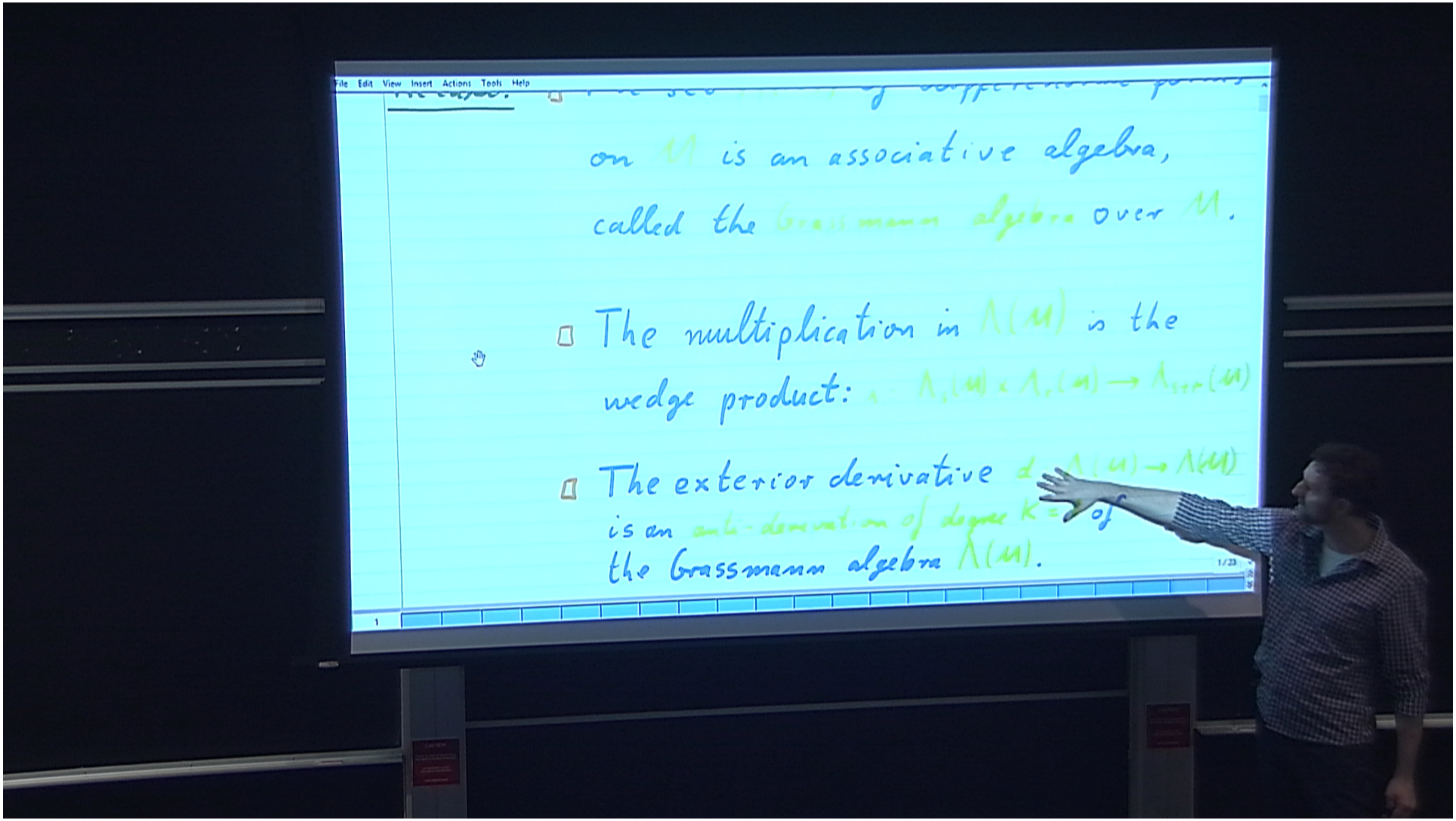
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on M is an associative algebra,
called the Grassmann algebra over M.
The multiplication in Λ(M) is the
wedge product: ∧ : Λ_s(M) × Λ_r(M) → Λ_{s+r}(M)
The exterior derivative d : Λ(M) → Λ(M)
is an anti-derivation of degree k = of
the Grassmann algebra Λ(M).

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Recall: Tangent vectors ξ are directional derivatives on $\Lambda_0(M)$!

Plan now:

A. Define an anti-derivation i_ξ of degree $K=-1$: the inner derivation.

(i_ξ will generalize feeding a tangent vector ξ to a 1-form to feeding it to a p-form.)

B. Combine d, i_ξ to obtain a derivation of degree $K=0$: the Lie derivative

(And the Lie derivative is going to be the directional derivative for differential forms and ten...)

