

Title: AMATH 875/PHYS 786 - Fall 2015 - Lecture 4

Date: Sep 25, 2015 01:30 PM

URL: <http://pirsa.org/15090005>

Abstract: <p>Course Description coming soon.</p>

GR for Cosmology, Achim Kempf, Fall 2015, Lecture 4

Differential forms (also called "exterior differential forms")

Why? Every p -dim. integration is an integration over a differential p -form!

Preparation:

Consider the cotangent space $T_p(M)^*$ at p :

□ Each $\omega \in T_p(M)^*$ is a lin. map:

$$\omega: T_p(M) \rightarrow \mathbb{R}$$

$$\omega: \xi \rightarrow \omega(\xi)$$

Preparation:

Consider the cotangent space $T_p(M)^*$ at p :

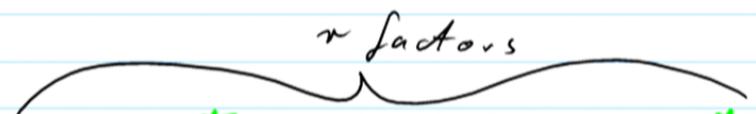
Each $\omega \in T_p(M)^*$ is a lin. map:

$$\omega: T_p(M) \rightarrow \mathbb{R}$$

$$\omega: \xi \rightarrow \omega(\xi)$$

Each such ω is a covariant tensor, of rank $(0, 1)$

More generally, consider the covariant tensors of rank $(0, r)$:



More generally, consider the covariant tensors of rank $(0, r)$:

□ Recall: $T_p(\mathcal{M})_r := \overbrace{T_p(\mathcal{M})^* \otimes \dots \otimes T_p(\mathcal{M})^*}^{r \text{ factors}}$

□ Each $v \in T_p(\mathcal{M})_r$ is a multi-linear map:

$$v: T_p(\mathcal{M})^* \rightarrow \mathbb{R}$$

□ In particular, if $\xi_1, \dots, \xi_r \in T_p(\mathcal{M})$ then:

$$v: \xi_1 \times \dots \times \xi_r \rightarrow v(\xi_1, \dots, \xi_r)$$

Definition:

$r > 1$ and $v \in T_p(M)_r$, then we define the "anti-symmetric part of v " as the image $\tilde{v} = A(v)$ of v under the linear antisymmetrization map A :

$$\tilde{v}(\xi_1, \dots, \xi_r) = A(v)(\xi_1, \dots, \xi_r)$$

$$:= \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) v(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)})$$

the sign (± 1) of the permutation σ

\leftarrow group of all $r!$ permutations of $(1, 2, \dots, r)$

Why consider these?

They will be key for integration! Only antisym. cov. tensors transform under chart change so as to match the Jacobian determinant arising in integrals when changing charts

Definition:

of $r > 1$ and $v \in T_p(M)_r$, then we

define the "anti-symmetric part of v "

as the image $\tilde{v} = A(v)$ of v under the linear antisymmetrization map A :

$$\tilde{v}(\xi_1, \dots, \xi_r) = A(v)(\xi_1, \dots, \xi_r)$$

the sign (± 1) of the permutation σ

$$:= \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) v(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)})$$

group of all $r!$ permutations of $(1, 2, \dots, r)$

Why consider these?

They will be key for integration! Only antisym. cov. tensors transform under chart change so as to match the Jacobian determinant arising in integrals when changing charts.

Concretely:

□ Consider $v := df \otimes dg$, for $f, g \in \mathcal{F}_p(U)$

□ Then $v(\xi_1, \xi_2) = df(\xi_1) dg(\xi_2)$ (which is $= \xi_1(f) \xi_2(g)$)

□ Apply A :

$$\tilde{v}(\xi_1, \xi_2) = Av(\xi_1, \xi_2) = \frac{1}{2} (df(\xi_1) dg(\xi_2) - df(\xi_2) dg(\xi_1))$$

□ \Rightarrow We can also write:

$$A(df \otimes dg) = \frac{1}{2} (df \otimes dg - dg \otimes df)$$

Proposition: A is a projector, i.e., it obeys

$$A \circ A = A$$

Check in above example:

$$A \circ A(df \otimes dg) = A\left(\frac{1}{2}df \otimes dg - \frac{1}{2}dg \otimes df\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}df \otimes dg - \frac{1}{2}dg \otimes df\right) - \frac{1}{2}\left(\frac{1}{2}dg \otimes df - \frac{1}{2}df \otimes dg\right)$$

$$= \frac{1}{2}(df \otimes dg - dg \otimes df)$$

$$= A(df \otimes dg)$$

Definition: \square For $r=0$ we define the set of differential 0-forms at $p \in M$ as:

$$\Lambda_0(p) := \mathbb{R} \quad \left(\begin{array}{l} \text{for 0 forms on the entire} \\ \text{manifold we will have } \Lambda_0 := \mathbb{F}(M) \end{array} \right)$$

\square For $r=1$ we define the set of

differential 1-forms (or "Pfaffian forms")

at $p \in M$ through:

$$\Lambda_1(p) := T_p^*(M),$$

Strategy now:

Define multiplication \rightarrow obtain algebra \rightarrow obtain derivatio-

The wedge product :

Def: If $\omega \in \Lambda_s(p)$, $\nu \in \Lambda_r(p)$, and $r, s \neq 0$ then the wedge product \wedge yields a new differential form:

$$\wedge : \Lambda_r(p) \times \Lambda_s(p) \rightarrow \Lambda_{r+s}(p)$$

$$\wedge : (\omega, \nu) \rightarrow \omega \wedge \nu = \underbrace{\frac{(s+r)!}{s!r!}}_{\text{a normalization factor}} A(\omega \otimes \nu)$$

Def: For $c \in \Lambda_0$, $\omega \in \Lambda_s$ we have $c \wedge \omega = c\omega$

Note: $dx^i \wedge dx^i = 0 \quad \forall i$
 \Uparrow

Example: For dx^i, dx^j we obtain:

The wedge product.

Def: If $\omega \in \Lambda_s(p)$, $\nu \in \Lambda_r(p)$, and $r, s \neq 0$ then the wedge product \wedge yields a new differential form:

$$\wedge : \Lambda_r(p) \times \Lambda_s(p) \rightarrow \Lambda_{r+s}(p)$$

$$\wedge : (\omega, \nu) \rightarrow \omega \wedge \nu = \underbrace{\frac{(s+r)!}{s!r!}}_{\text{a normalization factor}} A(\omega \otimes \nu)$$

Def: For $c \in \Lambda_0$, $\omega \in \Lambda_s$ we have $c \wedge \omega = c\omega$

Note: $dx^i \wedge dx^i = 0 \quad \forall i$

Example: For dx^i, dx^j we obtain:

$$dx^i \wedge dx^j = (dx^i \otimes dx^j - dx^j \otimes dx^i)$$

Properties of \wedge :

□ bi-linear:

$$(\omega + \nu) \wedge \gamma = \omega \wedge \gamma + \nu \wedge \gamma$$

$$(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \text{ if } a \in \mathbb{R}$$

□ associative:

$$(\omega \wedge \nu) \wedge \gamma = \omega \wedge (\nu \wedge \gamma)$$

□ "graded" commutative:

$$\omega \wedge \nu = (-1)^{rs} \nu \wedge \omega \text{ if } \omega \in \Lambda_r, \nu \in \Lambda_s$$

(E.g. for $dx^i \in \Lambda_1(p)$:
 $dx^i \wedge dx^i = -dx^i \wedge dx^i$)



□ bi-linear:

$$(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta$$

$$(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \text{ if } a \in \mathbb{R}$$

□ associative:

$$(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)$$

□ "graded" commutative:

(E.g. for $dx^i \in \Lambda_1(p)$:
 $dx^i \wedge dx^j = -dx^j \wedge dx^i$
 since $r=s=1$.)

$$\omega \wedge \nu = (-1)^{rs} \nu \wedge \omega \text{ if } \omega \in \Lambda_r, \nu \in \Lambda_s$$

□ We can use \wedge to build bases of $\Lambda_r(p)$:

Assume: $\{\theta^i\}_{i=1}^n$ is a basis of $\Lambda_1 = T_p(M)^*$.