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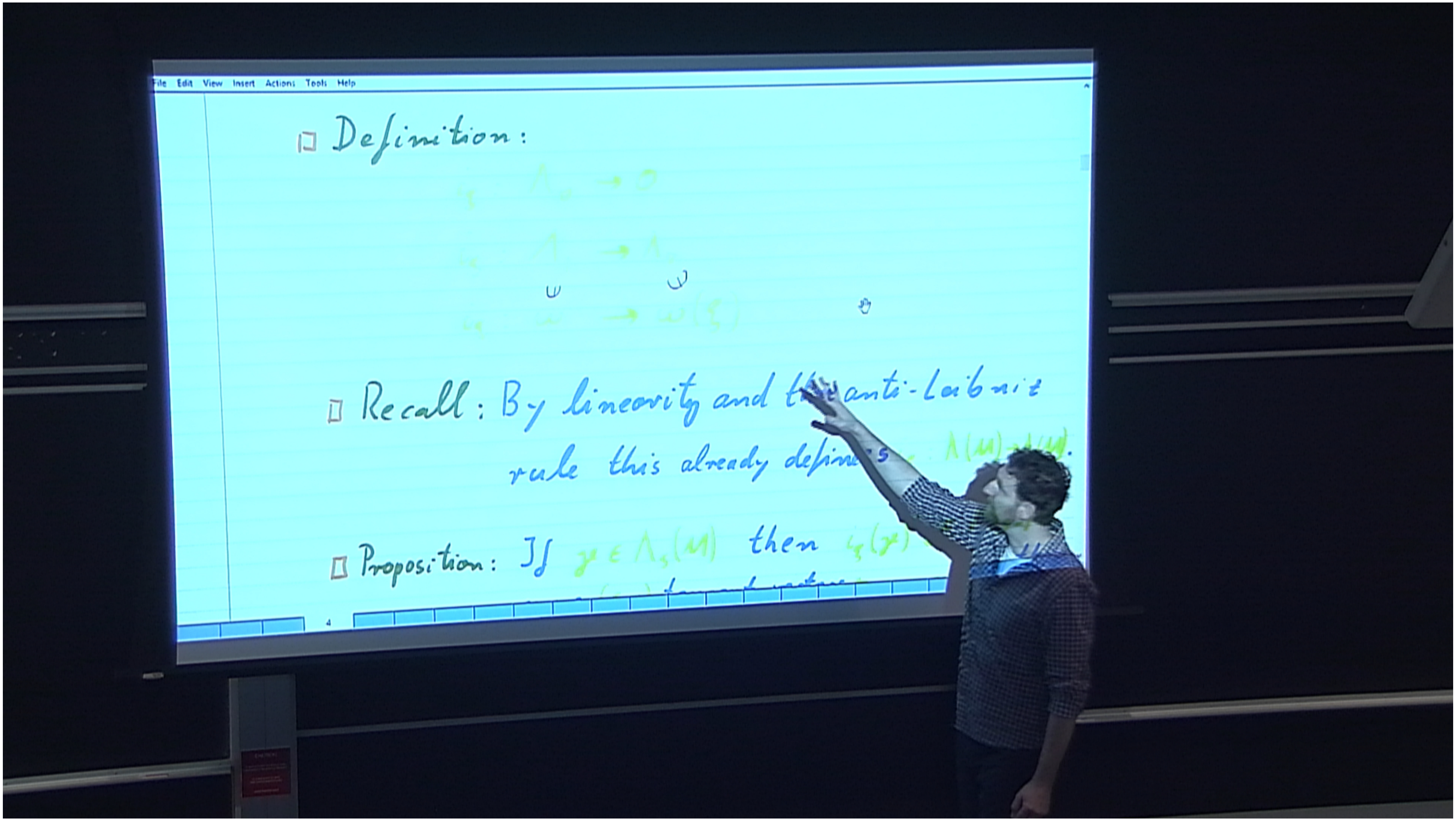
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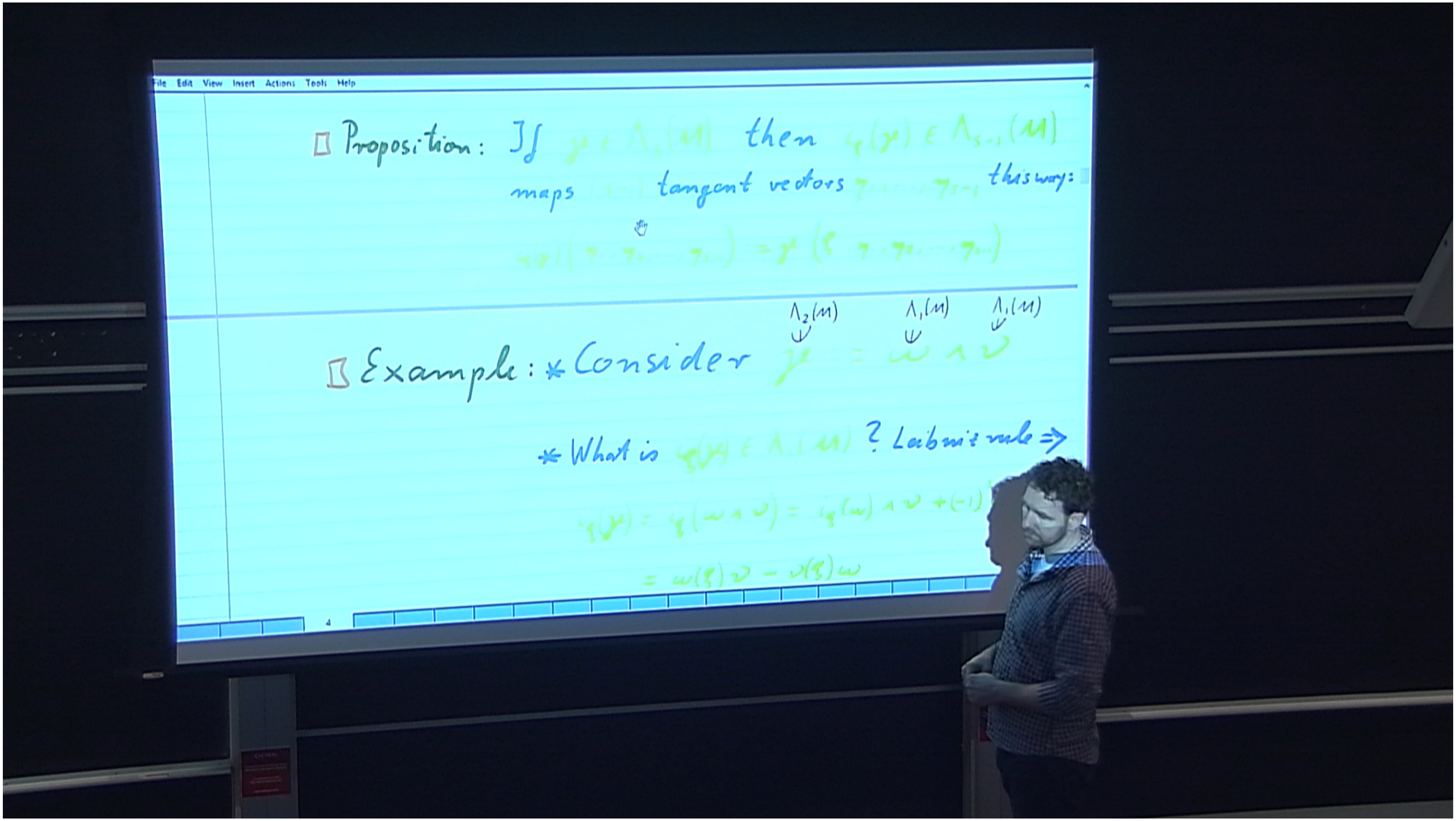
Definition:
Recall: By linearity and the anti-Leibniz
rule this already defines
Proposition: If
then



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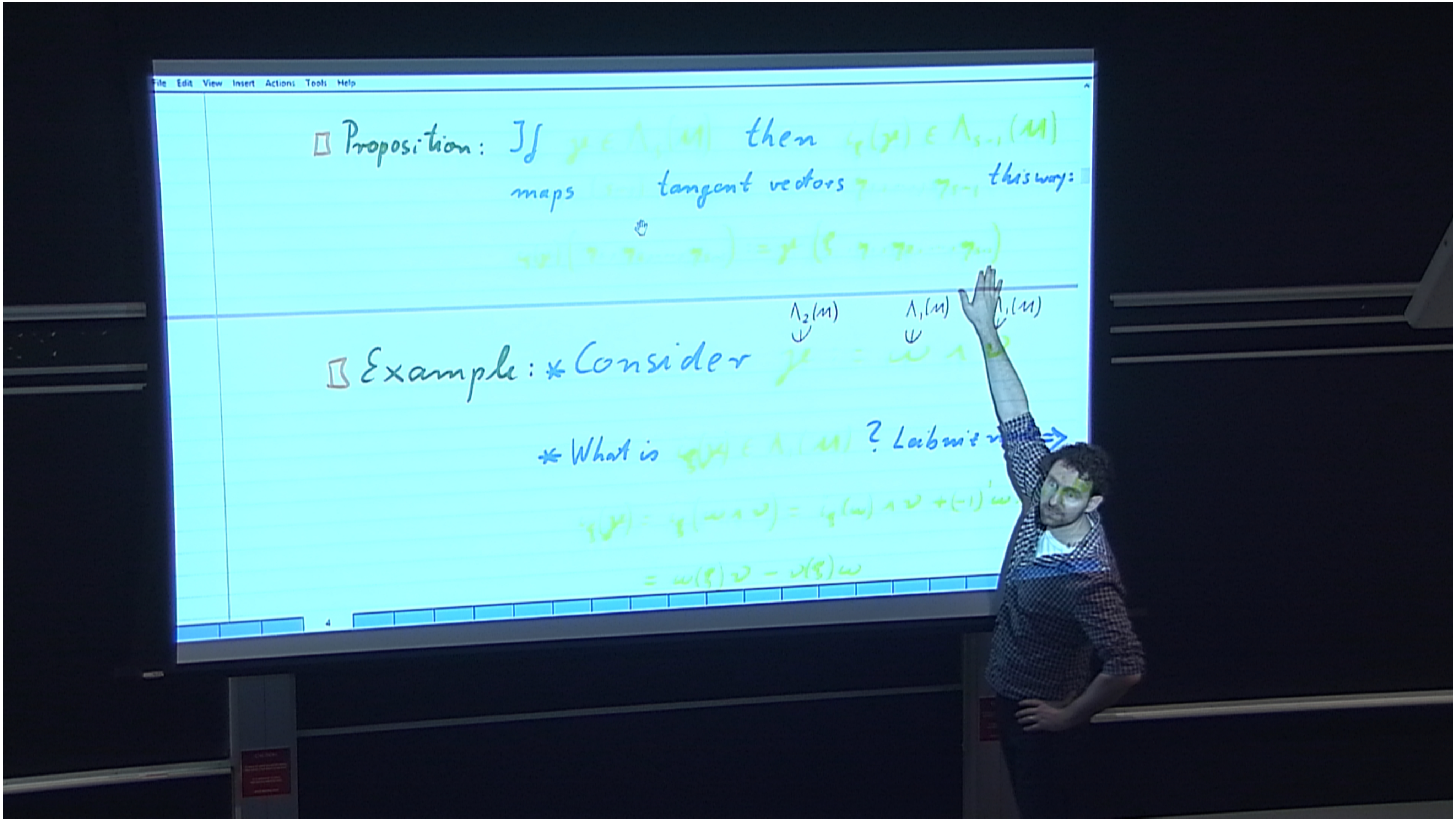
Proposition: If
then
maps
tangent vectors
this way:
Λ₂(M)
Λ₁(M)
Λ₁(M)
Example: * Consider
* What is
? Leibniz rule ⇒
= ω(ξ)ν − ν(ξ)ω
4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Proposition: If then
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* What is ? Leibniz rule ⇒

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Properties of i_ξ :

- $i_{\xi_1} \circ i_{\xi_2} = - i_{\xi_2} \circ i_{\xi_1}$

(Exercise : prove this)

- Thus, in particular:

$$i_\xi \circ i_\xi = 0$$

- Recall: We also have $d \circ d = 0$

(Simply the evaluation of a dual vector applied to a vector in the vector space)

Recall: For $\xi \in T_p(M)$, $\gamma \in T_p^*(M)$, we have $i_\xi(\gamma) = \gamma(\xi) = \xi(\gamma)$

Definition: The inner derivation, $i_\xi(\gamma)$, of a $\gamma \in \Lambda(M)$

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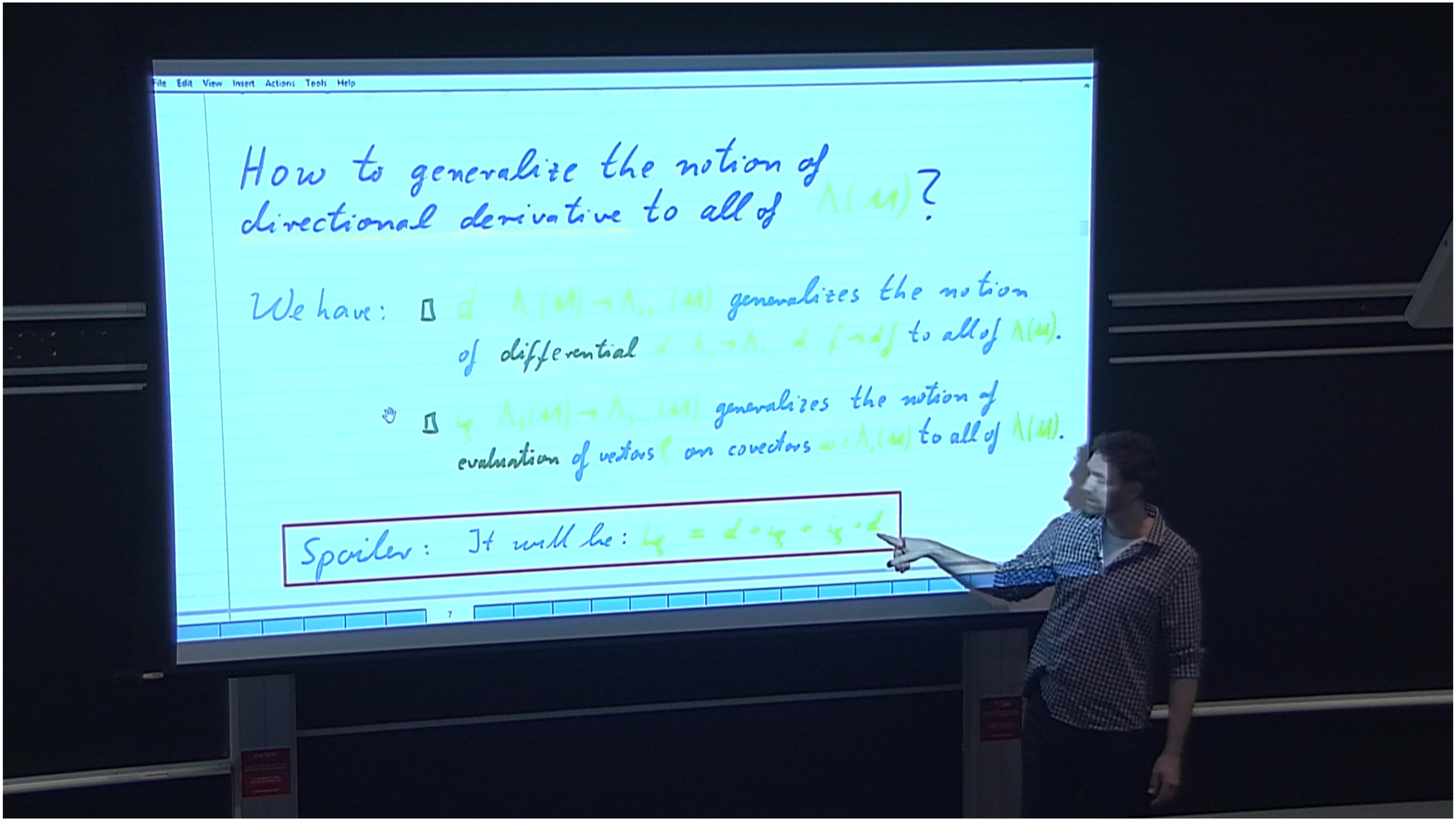
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How to generalize the notion of directional derivative to all of Λ(M)?
We have:
generalizes the notion of differential
to all of Λ(M).
generalizes the notion of
evaluation of vectors on covectors
to all of Λ(M).
Spoiler: It will be:

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Then it is natural to define the directional derivative of a gradient field of a function to be the gradient of the directional derivative of the function: (because derivatives ought to commute and the gradient is a derivative too.)

$$L_\xi : \Lambda_1(M) \to \Lambda_1(M)$$

$$L_\xi : \quad df \to \underbrace{d(\underbrace{\xi(f)}_{\in \Lambda_0(M)})}_{\in \Lambda_1(M)}$$

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i.e.: $\underbrace{L_\xi(df)}_{\text{directional derivative of gradient}} = \underbrace{d(\xi(f))}_{\text{gradient of directional derivative}}$ (D)

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Question: Now that L_ξ is a fully defined derivation

$$L_\xi : \Lambda(M) \to \Lambda(M),$$

can we relate it to d and i_ξ? Yes:

Cartan's equation:

$$\boxed{\overbrace{L_\xi = d \circ i_\xi + i_\xi \circ d}^{\text{Exercise: show it is a derivation}}}$$

Proof:

check on $\Lambda_0(M)$: $L_\xi \underset{\Lambda_0(M)}{f} = d \circ \underbrace{i_\xi(f)}_{\substack{=0 \text{ because} \\ f \in \Lambda_0(M)}} + i_\xi(df) = 0 + df(\xi) = \xi(f)$ ✓

because: $d^2 = 0$

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check on $\Lambda_0(M)$: $\underset{\Lambda_0(M)}{L_\xi f} = d \circ \underbrace{i_\xi(f)}_{\substack{=0 \text{ because} \\ f \in \Lambda_0(M)}} + i_\xi(df) = 0 + df(\xi) = \xi(f)$ ✓

check on basis of $\Lambda_1(M)$, e.g. $df = dx^i$: $L_\xi df = d \circ \underbrace{i_\xi(df)}_{= df(\xi) = \xi(f)} + i_\xi \circ \underbrace{ddf}_{\substack{0 \\ \text{because: } d^2 = 0}} = d(\xi(f))$ ✓

I.e., indeed, as in (D): directional derivative of gradient = gradient of directional derivative

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can we relate it to d and i_ξ? **Yes:**

Exercise: show it is a derivation

Cartan's equation: $$\boxed{\overbrace{L_\xi}^{\text{Exercise: show it is a derivation}} = d \circ i_\xi + i_\xi \circ d}$$

Proof:

check on $\Lambda_0(M)$: $\underset{\Lambda_0(M)}{L_\xi f} = d \circ \underbrace{i_\xi(f)}_{\substack{=0 \text{ because} \\ f \in \Lambda_0(M)}} + i_\xi(df) = 0 + df(\xi) = \xi(f)$ ✓