(for example $\theta^i = dx^i$)

Then: $\{\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_r}\}_{1 \leq i_1 < i_2 < \dots < i_r \leq n}$

Exercise: show this \nearrow

is a basis of $\Lambda_r(p)$ for $r > 1$.

□ Therefore:

$$\dim(\Lambda_r(p)) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

\Rightarrow no diff. forms of degree $r > n$!

Example: For $p \in M = \mathbb{R}^3$ we have bases:

$$\Lambda_1 = \text{span}(dx^1, dx^2, dx^3)$$

$$\Lambda_2 = \text{span}(dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3)$$

$$\Lambda_3 = \text{span}(dx^1 \wedge dx^2 \wedge dx^3)$$

Definition:

$(\dim(\Lambda) = 2^n)$

$\Lambda(p) := \bigoplus_{i=0}^n \Lambda_i(p)$ equipped with the multiplication \wedge , is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$

Definition:

$(\dim(\Lambda) = 2^n)$

$\Lambda(p) := \bigoplus_{i=0}^n \Lambda_i(p)$ equipped with the multiplication \wedge , is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$.

Generalization to fields:

□ A differential form field is a mapping

□ A covariant tensor field of rank r on M is a mapping

Generalization to fields:

- ▢ A differential form field is a mapping that associates to each $p \in M$ an element:
$$\omega(p) \in \Lambda(p)$$

It is usually also called simply a differential form and denoted ω .

- ▢ These fields form the Grassmann algebra of differential forms, $\Lambda(M)$.

Recall:

Given an algebra, it is often useful to consider derivations of the algebra, i.e., to consider linear maps that obey the Leibniz rule. (similar to how we defined tangent vectors as derivations of $F(M)$)

Here: For the algebra $\Lambda(M)$, let us consider the exterior and the inner derivations:

Definition:

A linear map $\Phi: \Lambda(M) \rightarrow \Lambda(M)$ is called a derivation of degree k , if:

$$\Phi: \Lambda_s(M) \rightarrow \Lambda_{s+k}(M) \quad \text{for all } s$$

$$\Phi: \alpha \wedge \beta \rightarrow \Phi(\alpha) \wedge \beta + \alpha \wedge \Phi(\beta)$$

for all $\alpha, \beta \in \Lambda(M)$.

Leibniz rule 

Also:

Definition:

A linear map $\Phi: \Lambda(M) \rightarrow \Lambda(M)$ is called an anti-derivation of degree k , if for all $\alpha \in \Lambda^1(M)$, $\beta \in \Lambda(M)$:

$$\Phi: \Lambda_s(M) \rightarrow \Lambda_{s+k}(M) \quad \text{for all } s \text{ and}$$

$$\Phi: \alpha \wedge \beta \rightarrow \Phi(\alpha) \wedge \beta + (-1)^s \alpha \wedge \Phi(\beta)$$

"Anti-Leibniz rule"

Proposition: (as we will show constructively)

Because of the Leibniz rule and linearity, any (anti-)derivation

$$\Phi: \Lambda(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$$

is already fully determined by its action only on $\Lambda_0(\mathcal{M})$ and on a basis of $\Lambda_1(\mathcal{M})$.

The exterior derivative

The exterior derivative,

$$d: \Lambda(M) \rightarrow \Lambda(M)$$

is the anti-derivation of degree $k=1$ which is defined through:

$$a) \quad \left. \begin{array}{l} d: \Lambda_0(M) \rightarrow \Lambda_1(M) \\ d: f \rightarrow df \end{array} \right\} \text{action of } d \text{ on } \Lambda_0(M)$$

$$b) \quad \underline{d: dx^i \rightarrow 0} \text{ for all } i. \quad \left. \right\} \text{action of } d \text{ on a basis of } \Lambda_1(M)$$

In a chart:

$$\text{We had: } d: f(x) \rightarrow df(x) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Now we have more generally:

$$\beta = \sum_{i_1 < \dots < i_s} \beta_{i_1, \dots, i_s}(x) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s} \in \Lambda_s(M)$$

\uparrow recall: $f \wedge w = fw$ when $f \in \Lambda_0$ and $w \in \Lambda$

Q: So how to carry out $d: \beta \rightarrow d\beta$?