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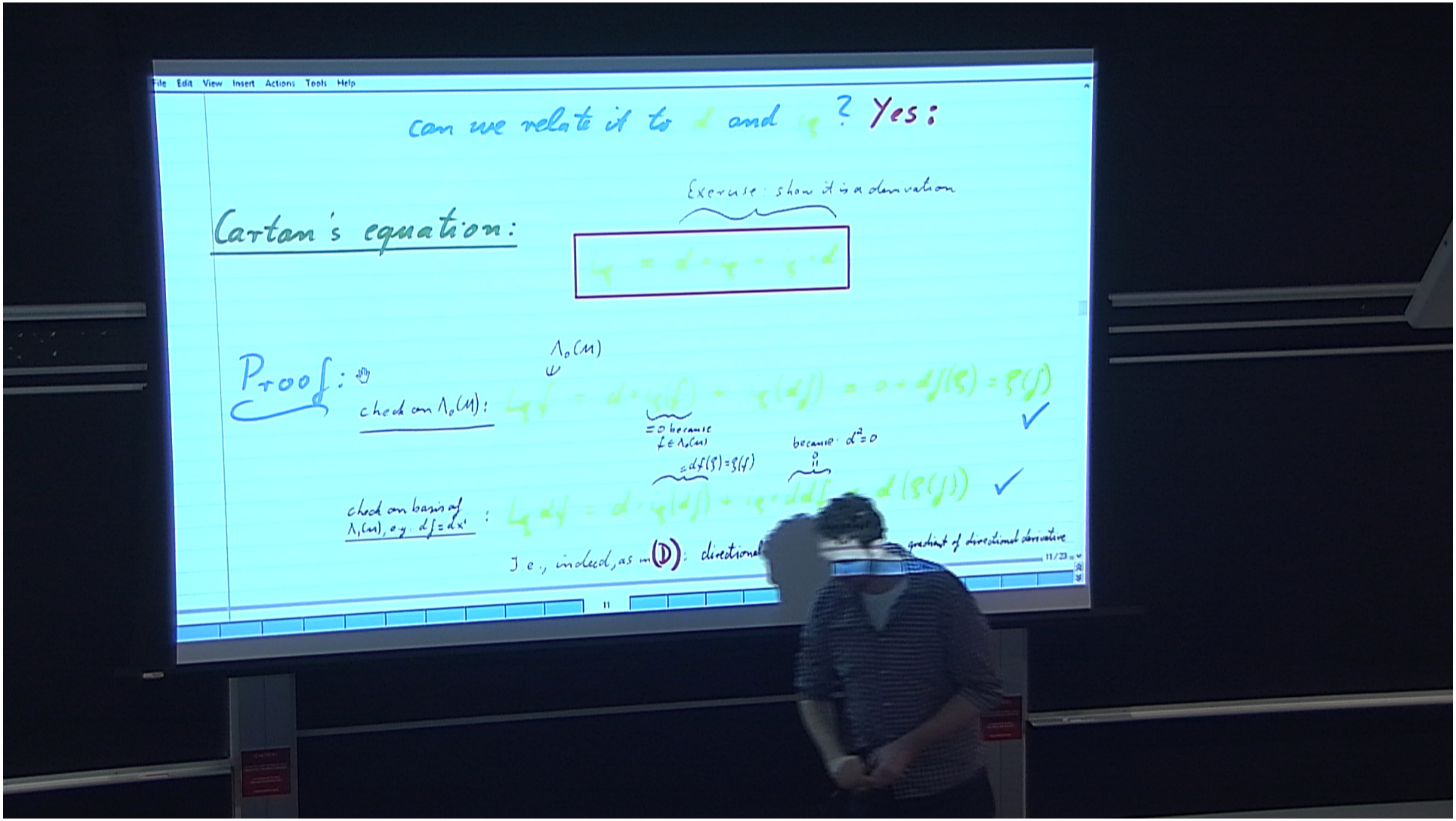
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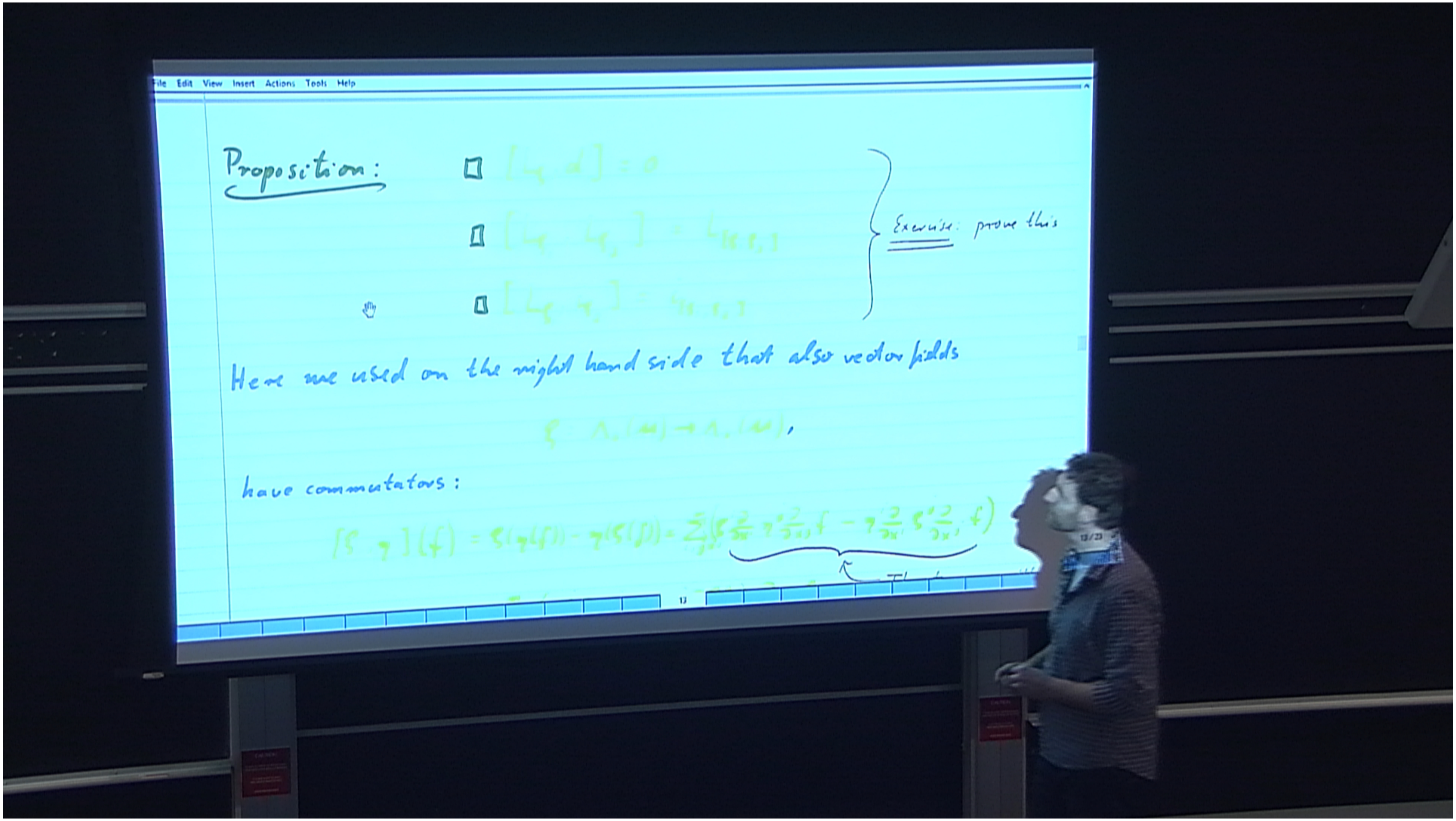
can we relate it to d and i_ξ ? Yes:
Exercise: show it is a derivation
Cartan's equation:
L_ξ = d∘i_ξ + i_ξ∘d
Proof:
Λ0(M)
check on Λ0(M):
L_ξ f = d∘i_ξ(f) + i_ξ(df) = 0 + df(ξ) = ξ(f)
= 0 because f ∈ Λ0(M)
because d² = 0
= df(ξ) = ξ(f)
check on basis of Λ1(M), e.g. df = dx^i :
L_ξ df = d∘i_ξ(df) + i_ξ∘ddf = d(ξ(f))
I.e., indeed, as in (D): directional
gradient of directional derivative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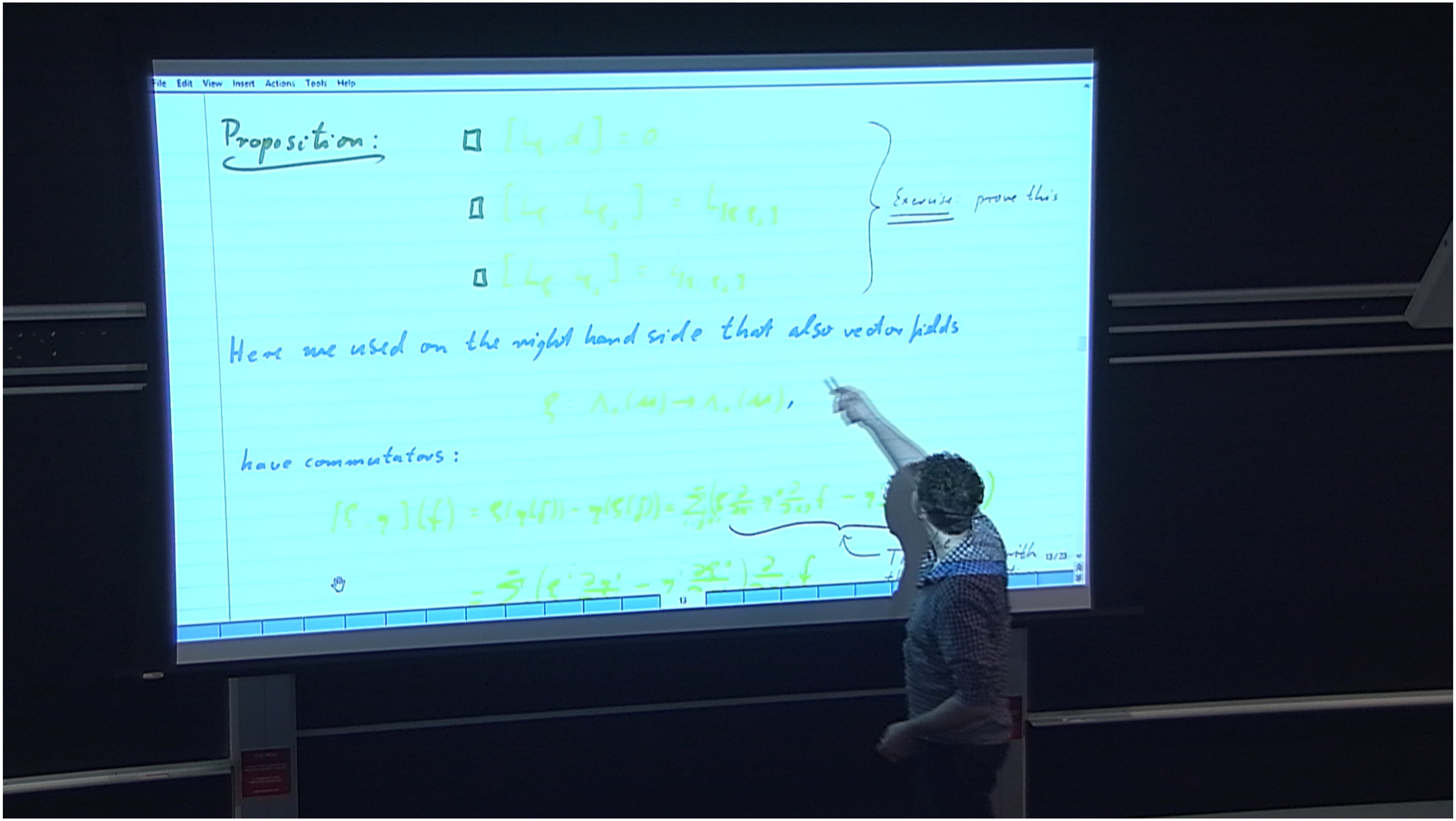
Proposition:
Exercise: prove this
Here we used on the right hand side that also vector fields
have commutators: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Here we used on the right hand side that also vector fields
have commutators:

$$\square \ [L_{\xi_1}, i_{\xi_2}] = i_{[\xi_1, \xi_2]}$$

Here we used on the right hand side that also vector fields

$$\xi : \Lambda_0(M) \to \Lambda_0(M),$$

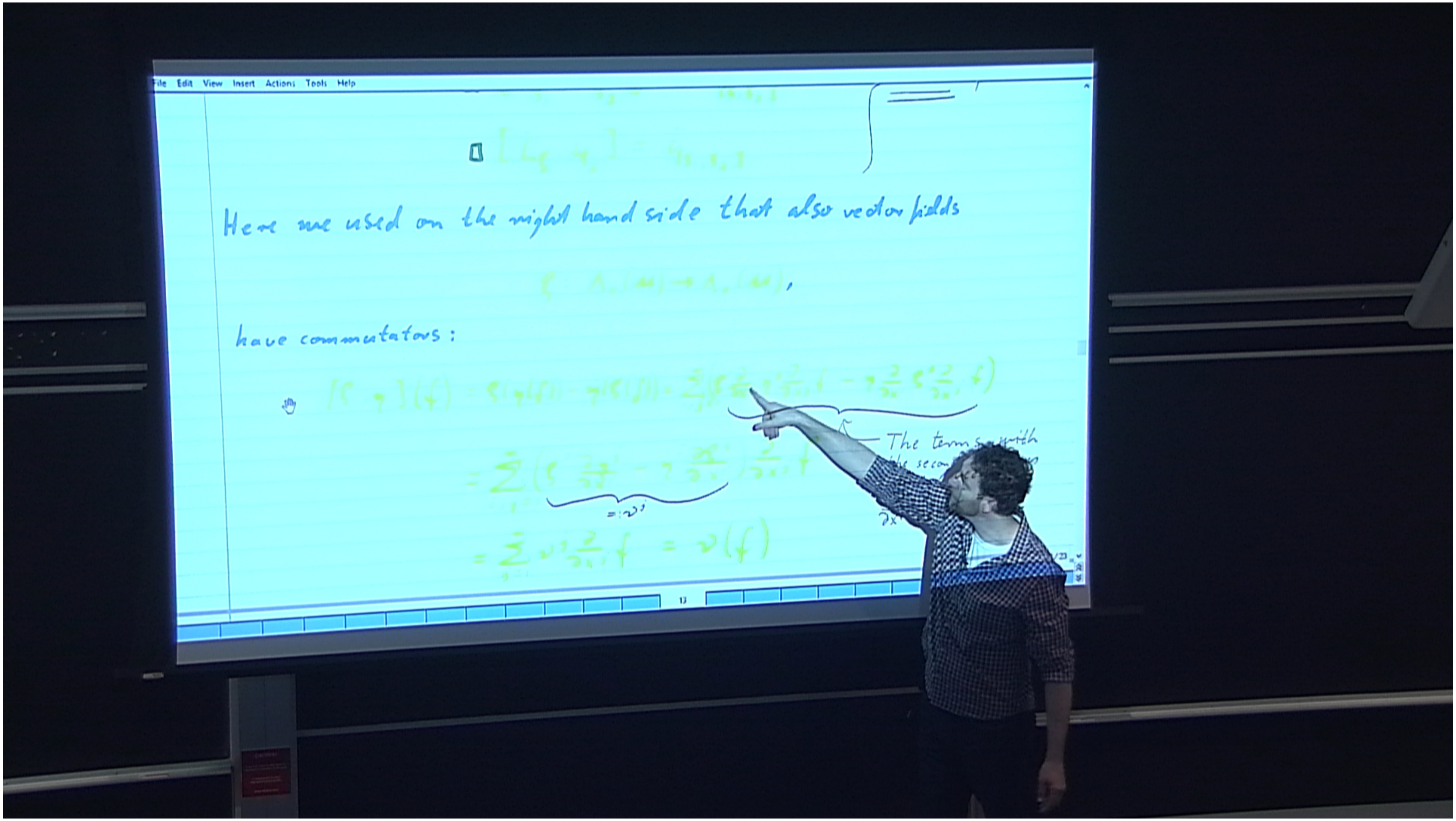
have commutators:

$$[\xi, \eta](f) = \xi(\eta(f)) - \eta(\xi(f)) = \sum_{i,j=1}^{m} \Big(\underbrace{\xi^i \frac{\partial}{\partial x^i} \eta^j \frac{\partial}{\partial x^j} f - \eta^i \frac{\partial}{\partial x^i} \xi^j \frac{\partial}{\partial x^j} f}\Big)$$

The terms with the second derivatives cancel because: $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} f$

$$= \sum_{i,j=1}^{m} \underbrace{\Big(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \Big)}_{=: \nu^j} \frac{\partial}{\partial x^j} f$$

$$= \sum_{j=1}^{m} \nu^j \frac{\partial}{\partial x^j} f = \nu(f)$$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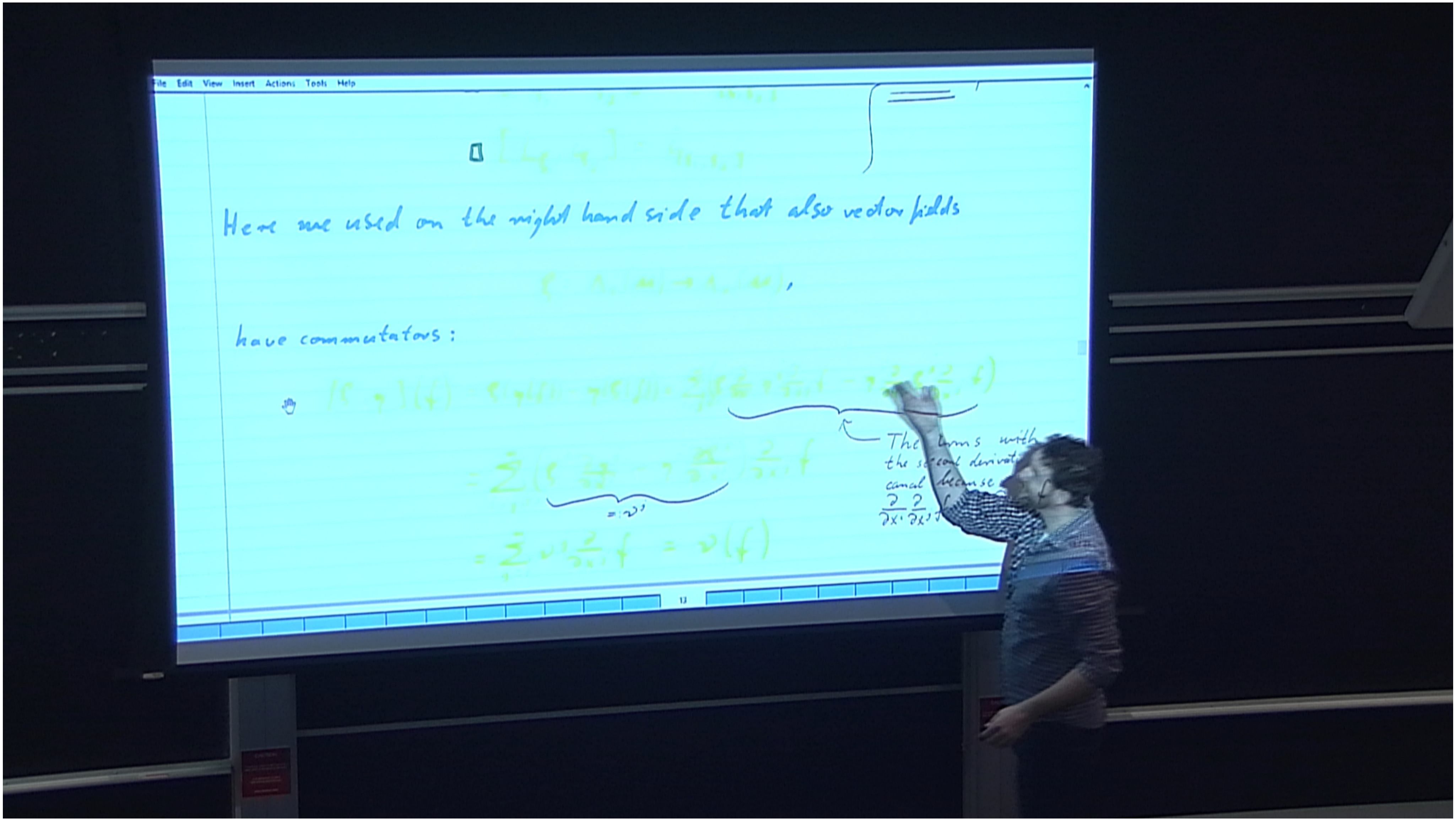
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Here we used on the right hand side that also vector fields
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Questions:

Since L_ξ is the directional derivative on $\Lambda(M)$:

- Can L_ξ be extended to a directional derivative for all tensor fields? Yes!

- Can L_ξ be expressed as a Newton-Leibniz limit similar to

need an analog: a shift on a manifold, in the direction given by ξ.

$$f'(x) = \lim \frac{f(\overbrace{x+\varepsilon}) - f(x)}{\ }$$? Yes!

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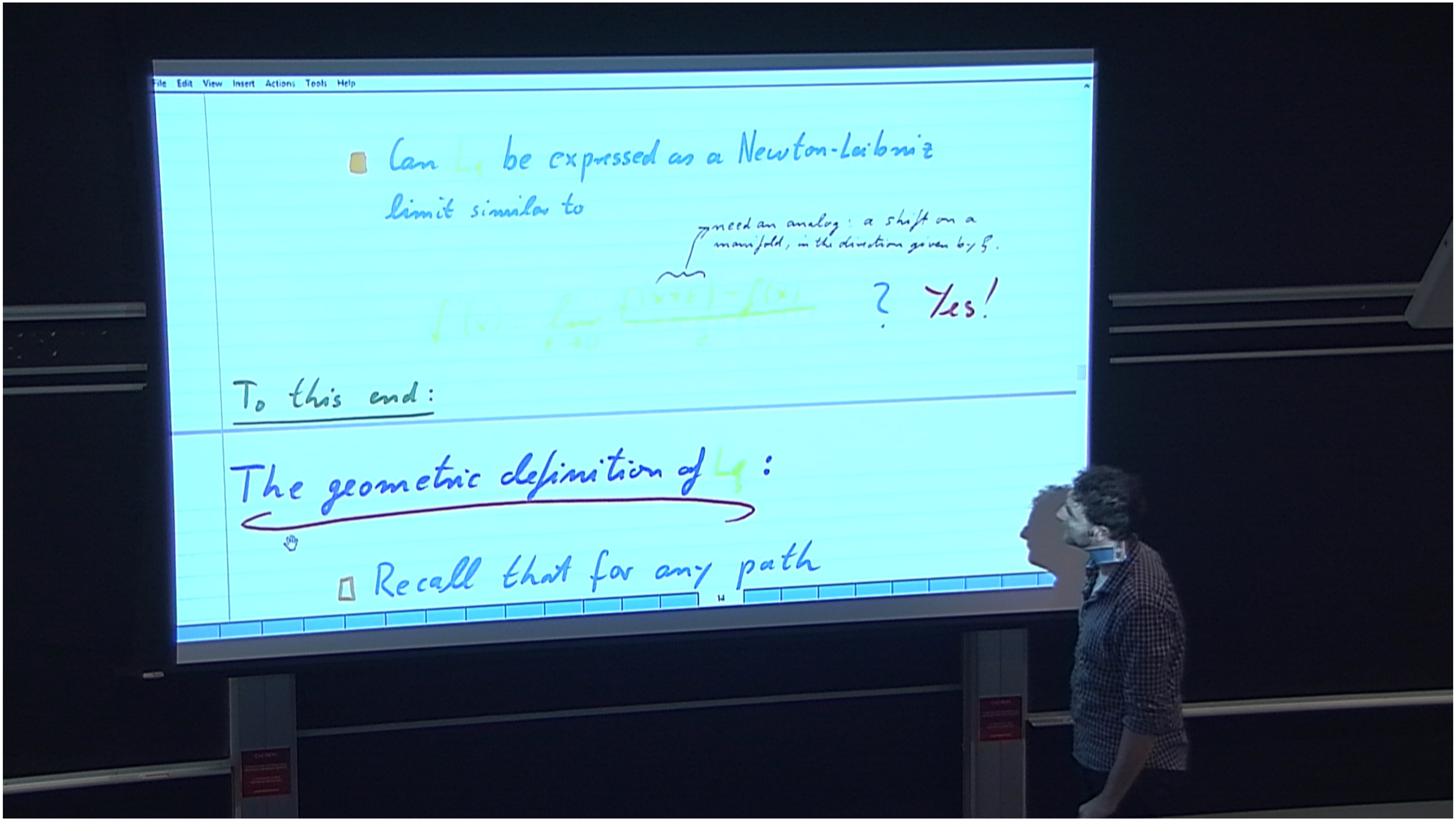
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need an analog: a shift on a manifold, in the direction given by ξ.
$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$? Yes!
To this end:
The geometric definition of L_ξ :
Recall that for any path
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The geometric definition of L_ξ:

- Recall that for any path

an open interval of $\mathbb{R}$

$$\gamma : \mathbb{R} \supset J \to M$$
$$\quad\quad\;\; \cup \quad\;\; \cup$$
$$\gamma : \quad t \to \gamma(t)$$

we have a tangent vector $\bar{\gamma}(t_0) \in T_{\gamma(t_0)}(M)$ at each point $\gamma(t_0)$ of the path:

$$\bar{\gamma}(t_0) : f \to \bar{\gamma}(t_0)(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=t_0}$$

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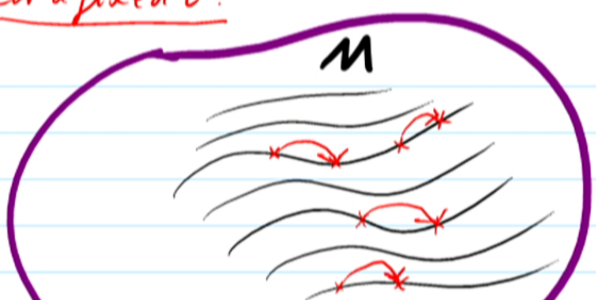
- From theory of ODEs:

  For every $p \in M$ there exists a maximal (i.e. inextendible) C^∞ integral curve through p.