A: By applying the anti-Leibniz rule:

$$d\beta = \sum_{i_1 < \dots < i_s} \frac{\partial \beta_{i_1, \dots, i_s}(x)}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

$$\beta = \sum_{i_1 < \dots < i_s} \beta_{i_1, \dots, i_s}(x) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s} \in \Lambda_s(M)$$

↑ recall: $f \wedge w = fw$ when $f \in \Lambda_0$ and $w \in \Lambda$

Q: So how to carry out $d: \beta \rightarrow d\beta$?

A: By applying the anti-Leibniz rule:

$$d\beta = \sum_{i_1 < \dots < i_s} \frac{\partial \beta_{i_1, \dots, i_s}(x)}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

+ terms of the form $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$

= 0 because when applying Leibniz rule to $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$ we eventually arrive at $d(dx^i) = 0 \dots$

Proposition:

+ terms of the form $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$

= 0 because when applying Leibniz rule to $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$ we eventually arrive at $d(dx^i) = 0 \dots$!

Proposition:

$d: \Lambda(M) \rightarrow \Lambda(M)$ obeys:

$$d \circ d = 0$$

Proof:

$$d \circ d(\beta) = \sum_{\substack{i_1 < \dots < i_s \\ j, k}} \frac{\partial^2 \beta_{i_1, \dots, i_s}(x)}{\partial x^j \partial x^k} dx^k \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

$\underbrace{\hspace{10em}}_{\text{Sym. in } j, k} \quad \underbrace{dx^k \wedge dx^j = -dx^j \wedge dx^k}_{\text{i.e. anti-sym. in } j, k}$

$$= 0 \quad \Rightarrow \sum = 0$$

Example:

□ For $M = \mathbb{R}^3$ and $f \in \mathcal{F}(M)$ we have:

$$df = \sum_{i=1}^3 \frac{\partial f}{\partial x^i} dx^i$$

e.g.: electric potential

□ Notice: $\left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3} \right)$ is the
"Gradient field ∇f of f "

□ Example: Electric field $E = -d\phi$ from potential ϕ .

□ Now assume $\gamma \in \Lambda_1(M)$ is an arbitrary (i.e. not necessarily gradient) covariant vector field:

$$\gamma = \sum_{i=1}^3 \gamma_i(x) dx^i \in \Lambda_1(\mathbb{R}^3)$$

□ Then:

$$\beta := d\gamma = \sum_{i,j} \frac{\partial \gamma_i(x)}{\partial x^j} dx^j \wedge dx^i$$

$i=j$ does not occur because $dx^i \wedge dx^i = 0$

$$= \sum_{i < j} \left(\frac{\partial \gamma_i(x)}{\partial x^j} - \frac{\partial \gamma_j(x)}{\partial x^i} \right) dx^j \wedge dx^i$$

from $dx^j \wedge dx^i = -dx^i \wedge dx^j$

$$\gamma = \sum_{i=1}^3 \gamma_i(x) dx^i \in \Lambda_1(\mathbb{R}^3)$$

□ Then:

$$\beta := d\gamma = \sum_{i,j} \frac{\partial \gamma_i(x)}{\partial x^j} dx^j \wedge dx^i$$