- Thus, ξ yields a "flow": (at least for small t, locally):

for a fixed t:



i.e., for any fixed value of the flow parameter t each point of M is mapped

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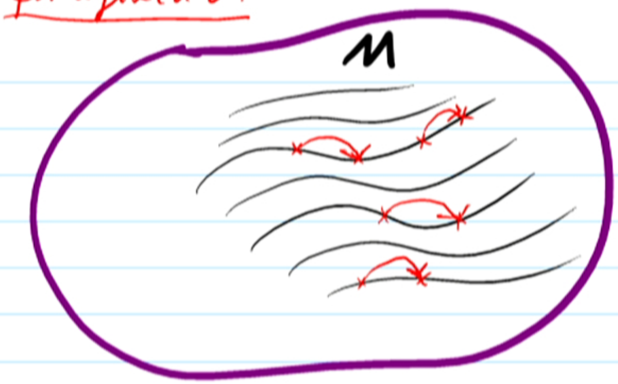
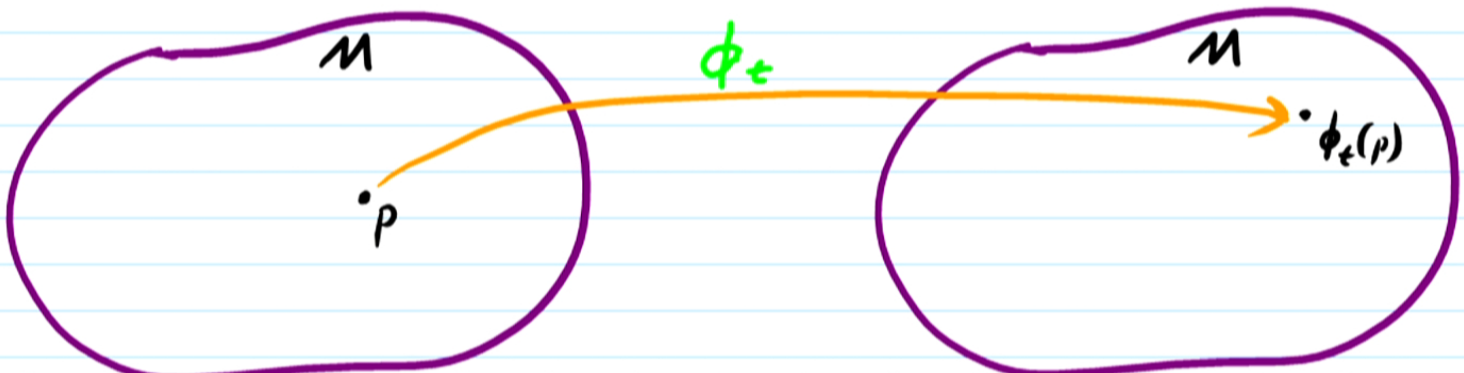
for a fixed t:



i.e., for any fixed value of the flow parameter t each point of M is mapped into another point of M.

- The flow is a diffeomorphism "$\phi_t : M \to M$":



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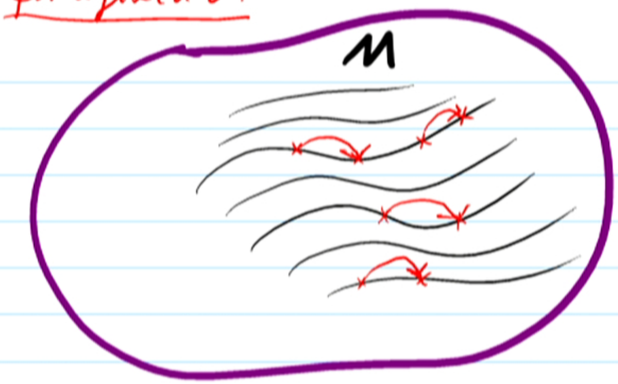

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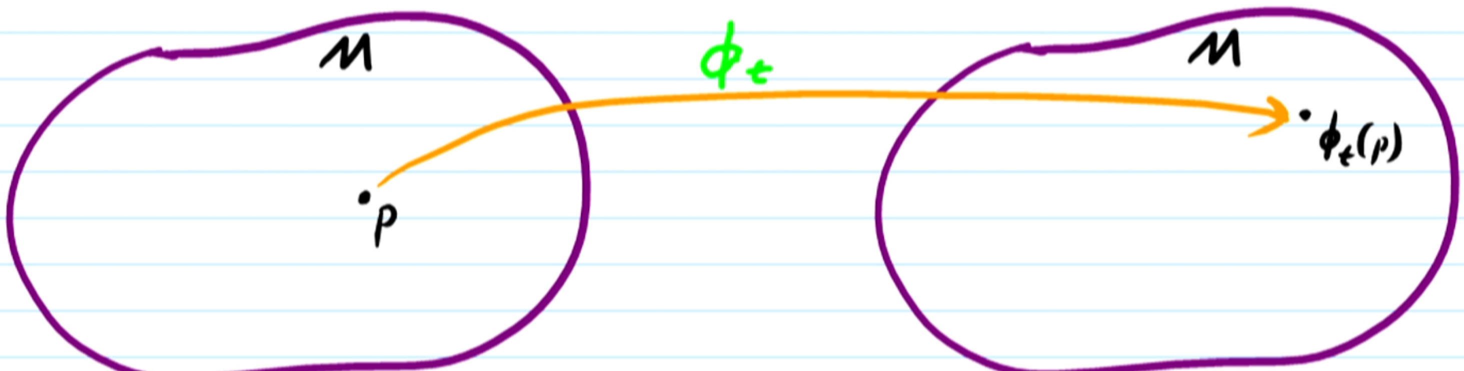
□ The flow is a diffeomorphism "$\phi_t : M \to M$":



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Definition:

geom. definition

The Lie derivative of any tensor field τ at the point $p = \gamma(0) \in \mathcal{M}$ with respect to the flow induced by a vector field ξ is defined through:

$$\mathcal{L}_\xi \tau := \lim_{t \to 0} \frac{1}{t} \left(\phi_t^{*-1}(\tau) - \tau \right)$$

i.e. $\mathcal{L}_\xi(\tau)(p) = \lim \frac{1}{t} \Big[(\phi^*)^{-1} \big(\overbrace{\tau(\gamma(t))}^{\text{tensor field value at image of } p, \text{ i.e. } \in T_{\gamma(t)}(\mathcal{M})^r_s} \big) - \tau(p) \Big]$

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Explicitly, in a chart:

- $\phi: x \to \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t\,\xi^i(x) + \mathcal{O}(t^2)$

- Jacobian matrix: $\dfrac{\partial \tilde{x}^i}{\partial x^j} = \delta^i_j + t\,\dfrac{\partial \xi^i(x)}{\partial x^j} + \mathcal{O}(t^2)$

  we write $= \xi^i{}_{,j}$

- Inverse Jacobian: $\dfrac{\partial x^i}{\partial \tilde{x}^j} = \delta^i_j - t\,\dfrac{\partial \xi^i(x)}{\partial x^j} + \mathcal{O}(t^2)$

- Image of tensor at $\tau(\tilde{x})^{i_1 \dots i_r}_{j_1 \dots j_s}$ under flow, backwards, $\tilde{x} \to x$, has the

From now, we will omit writing Σ: Twice occurring indices are always to be summed over (Einstein convention)

components:

$$\phi^{*}{}^{-1}(\tau(x))^{i_1 \dots i_r}_{j_1 \dots j_s} = \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(\tilde{x})\,\frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}}\,\frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

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- $\phi: x \to \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t\xi^i(x) + O(t^2)$

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$$= \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(x + t\xi)\left(\delta^{i_1}_{\bar{i}_1} - t\xi^{i_1}_{,\bar{i}_1}\right) \cdots \left(\delta^{i_r}_{\bar{i}_r} - t\xi^{i_r}_{,\bar{i}_r}\right)$$

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■ Image of tensor at $\tau(\tilde{x})^{i_1\dots i_r}_{j_1\dots j_s}$ under flow, backwards, $\tilde{x}\to x$, has the

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$$\phi^{*}\left(\tau(\tilde{x})\right)^{i_1\dots i_r}_{j_1\dots j_s} = \tau^{\bar{i}_1\dots\bar{i}_r}_{\bar{j}_1\dots\bar{j}_s}(\tilde{x})\,\frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}}\cdots\frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}}\,\frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}}\cdots\frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar{i}_1\dots\bar{i}_r}_{\bar{j}_1\dots\bar{j}_s}(x+t\xi)\left(\delta^{i_1}_{\bar{i}_1}-t\,\xi^{i_1}_{,\bar{i}_1}\right)\cdots\left(\delta^{i_r}_{\bar{i}_r}-t\,\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\cdot\left(\delta^{\bar{j}_1}_{j_1}+t\,\xi^{\bar{j}_1}_{,j_1}\right)\cdots\left(\delta^{\bar{j}_s}_{j_s}+t\,\xi^{\bar{j}_s}_{,j_s}\right)+O(t^2)$$

$$f_{,k} := \frac{\partial}{\partial x^k} f$$

$$= \tau^{i_1\dots i_r}_{j_1\dots j_s}(x) + t\,\tau^{i_1\dots i_r}_{j_1\dots j_s\,,k}(x)\,\xi^k(x)$$