$i=j$ does not occur because $dx^i \wedge dx^i = 0$

$$= \sum_{i < j} \left(\frac{\partial \gamma_i(x)}{\partial x^j} - \frac{\partial \gamma_j(x)}{\partial x^i} \right) dx^j \wedge dx^i$$

from $dx^j \wedge dx^i = -dx^i \wedge dx^j$

$$= - \left(\frac{\partial \gamma_1}{\partial x^2} - \frac{\partial \gamma_2}{\partial x^1} \right) dx^1 \wedge dx^2 - \left(\frac{\partial \gamma_1}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^1} \right) dx^1 \wedge dx^3$$

$i=j$ does not occur because $dx^i \wedge dx^i = 0$

$i, j \in \{1, 2, 3\}$

from $dx^j \wedge dx^i = -dx^i \wedge dx^j$

$$= \sum_{i < j} \left(\frac{\partial \gamma_i(x)}{\partial x^j} - \frac{\partial \gamma_j(x)}{\partial x^i} \right) dx^j \wedge dx^i$$

$$= - \overbrace{\left(\frac{\partial \gamma_1}{\partial x^2} - \frac{\partial \gamma_2}{\partial x^1} \right)}^{\beta_3} dx^1 \wedge dx^2$$

$$- \overbrace{\left(\frac{\partial \gamma_1}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^1} \right)}^{\beta_2} dx^1 \wedge dx^3$$

$$- \overbrace{\left(\frac{\partial \gamma_2}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^2} \right)}^{\beta_1} dx^2 \wedge dx^3$$

Notice: $(\beta_1(x), \beta_2(x), \beta_3(x))$ are the

components of the curl

$$\begin{aligned}
 &= - \left(\frac{\partial r_1}{\partial x^2} - \frac{\partial r_2}{\partial x^1} \right) dx^1 \wedge dx^2 \\
 &\quad - \overbrace{\left(\frac{\partial r_1}{\partial x^3} - \frac{\partial r_3}{\partial x^1} \right)}^{\beta_2 :=} dx^1 \wedge dx^3 \\
 &\quad - \underbrace{\left(\frac{\partial r_2}{\partial x^3} - \frac{\partial r_3}{\partial x^2} \right)}_{\beta_1 :=} dx^2 \wedge dx^3
 \end{aligned}$$

□ Notice: $(\beta_1(x), \beta_2(x), \beta_3(x))$ are the components of the curl

$$\beta = \nabla \times \gamma$$

It is called a "pseudo vector field"

It is really a 2-form field in 3 dim.

□ Recall:

Gradient vector fields

are curl free: $\nabla \times (\nabla f) = 0$

This is a special case of

$$d \circ d = 0$$

because if $\beta = d\gamma$ then:

$$d\beta = d^2\gamma = 0$$

Definition:

□ A differential form ω is called closed if:

$$d\omega = 0$$

E.g.: We saw that $\beta := dy$ is closed. Is this example typical?

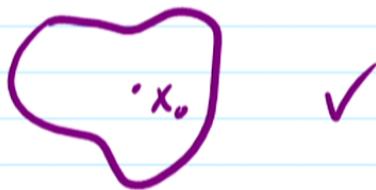
□ A differential form ω is called exact if there exists a v so that

$$\omega = dv \quad (v \text{ is like an anti-derivative!})$$

How are closedness and exactness related?

This actually depends on the global topology of the manifold! (because anti-derivatives are in a sense global)

Simplest case: Assume M is contractible



i.e., $\exists \overset{\text{continuous}}{F}: [0, 1] \times M \rightarrow M$

so that $F(0, x) = x \quad \forall x$

$F(1, x) = x_0 \quad \forall x$

\uparrow some fixed pt e , 25 / 32

Poincaré lemma:

On any contractible manifold:

$$\gamma \text{ exact} \iff \gamma \text{ closed}$$

E.g.

□ \mathbb{R}^n is contractible

□ $\mathbb{R}^n \setminus \{p\}$ is not contractible

↑ some arbitrary point $p \in M$

In general: We only have

$$\gamma \text{ exact} \Rightarrow \gamma \text{ closed}$$

which is because $d^2 = 0$.

\Rightarrow We obtain a tool for classifying the "global topology" of differentiable manifolds (checking for holes, handles etc.)