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$$= \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(x + t\xi)\left(\delta^{i_1}_{\bar{i}_1} - t\xi^{i_1}_{,\bar{i}_1}\right) \cdots \left(\delta^{i_r}_{\bar{i}_r} - t\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\cdot \left(\delta^{\bar{j}_1}_{j_1} + t\xi^{\bar{j}_1}_{,j_1}\right) \cdots \left(\delta^{\bar{j}_s}_{j_s} + t\xi^{\bar{j}_s}_{,j_s}\right) + O(t^2)$$

$$f_{,k} := \frac{\partial}{\partial x^k} f$$

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are always to be summed over (Einstein convention)

components:

$$\phi^*(\tau(x))^{i_1\dots i_r}_{j_1\dots j_s} = \tau^{\bar i_1\dots\bar i_r}_{\bar j_1\dots \bar j_s}(\tilde x)\,\frac{\partial x^{i_1}}{\partial\tilde x^{\bar i_1}}\cdots\frac{\partial x^{i_r}}{\partial \tilde x^{\bar i_r}}\,\frac{\partial\tilde x^{\bar j_1}}{\partial x^{j_1}}\cdots\frac{\partial \tilde x^{\bar j_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar i_1\dots\bar i_r}_{\bar j_1\dots\bar j_s}(x+t\xi)\left(\delta^{i_1}_{\bar i_1} - t\,\xi^{i_1}_{,\bar i_1}\right)\cdots\left(\delta^{i_r}_{\bar i_r}-t\,\xi^{i_r}_{,\bar i_r}\right)$$

$$\cdot\left(\delta^{\bar j_1}_{j_1}+t\,\xi^{\bar j_1}_{j_1}\right)\cdots\left(\delta^{\bar j_s}_{j_s}+t\,\xi^{\bar j_s}_{j_s}\right)+O(t^2)$$

$f_{,k} := \frac{\partial}{\partial x^k} f$

$$= \tau^{i_1\dots i_r}_{j_1\dots j_s}(x) + t\,\tau^{i_1\dots i_r}_{j_1\dots j_s,k}(x)\,\xi^k(x)$$

$$- t\,\tau^{\bar i_1\dots i_r}_{j_1\dots j_s}(x)\,\xi^{i_1}_{,\bar i_1}(x) - \dots - t\,\tau^{i_1\dots \bar i_r}_{j_1\dots j_s}(x)\,\xi^{i_r}_{,\bar i_r}(x)$$

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components:

$$\phi^{*}(\tau(\tilde{x}))^{i_1 \dots i_r}_{j_1 \dots j_s} = \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(\tilde{x}) \frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}} \frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(x + t\xi)\left(\delta^{i_1}_{\bar{i}_1} - t\xi^{i_1}_{,\bar{i}_1}\right) \cdots \left(\delta^{i_r}_{\bar{i}_r} - t\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\cdot \left(\delta^{\bar{j}_1}_{j_1} + t\xi^{\bar{j}_1}_{,j_1}\right) \cdots \left(\delta^{\bar{j}_s}_{j_s} + t\xi^{\bar{j}_s}_{,j_s}\right) + O(t^2)$$

$$f_{,k} := \frac{\partial}{\partial x^k} f$$

$$= \tau^{i_1 \dots i_r}_{j_1 \dots j_s}(x) + t\, \tau^{i_1 \dots i_r}_{j_1 \dots j_s, k}(x)\, \xi^k(x)$$

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$$\phi^{*}(\tau(\tilde{x}))^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \tau^{\bar{i}_1 \ldots \bar{i}_r}_{\bar{j}_1 \ldots \bar{j}_s}(\tilde{x}) \frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}} \frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

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$$- t\,\tau^{\bar{i}_1 \ldots i_r}_{j_1 \ldots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x) - \ldots - t\,\tau^{i_1 \ldots \bar{i}_r}_{j_1 \ldots j_s}(x)\,\xi^{i_r}_{,\bar{i}_r}(x)$$

$$+ t\,\tau^{i_1 \ldots i_r}_{\bar{j}_1 \ldots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x) + \ldots + t\,\tau^{i_1 \ldots i_r}_{j_1 \ldots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

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$$\partial x^k$$

$$= \tau^{i_1\cdots i_r}_{j_1\cdots j_s}(x) + t\,\tau^{i_1\cdots i_r}_{j_1\cdots j_s,k}(x)\,\xi^k(x)$$

$$- t\,\tau^{\bar i_1\cdots i_r}_{j_1\cdots j_s}(x)\,\xi^{i_1}_{,\bar i_1}(x) - \cdots - t\,\tau^{i_1\cdots \bar i_r}_{j_1\cdots j_s}(x)\,\xi^{i_r}_{,\bar i_r}(x)$$

$$+ t\,\tau^{i_1\cdots i_r}_{\bar j_1\cdots j_s}(x)\,\xi^{\bar j_1}_{,j_1}(x) + \cdots + t\,\tau^{i_1\cdots i_r}_{j_1\cdots \bar j_s}(x)\,\xi^{\bar j_s}_{,j_s}(x)$$

$$\Rightarrow (L_\xi \tau)^{i_1\cdots i_r}_{j_1\cdots j_s}(x) = \lim_{t\to 0}\frac{1}{t}\left(\phi^{*-1}(\tau(\tilde x))^{i_1\cdots i_r}_{j_1\cdots j_s} - \tau^{i_1\cdots i_r}_{j_1\cdots j_s}(\overbrace{x(0)}^{x \,=})\right)$$

$$= \tau^{i_1\cdots i_r}_{j_1\cdots j_s,k}(x)\,\xi^k(x) - \tau^{\bar i_1\cdots i_r}_{j_1\cdots j_s}(x)\,\xi^{i_1}_{,\bar i_1}(x) - \cdots$$

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$$+t\,\tau^{i_1\cdots i_r}_{\bar{j}_1\cdots j_s}(x)\,\xi^{\bar{j}_1}{}_{,j_1}(x)+\dots+t\,\tau^{i_1\cdots i_r}_{j_1\cdots \bar{j}_s}(x)\,\xi^{\bar{j}_s}{}_{,j_s}(x)$$

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$$=\tau^{i_1\cdots i_r}_{j_1\cdots j_s,k}(x)\,\xi^{k}(x)-\tau^{\bar{i}_1\cdots i_r}_{j_1\cdots j_s}(x)\,\xi^{i_1}{}_{,\bar{i}_1}(x)-\dots$$

$$+\tau^{i_1\cdots i_r}_{\bar{j}_1\cdots j_s}(x)\,\xi^{\bar{j}_1}{}_{,j_1}(x)+\dots\tau^{i_1\cdots i_r}_{j_1\cdots \bar{j}_s}(x)\,\xi^{\bar{j}_s}{}_{,j_s}(x)$$

Equivalent to algebraic definition of L_ξ?

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□ Equivalent to algebraic definition of L_ξ ?

Yes: Check, e.g., that action on $\Lambda_0(M)$ and $\Lambda_1(M)$ is the same:

□ For $\tau \in \Lambda_0(M)$ we have $\tau = \tau(x)$

$$L_\xi \tau(x) = \xi^k \tau_{,k} = \xi^k \frac{\partial}{\partial x^k} \tau(x) \text{ is gradient } \checkmark$$

□ Co-Vector field: $\tau = \tau_j(x)\, dx^j \in \Lambda_1(M)$

$$L_\xi \tau(x) = \left(\xi^k(x)\, \tau_{j,k}(x) + \tau_k(x)\, \xi^k_{,j}(x)\right) dx^j$$

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$L_\xi : T_p^{(r,s)} \to T_p^{(r,s)}$ (i.e. not just ...)

- In particular, the Lie derivative of a vector field η is:

$$L_\xi : \eta \to L_\xi(\eta) = [\xi, \eta]$$

- One also finds:

$$L_{\xi+\eta} = L_\xi + L_\eta$$

$$L_{[\xi,\eta]} = [L_\xi, L_\eta] \quad \left(= L_\xi \circ L_\eta - L_\eta \circ L_\xi \right)$$

- Does it still obey a Leibniz rule?

Yes: $$L_\xi(\tau \otimes \sigma) = L_\xi(\tau) \otimes \sigma + \tau \otimes L_\xi(\sigma)$$

(tensors form an algebra w. respect to multiplication $\otimes$)

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