Def: This is called the "de Rham cohomology theory"

There are also other cohomology theories!

(Which is good because none yields a complete classification)

Example: K -theory

- This cohomology theory uses the fact that:
- There is only one way to put a vector bundle on a contractible mfd but others can have many non-isomorphic vector bundles.
- Recall that, e.g., for a suitable vector bundle B :



$$\pi^{-1}(U_1) \simeq U_1 \times \mathbb{R}^n$$

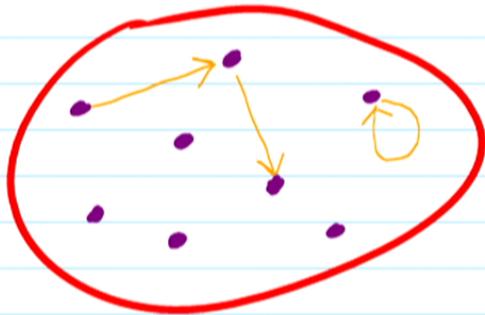
$$\pi^{-1}(U_2) \simeq U_2 \times \mathbb{R}^n$$

But: $B \neq M \times \mathbb{R}^n$

Comment: All cohomology theories are:

"Natural Transformations" between two "Categories".

Def: A category is a set of objects and morphisms:



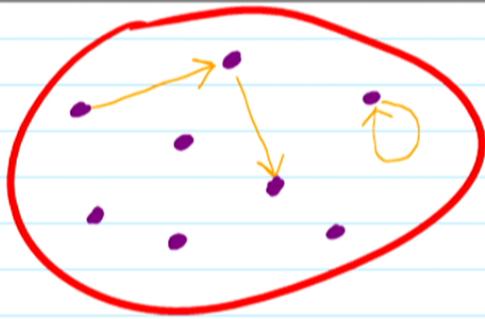
Axioms:

$$\square \exists A \rightarrow B \text{ and } B \rightarrow C \Rightarrow \exists A \rightarrow C$$

\square Associativity:

$$(\cdot \rightarrow \cdot \rightarrow \cdot) \rightarrow \cdot = \cdot \rightarrow (\cdot \rightarrow \cdot) \rightarrow \cdot$$

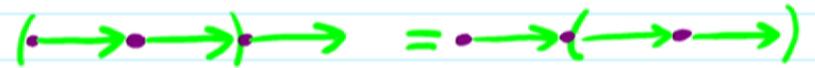
$$\square \forall A \cdot \exists \text{ loop}$$



Axioms:

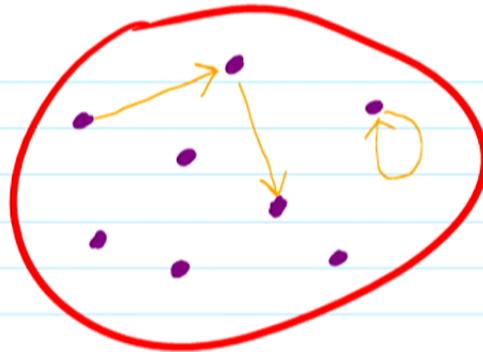
$$\square \exists A \rightarrow B \text{ and } B \rightarrow C \Rightarrow \exists A \rightarrow C$$

\square Associativity:



$$\square \forall v \exists \lambda$$

Examples:

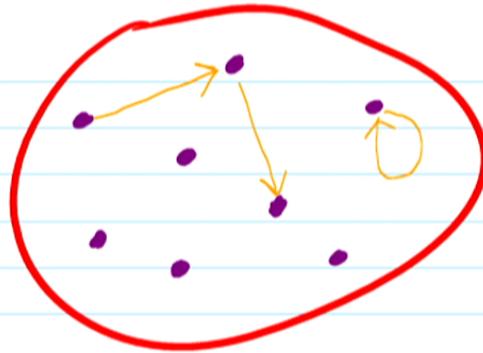


Category of vector spaces Vec:

Objects: all vector spaces

Morphisms: linear transformations

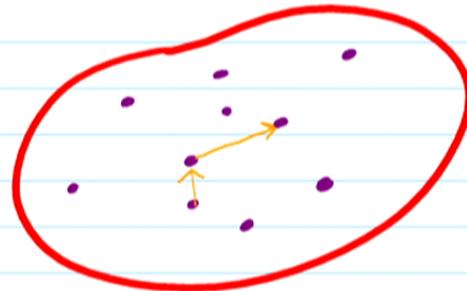
Examples:



Category of vector spaces Vec:

Objects: all vector spaces

Morphisms: linear transformations

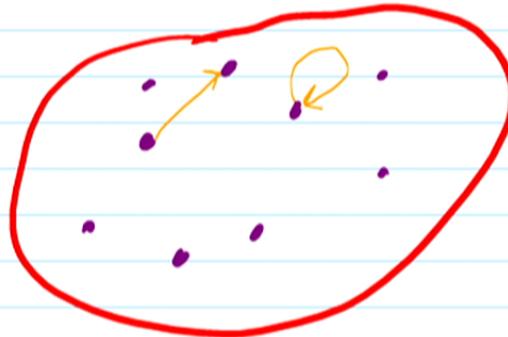


Category of associative algebras Alg:

Objects: all assoc. algebras

Morphisms: algebra homomorphisms

But also:

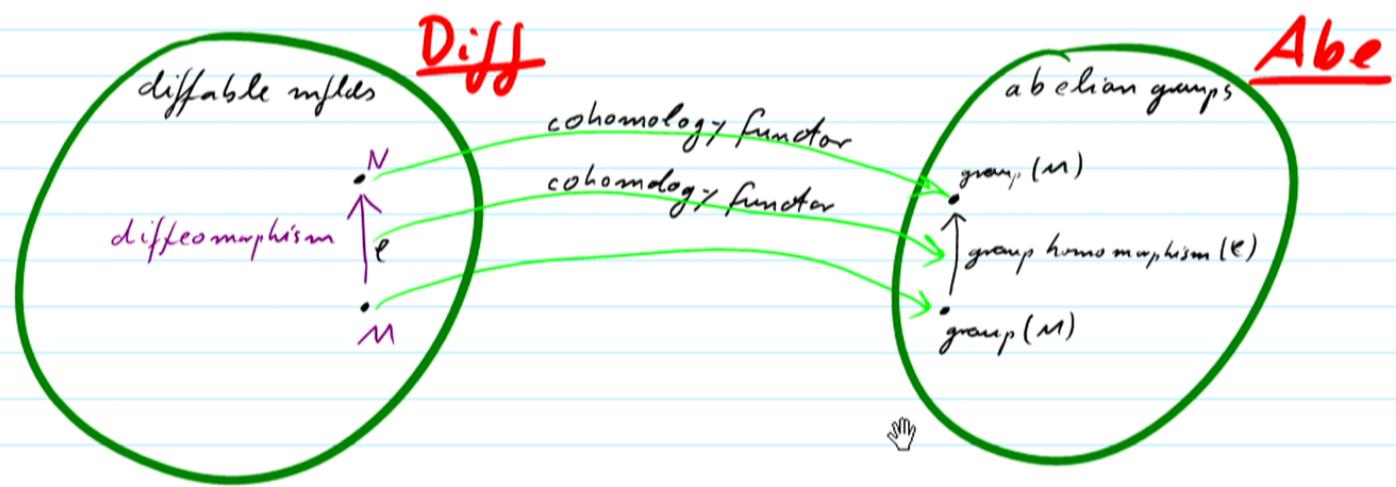


Category of categories Cat:

Objects: all categories

Morphisms: natural transformations
also called functors.

A cohomology theory is a morphism between two objects in Cat, namely Diff and Abe:



Crucial: $\text{group}(M)$ not homomorph $\text{group}(N) \Rightarrow M$ not diffeomorph to N

In this course:

We'll focus on the local properties of the manifolds, such as curvature.

→ We mention cohomology theory issues only on the side in this course.

Next: The inner  derivation of $\Lambda(M)$.